Discrete basis $\{|i\rangle\}$ $\qquad |\psi\rangle = \sum_i \psi_i |i\rangle$

$\psi_i = \langle i|\psi\rangle$

Vector : $\underline{\psi} = \begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\end{pmatrix}$ $\qquad \langle\psi| = \sum_i \langle i|\,\psi_i^*$

Operators: $\hat A = \sum_{i,j} A_{ij}\,|i\rangle\langle j|$ $\qquad A_{ij} = \langle i|\hat A|j\rangle$

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \cdots \\ \vdots & & \ddots \end{pmatrix}$$

$$|\phi\rangle = \hat A|\psi\rangle = \sum_{i,j} A_{ij}\,|i\rangle\langle j| \sum_k \psi_k |k\rangle = \sum_{i,j,k} A_{ij}\,\psi_k\,|i\rangle \underbrace{\langle j|k\rangle}_{\delta_{jk}}$$

$$= \sum_{i,j} \underbrace{A_{ij}\psi_j}_{\phi_i}\,|i\rangle = \sum_i \phi_i\,|i\rangle \qquad \phi_i = \sum_j A_{ij}\psi_j$$

$$\underline{\phi} = \begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\end{pmatrix} = A\,\underline{\psi} = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ \vdots & & \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\end{pmatrix}$$

Expectation:

$$\langle A\rangle = \langle\psi|\hat A|\psi\rangle = \sum_i \psi_i^*\langle i| \sum_{j,k} A_{jk}\,|j\rangle\langle k| \sum_l \psi_l\,|l\rangle$$

$$= \sum_{i,j,k,l} \psi_i^* A_{jk}\psi_l \underbrace{\langle i|j\rangle}_{\delta_{ij}}\underbrace{\langle k|l\rangle}_{\delta_{kl}} = \sum_{i,j} \psi_i^* A_{ik}\psi_k = \underline{\psi}^\dagger A\,\underline{\psi}$$

$(\psi_1^*, \psi_2^* \cdots)$

Continuous basis $\qquad$ position (1D)

$\hat q$ position operator $\qquad \hat q|q\rangle = q|q\rangle$

$\langle q'|q\rangle = \delta(q-q')$ $\qquad \psi(q) = \langle q|\psi\rangle$

$$|\psi\rangle = \int dq\, \underbrace{\psi(q)}_{\psi_q}\, |q\rangle \qquad |\psi\rangle = \sum_i \psi_i\,|i\rangle$$

$\hat A$ $\qquad A(q',q) = \langle q'|A|q\rangle$

$|\phi\rangle = \hat A|\psi\rangle$ $\qquad \mathbb{1} = \int dq'\,|q'\rangle\langle q'|$

$$\phi(q) = \langle q|\phi\rangle = \langle q|\hat A|\psi\rangle = \int dq' \underbrace{\langle q|\hat A|q'\rangle}_{A(q,q')}\underbrace{\langle q'|\psi\rangle}_{\psi(q')} = \int dq'\, A(q,q')\,\psi(q')$$

Ex a) Potential $V(q',q) = V(q)\,\delta(q'-q) = \langle q'|\hat{V}|q\rangle$

$$\langle q|\hat{V}\underset{\mathbb{1} = \int dq''|q''\rangle\langle q''|}{\,}|\psi\rangle = \int dq'' \underbrace{\langle q|\hat{V}|q''\rangle}_{V(q)\delta(q-q'')}\underbrace{\langle q''|\psi\rangle}_{\psi(q'')}$$

$$= V(q)\,\psi(q)$$

b) Momentum $\langle q|\hat{p}|\psi\rangle = -i\hbar \frac{\partial}{\partial q}\psi(q)$

$$\underbrace{p(q',q)} = i\hbar\,\delta'(q'-q)$$

$$\langle q|\hat{p}\underset{\mathbb{1} = \int dq'|q'\rangle\langle q'|}{\,}|\psi\rangle = \int dq' \underbrace{\langle q|\hat{p}|q'\rangle}_{i\hbar\delta'(q-q')}\underbrace{\langle q'|\psi\rangle}_{\psi(q')} = i\hbar\int dq'\,\delta'(q-q')\psi(q')$$

$$= -i\hbar\int dq'\,\delta(q-q')\frac{\partial}{\partial q'}\psi(q') = -i\hbar\frac{\partial}{\partial q}\psi(q)$$

To basis sets: $\{|a_n\rangle\}$ $\{|b_m\rangle\}$

$$|b_m\rangle = \sum_k \underbrace{U_{km}}_{\hookrightarrow \langle a_k|b_m\rangle}|a_k\rangle \qquad U_{km} \text{ is unitary.}$$

$$\hat{U}\hat{U}^+ = \mathbb{1} \quad \sum_m U_{nm}U^*_{mn'} = \sum_m \langle a_n|b_m\rangle\langle b_m|a_{n'}\rangle = \langle a_n|\Big(\underbrace{\sum_m |b_m\rangle\langle b_m|}_{\mathbb{1}}\Big)|a_{n'}\rangle = \delta_{nn'}$$

