UNIVERSITY OF OSLO

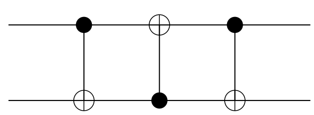
Faculty of Mathematics and Natural Sciences

Exam in:	**FYS4110/9110 Modern Quantum Mechanics**
Day of exam:	**3. December 2020**
Exam hours:	**15.00-19.00, 4 hours**
This examination paper consists of 4 pages	
Permitted materials:	**Approved electronic calculator.**
	Angell and Lian: Størrelser og enheter i fysikken
	Rottmann: Matematisk formelsamling

Language: The solutions may be written in Norwegian or English depending on your own preference.
Make sure that your copy of this examination paper is complete before answering.
All answers should be justified

PROBLEM 1
SWAP gate

a) A useful quantum gate is the SWAP gate which interchanges the state of two qubits. That is, if the input state is $|\psi\rangle \otimes |\phi\rangle$ the output will be $SWAP|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$. Show that the following quantum circuit on two qubits gives a decomposition of the SWAP gate using three CNOT gates.



b) Write the matrix for the SWAP gate. Describe which basis you use.

c) Generalize the above circuit to implement the SWAP gate on two n-qubit registers using only CNOT gates. How many CNOT gates do you need?

PROBLEM 2
Sending information with entangled photons?

The violation of Bell's inequality by certain quantum states is often interpreted as an expression of quantum non-locality. Entangled states are non-local in the sense that one can not assign a pure quantum state to individual particles, but only to the system as a whole. However, this non-locality is of a special form that prevents any information to be transmitted using quantum entanglement. Consider a bipartite system with two parties

A and B. The full system is in a pure quantum state $|\psi\rangle$. Party B could try to transmit information to A in two ways, either making a unitary transformation on subsystem B or a measurement on subsystem B. We will show that neither of these changes the expectation values of observables on subsystem A. We remind you of the Schmidt decomposition of a pure quantum state. For any $|\psi\rangle$ there exist orthonormal bases $|n_i^A\rangle$ and $|n_i^B\rangle$ for the Hilbert spaces of A and B such that

$$|\psi\rangle = \sum_i d_i |n_i^A\rangle \otimes |n_i^B\rangle.$$

a) Define the reduced density matrix of subsystem A and show that the expectation value of all possible observables on subsytem A can be found from the reduced density matrix.

b) Show that the reduced density matrix of A does not change when applying a unitary transformation to B.

c) Show that the reduced density matrix of A does not change when making a measurement on B, as long as we do not know the result of the measurement.

d) What happens with the density matrix for A if we get to know the outcome of the measurement on B?

PROBLEM 3

Charge transfer by adiabatic passage

We have three quantum dots in a row and one electron. Each dot has one state for an electron, so that the electron has three possible states, $|1\rangle$, $|2\rangle$ and $|3\rangle$ (and it can of course also be in superpositions of these). The three basis states are orthogonal and normalized. The motion of the electron can be controlled by gates which change the tunneling amplitude between the dots. The system is described by the Hamiltonian

$$H = -\hbar \begin{pmatrix} 0 & \Omega_1 & 0 \\ \Omega_1 & 0 & \Omega_2 \\ 0 & \Omega_2 & 0 \end{pmatrix}.$$

Here Ω_1 is the tunneling amplitude between dots 1 and 2 while Ω_2 is the tunneling amplitude between dots 2 and 3. Both amplitudes are controllable and can be time dependent. The initial state of the electron is $|1\rangle$, which means that the electron is localized on the first dot.

a) Consider first the situation where $\Omega_1 > 0$ is constant and $\Omega_2 = 0$. Find the time dependent state $|\psi(t)\rangle$ if the initial state is $|\psi(0)\rangle = |1\rangle$. Explain in words what this means physically.

b) We now let the matrix elements $\Omega_i(t)$ be time dependent, which means that the Hamiltonian also is time dependent. We define the instantaneous eigenstates and eigenvalues of the Hamiltonian as

$$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle$$

with $n = 1, 2, 3$. Find the states $|n(t)\rangle$ and energies $E_n(t)$ expressed in terms of the matrix elements $\Omega_i(t)$.

c) To study the dynamics of the system we will use a transformed representation of the state. Define the time dependent unitary tranformation $T(t)$ by

$$T(t)|n(0)\rangle = |n(t)\rangle.$$

We define the transformed representation of the state as

$$|\psi'(t)\rangle = T(t)^\dagger|\psi(t)\rangle.$$

Show that the time dependence of the state $|\psi'(t)\rangle$ is given by the Schrödinger equation with a transformed Hamiltonian $H'(t)$ and derive the expression for $H'(t)$ in terms of $H(t)$ and $T(t)$.

d) We now choose the time dependence

$$\Omega_1(t) = \Omega_m e^{-\frac{(t-(t_m+\sigma)/2)^2}{2\sigma^2}}$$
$$\Omega_2(t) = \Omega_m e^{-\frac{(t-(t_m-\sigma)/2)^2}{2\sigma^2}}$$

We want to start the dynamical evolution at $t = 0$ and stop at $t = t_m$. At both these times, we want to have $\Omega_i \approx 0$ so that no tunnelling takes place. This means that we must choose $\sigma \ll t_m$, which implies that

$$\frac{\Omega_1(0)}{\Omega_2(0)} = e^{-t_m/2\sigma} \ll 1.$$

Show that the transformed Hamiltonian $H'(t)$ is

$$H'(t) = -\hbar\Omega(t)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + i\hbar\frac{d\theta}{dt}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{1}$$

where

$$\tan\theta(t) = \frac{\Omega_1(t)}{\Omega_2(t)} \quad \text{and} \quad \Omega(t) = \sqrt{\Omega_1(t)^2 + \Omega_2(t)^2}.$$

e) $\frac{d\theta}{dt}$ can be made arbitrarily small by changing the Hamiltonian slowly. In the following we will assume that the change is so small that we can neglect the final term in (1). We start from $|1\rangle$ at $t = 0$ and evolve slowly in time, find the final state at $t = t_m$.

f) What is the probability of finding the electron in the state $|2\rangle$ during the process? Comment on the result.