

**FYS 4110/9110 Modern Quantum Mechanics  
Exam, Fall Semester 2021. Solution**

**Problem 1: SWAP gate**

a) We write  $|\psi\rangle = a|0\rangle + b|1\rangle$  and  $|\phi\rangle = c|0\rangle + d|1\rangle$  and get

$$\begin{aligned} |\psi\rangle \otimes |\phi\rangle &= (a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) \\ &\xrightarrow{CNOT} a|0\rangle(c|0\rangle + d|1\rangle) + b|1\rangle(c|1\rangle + d|0\rangle) \\ &\xrightarrow{CNOT} ac|00\rangle + ad|11\rangle + bc|01\rangle + bd|10\rangle \\ &\xrightarrow{CNOT} ac|00\rangle + ad|10\rangle + bc|01\rangle + bd|11\rangle \\ &= (c|0\rangle + d|1\rangle)(a|0\rangle + b|1\rangle) = |\phi\rangle \otimes |\psi\rangle. \end{aligned}$$

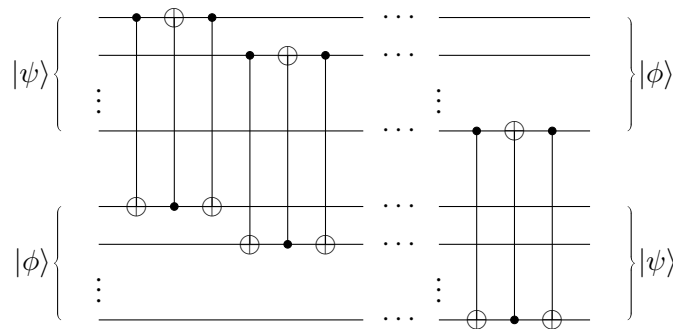
b) In the basis  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$  the action of SWAP on the basis vectors is

$$|00\rangle \xrightarrow{SWAP} |00\rangle, \quad |01\rangle \xrightarrow{SWAP} |10\rangle, \quad |10\rangle \xrightarrow{SWAP} |01\rangle, \quad |11\rangle \xrightarrow{SWAP} |11\rangle,$$

which gives the matrix

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c) We can SWAP multi-qubit registers one qubit at a time



We need  $3n$  CNOT gates.

## Problem 2: Sending information with entangled photons?

- a) The reduced density matrix of system A is given by the partial trace of the full density matrix over system B. The full density matrix is given by

$$\rho = |\phi\rangle\langle\phi| = \sum_{ij} d_i d_j^* |n_i^A\rangle\langle n_j^A| \otimes |n_i^B\rangle\langle n_j^B|.$$

Calculating the partial trace in the basis  $|n_i^B\rangle$  we see that only terms with  $i = j$  contribute, so the reduced density matrix is

$$\rho_A = \sum_i |d_i|^2 |n_i^A\rangle\langle n_i^A|.$$

The expectation value of an operator  $A \otimes \mathbb{1}$  on A is

$$\begin{aligned} \langle A \rangle &= \text{Tr}(A \otimes \mathbb{1} \rho) = \sum_{kl} \langle n_k^A n_l^B | A \otimes \mathbb{1} \rho | n_k^A n_l^B \rangle = \sum_{kl} \langle n_k^A n_l^B | \sum_{ij} d_i d_j^* A | n_i^A \rangle \langle n_j^A | \otimes | n_i^B \rangle \langle n_j^B | | n_k^A n_l^B \rangle \\ &= \sum_k \langle n_k^A | A \sum_i |d_i|^2 |n_i^A\rangle\langle n_i^A| | n_k^A \rangle = \text{Tr}(A \rho_A). \end{aligned}$$

- b) Applying the unitary transformation  $U$  to system B means applying  $U = \mathbb{1} \otimes U_B$  to the full system. We have the reduced density matrix for A after the transformation

$$\begin{aligned} \rho'_A &= \text{Tr}_B[\mathbb{1} \otimes U_B \rho \mathbb{1} \otimes U_B^\dagger] = \sum_{ijk} d_i d_j^* |n_i^A\rangle\langle n_j^A| \langle n_k^B | U_B | n_i^B \rangle \langle n_j^B | U_B^\dagger | n_k^B \rangle \\ &= \sum_{ijk} d_i d_j^* |n_i^A\rangle\langle n_j^A| \langle n_j^B | U_B^\dagger | n_k^B \rangle \langle n_k^B | U_B | n_i^B \rangle \\ &= \sum_i |d_i|^2 |n_i^A\rangle\langle n_i^A| = \rho_A. \end{aligned}$$

So the reduced density matrix does not change.

- c) An observable on system B has the form  $\mathbb{1} \otimes B$ . Let the eigenstates of  $B$  be given by

$$B|\phi_i^B\rangle = \lambda_i|\phi_i^B\rangle.$$

Similarly to the Schmidt decomposition we can write the full state as

$$|\psi\rangle = \sum_i \sqrt{p_i} |\phi_i^A\rangle \otimes |\phi_i^B\rangle.$$

The only difference is that when choosing the basis  $|\phi_i^B\rangle$  for B we are not guaranteed that the corresponding states  $|\phi_i^A\rangle$  are orthogonal. Here  $p_i$  are the probabilities of the different measurement outcomes. We have that the reduced density matrix for A is

$$\rho_A = \sum_i p_i |\phi_i^A\rangle\langle\phi_i^A|.$$

We measure the outcome  $\phi_i^B$  with probability  $p_i$ , collapsing the wavefunction for A to  $|\phi_i^A\rangle$ . As long as we do not get to know the outcome of the measurement, the state of A is the mixed state

$$\rho'_A = \sum_i p_i |\phi_i^A\rangle\langle\phi_i^A|.$$

The state changes from an entangled state to a mixed state, but the density matrix is unchanged.