Ex: $\psi(q) = \langle q|\psi\rangle$ $\quad \psi(p) = \langle p|\psi\rangle$ $\quad \hat{p}|p\rangle = p|p\rangle$

$$\langle q|p\rangle \overset{\text{exercise}}{=} \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}qp} = U_{qp} = U(q,p)$$

$$\hat{U}\cdot\hat{U}^+ = \frac{1}{2\pi\hbar}\int dp\, e^{\frac{i}{\hbar}qp}e^{-\frac{i}{\hbar}q'p} = \frac{1}{2\pi\hbar}\int dp\, e^{\frac{i}{\hbar}p(q-q')} \underset{?}{=} \delta(q-q')$$

exercise

Two level systems

Two basis states $\{|0\rangle, |1\rangle\}$

$$|\psi\rangle = \sum_{k=0}^{1} c_k |k\rangle \qquad \underline{\Psi} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \qquad \text{"} |\psi\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \text{"}$$

Hermitian op:

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

$$A^+ = \begin{pmatrix} A_{00}^* & A_{10}^* \\ A_{01}^* & A_{11}^* \end{pmatrix} \qquad A^+ = A \Rightarrow \overset{a}{\overset{\shortparallel}{A_{00}}}, \overset{d}{\overset{\shortparallel}{A_{11}}} \in \mathbb{R}$$

$$A_{10}^* = A_{01} = b - ic \qquad A = \begin{pmatrix} a & b-ic \\ b+ic & d \end{pmatrix}$$

$$a, b, c, d \in \mathbb{R}$$

Basis for 2x2 Hermitian matrices

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0 \qquad \sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A = a_0 \mathbb{1} + \sum_m a_m \sigma_m = \sum_{m=0}^{3} a_m \sigma_m$$

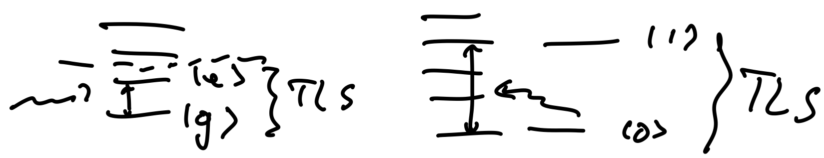
$$[\sigma_i, \sigma_j] = 2i \sum_k \epsilon_{ijk} \sigma_k$$

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if two index eq.} \\ 1 & \text{for even perm. of } 123 \\ -1 & \text{odd} \end{cases}$$

$$\epsilon_{123} = 1 \qquad \epsilon_{132} = -1 \qquad \epsilon_{312} = 1 \qquad \vdots$$

$$\sigma_i \sigma_j = i \sum_k \epsilon_{ijk} \sigma_k + \delta_{ij} \cdot \mathbb{1}$$

Physical systems:
- spin-$\frac{1}{2}$
- photon polarization
- Atom



$\vdots$

qubit:

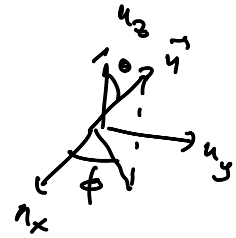
$|0\rangle$

$|1\rangle$

"Rotated" Pauli matrices

$\sigma_n = \vec{n}\cdot\vec{\sigma}$ $\quad |\vec{n}|=1$ $\quad \vec{n} = (n_x, n_y, n_z)$

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$



$n_x = \sin\theta\cos\phi$

$n_y = \sin\theta\sin\phi$

$n_z = \cos\theta$

$$\sigma_n = n_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + n_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + n_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}n_z & n_x - in_y\\ n_x + in_y & -n_z\end{pmatrix}$$

$$= \begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\phi}\\ \sin\theta\, e^{i\phi} & -\cos\theta\end{pmatrix}$$

Eigenvalues: $|\sigma_n - \lambda\mathbb{1}| = 0$ $\qquad \begin{vmatrix}\cos\theta-\lambda & \sin\theta\, e^{-i\phi}\\ \sin\theta\, e^{i\phi} & -\cos\theta-\lambda\end{vmatrix} =$

$$= \underbrace{-(\cos\theta-\lambda)(+\cos\theta+\lambda)}_{\lambda^2-\cos^2\theta} - \sin^2\theta = \lambda^2 - 1 = 0$$

$\lambda^2 = 1$

$\lambda = \pm 1$.

Exercise:

Eigenvectors: $\lambda = +1$: $\underline{\Psi_{+n}} = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}$

-1: $\Psi_{-n} = \begin{pmatrix}-e^{-i\phi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$

$$\langle\vec{\sigma}\rangle = \langle\psi_{+n}|\vec{\sigma}|\psi_{+n}\rangle = \vec{n}$$

General state: $\Psi = \begin{pmatrix}c_0\\ c_1\end{pmatrix}$ $\qquad \Psi^*\Psi = (c_0^*, c_1^*)\begin{pmatrix}c_0\\ c_1\end{pmatrix} = |c_0|^2 + |c_1|^2 = 1$

Claim: $\left.\begin{array}{l} c_0 = \cos\frac{\theta}{2}\\ c_1 = e^{i\phi}\sin\frac{\theta}{2}\end{array}\right\} \Rightarrow \theta, \phi$