- d) If we get to know the outcome of the measurement on B, the state collapses and the density matrix corresponds to that state. If the outcome is  $\phi_i^B$  the density matrix of A is

$$\rho_A^i = |\phi_i^A\rangle\langle\phi_i^A|.$$

### Problem 3: Charge transfer by adiabatic passage

We have three quantum dots in a row and one electron. Each dot has one state for an electron, so that the electron has three possible states,  $|1\rangle$ ,  $|2\rangle$  and  $|3\rangle$  (and it can of course also be in superpositions of these). The three basis states are orthogonal and normalized. The motion of the electron can be controlled by gates which change the tunneling amplitude between the dots. The system is described by the Hamiltonian

$$H = -\hbar \begin{pmatrix} 0 & \Omega_1 & 0 \\ \Omega_1 & 0 & \Omega_2 \\ 0 & \Omega_2 & 0 \end{pmatrix}.$$

Here  $\Omega_1$  is the tunneling amplitude between dots 1 and 2 while  $\Omega_2$  is the tunneling amplitude between dots 2 and 3. Both amplitudes are controllable and can be time dependent. The initial state of the electron is  $|1\rangle$ , which means that the electron is localized on the first dot.

- a) When  $\Omega_1 > 0$  is constant and  $\Omega_2 = 0$  the Hamiltonian is proportional to  $\sigma_x$  in the  $\{|1\rangle, |2\rangle\}$  subspace, and the corresponding eigenvectors are  $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$  with eigenvalues  $\mp\hbar\Omega_1$ . We have that the initial state  $|1\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle + |\psi^-\rangle)$ , so

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|1\rangle = \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}Ht}(|\psi^+\rangle + |\psi^-\rangle) = \frac{1}{\sqrt{2}}(e^{i\Omega_1 t}|\psi^+\rangle + e^{-i\Omega_1 t}|\psi^-\rangle) = \cos\Omega_1 t|1\rangle + i\sin\Omega_1 t|2\rangle.$$

This means that the electron is oscillating between quantum dots 1 and 2.

- b) The eigenvalues  $E = \hbar\lambda$  are found from

$$\begin{vmatrix} \lambda & \Omega_1 & 0 \\ \Omega_1 & \lambda & \Omega_2 \\ 0 & \Omega_2 & \lambda \end{vmatrix} = \lambda(\lambda^2 - \Omega_2^2) - \Omega_1^2\lambda = 0$$

which gives the energies

$$E_0 = 0, \quad E_{\pm} = \pm \hbar \Omega, \quad \Omega = \sqrt{\Omega_1^2 + \Omega_2^2}.$$

The corresponding eigenvectors are

$$\begin{aligned} |n_0\rangle &= \cos \theta |1\rangle - \sin \theta |3\rangle, \\ |n_{\pm}\rangle &= \frac{1}{\sqrt{2}} (\sin \theta |1\rangle \mp |2\rangle + \cos \theta |3\rangle). \end{aligned}$$

with

$$\sin \theta = \frac{\Omega_1}{\Omega}, \quad \cos \theta = \frac{\Omega_2}{\Omega}.$$

c) We have

$$i\hbar \frac{d}{dt} |\psi'\rangle = i\hbar \dot{T}^\dagger |\psi\rangle + T^\dagger i\hbar \frac{d}{dt} |\psi\rangle = (T^\dagger H T + i\hbar \dot{T}^\dagger T) |\psi'\rangle,$$

which is the Schrödinger equation with the transformed Hamiltonian

$$H' = T^\dagger H T + i\hbar \dot{T}^\dagger T.$$

d) The condition

$$\tan \theta(0) = \frac{\Omega_1(0)}{\Omega_2(0)} \ll 1$$

implies that  $\theta(0) \approx 1$ . This means that the eigenvectors at  $t = 0$  are approximately

$$|n_0(0)\rangle = |1\rangle, \quad |n_{\pm}(0)\rangle = \frac{1}{\sqrt{2}} (\mp |2\rangle + |3\rangle).$$

From this we see that the transformation

$$T(t) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

and we can calculate the Hamiltonian

$$H'(t) = -\hbar \Omega(t) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + i\hbar \frac{d\theta}{dt} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (1)$$

e) At  $t = t_m$  we have

$$\tan \theta(t_m) = \frac{\Omega_1(t_m)}{\Omega_2(t_m)} = e^{t_m/2\sigma} \gg 1$$

which means that  $\theta(t_m) \approx \frac{\pi}{2}$ . When neglecting the term proportional to  $\frac{d\theta}{dt}$  in the Hamiltonian we get that  $H'|1\rangle = 0$ , so the state will not change in time, giving  $|\psi'(t_m)\rangle \approx |1\rangle$ . We then get

$$|\psi(t_m)\rangle = T(t_m)|1\rangle = -|3\rangle.$$

The electron is transferred from dot 1 to dot 3.

f) At intermediate times, the state will be

$$|\psi(t)\rangle = T(t)|1\rangle = \cos\theta|1\rangle - \sin\theta|3\rangle.$$

The probability of finding the electron in state  $|2\rangle$  is zero during the process. This is a bit surprising, as the Hamiltonian only has terms for tunneling from dot 1 to 2 and from dot 2 to 3. So there is no term that allows the electron to tunnel directly from dot 1 to dot 3, it has to pass through dot 2 on the way. At a finite rate of change,  $\frac{d\theta}{dt}$ , we would not have the probability to be on dot 2 exactly zero, but it goes to zero as  $\frac{d\theta}{dt} \rightarrow 0$ . The tunneling rates are so adjusted in time, that as soon as the electron comes to dot 2 it is immediately tunneling on to dot 3.