F4110, Exam 2008

Solutions

Problem 1

a) $\hat{H}|0,+1\rangle = \frac{1}{2}\hbar(\omega_0+\omega_1)|0,+1\rangle + \lambda\hbar|1,-1\rangle$

$\hat{H}|1,-1\rangle = \frac{1}{2}\hbar(3\omega_0-\omega_1)|1,-1\rangle + \lambda\hbar|0,+1\rangle$

matrix form:

$$H = \hbar\begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad \text{with} \quad a = \tfrac{1}{2}(\omega_0+\omega_1), \quad b = \lambda, \quad c = \tfrac{1}{2}(3\omega_0-\omega_1)$$

written as:

$$H = \hbar\Delta\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} + \hbar\varepsilon\mathbb{1}$$

$\Rightarrow \quad a = \Delta\cos\theta + \varepsilon, \quad b = \Delta\sin\theta, \quad c = -\Delta\cos\theta + \varepsilon$

$\Rightarrow \quad \underline{\varepsilon = \frac{1}{2}(a+b) = \omega_0}, \quad \underline{\Delta\cos\theta = \frac{1}{2}(a-b) = \frac{1}{2}(\omega_1-\omega_0)}, \quad \underline{\Delta\sin\theta = \lambda}$

b) Write $H = \hbar\Delta M + \hbar\varepsilon\mathbb{1}$

Eigenvalue problem for M: $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \delta\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} \cos\theta-\delta & \sin\theta \\ \sin\theta & -\cos\theta-\delta \end{vmatrix} = 0 \Rightarrow \delta^2 - \cos^2\theta - \sin^2\theta = 0 \Rightarrow \underline{\delta = \pm 1}$$

Energy eigenvalues $\underline{E_\pm = \hbar(\varepsilon \pm \Delta)}$

Eigenvectors $(\cos\theta \mp 1)\alpha + \sin\theta\,\beta = 0 \Rightarrow \dfrac{\beta}{\alpha} = \mp\dfrac{1\pm\cos\theta}{\sin\theta}$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_\pm = N_\pm\begin{pmatrix} \mp\sin\theta \\ 1\pm\cos\theta \end{pmatrix} \quad \text{with } N_\pm^{-2} = \sin^2\theta + (1\pm\cos\theta)^2 = 2(1\pm\cos\theta)$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} \mp\dfrac{\sin\theta}{\sqrt{1\pm\cos\theta}} \\ \sqrt{1\pm\cos\theta} \end{pmatrix} = \underline{\frac{1}{\sqrt{2}}\begin{pmatrix} \mp\sqrt{1\mp\cos\theta} \\ \sqrt{1\pm\cos\theta} \end{pmatrix}}$$

or $\underline{|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(\mp\sqrt{1\mp\cos\theta}\,|0,+1\rangle + \sqrt{1\pm\cos\theta}\,|1,-1\rangle\right)}$

c) Density operator

$$\rho_\pm = \tfrac{1}{2}(1 \mp \cos\theta)\,|0\rangle\langle 0| \otimes |{+1}\rangle\langle{+1}| + \tfrac{1}{2}(1 \pm \cos\theta)\,|1\rangle\langle 1| \otimes |{-1}\rangle\langle{-1}|$$

$$\mp \tfrac{1}{2}\sin\theta \left(|0\rangle\langle 1| \otimes |{+1}\rangle\langle{-1}| + |1\rangle\langle 0| \otimes |{-1}\rangle\langle{+1}| \right)$$

Reduced density operators

position $\rho_\pm^P = \mathrm{Tr}_S\, \rho_\pm = \tfrac{1}{2}(1 \mp \cos\theta)\,|0\rangle\langle 0| + \tfrac{1}{2}(1 \pm \cos\theta)\,|1\rangle\langle 1|$

spin $\rho_\pm^S = \mathrm{Tr}_P\, \rho_\pm = \tfrac{1}{2}(1 \mp \cos\theta)\,|{+1}\rangle\langle{+1}| + \tfrac{1}{2}(1 \pm \cos\theta)\,|{-1}\rangle\langle{-1}|$

Entropies

$$S_\pm^P = S_\pm^S = -\left[\tfrac{1}{2}(1-\cos\theta)\log\left(\tfrac{1}{2}(1-\cos\theta)\right) + \tfrac{1}{2}(1+\cos\theta)\log\left(\tfrac{1}{2}(1+\cos\theta)\right)\right]$$

$$= \underline{-\left[\cos^2\tfrac{\theta}{2}\log\cos^2\tfrac{\theta}{2} + \sin^2\tfrac{\theta}{2}\log\sin^2\tfrac{\theta}{2}\right]} \equiv S$$

gives the measure of entanglement between spin and position

$\cos\theta = 0 \;(\theta = \tfrac{\pi}{2}) \Rightarrow \cos^2\tfrac{\theta}{2} = \sin^2\tfrac{\theta}{2} = \tfrac{1}{2} \Rightarrow \underline{S = \log 2}$ max. entanglement

$\cos\theta = \pm 1 \;(\theta = 0, \pi) \Rightarrow \cos^2\tfrac{\theta}{2} = 1,\ \sin^2\tfrac{\theta}{2} = 0$ or $\cos^2\tfrac{\theta}{2} = 0,\ \sin^2\tfrac{\theta}{2} = 1$

$\Rightarrow \underline{S = 0}$ minimal entanglement

Problem 2

a) $x_{BA} = y_{BA} = 0$ due to rotational invariance about the z-axis (vanish under φ-integration, since ψ_A and ψ_B are φ independent)

z-component : $z = r\cos\theta \Rightarrow$

$$z_{BA} = \frac{1}{\sqrt{32}}\frac{1}{\pi a_0^3}\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta \int_0^\infty dr\, r^2\, r\cos\theta\, \cos\theta\, \frac{r}{a_0}\, e^{-\frac{3}{2}\frac{r}{a_0}}$$

$$= \frac{1}{4\sqrt{2}}\,\frac{1}{\pi}\,2\pi \int_0^\pi d\theta \sin\theta\cos^2\theta\; a_0 \int_0^\infty \frac{dr}{a_0}\left(\frac{r}{a_0}\right)^4 e^{-\frac{3}{2}\frac{r}{a_0}}$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{2}{3}\right)^5 \int_{-1}^{1} du\, u^2 \int_0^\infty d\xi\, \xi^4 e^{-\xi}\, a_0 \qquad \left(u = \cos\theta,\ \xi = \frac{3}{2}\frac{r}{a_0}\right)$$

$= \underline{\nu a_0}$ $\quad\nu$ numerical factor

$$\nu = \frac{1}{2\sqrt{2}}\left(\frac{2}{3}\right)^5 \cdot \frac{2}{3}\cdot 4! = \underline{\frac{1}{\sqrt{2}}\,\frac{256}{243} = 0.745}$$

b) Probability per unit solid angle, for arbitrary polarization

$$p(\theta,\varphi) = N \sum_a |\langle B, 1_{\vec{k}a} | \hat{H}_{emis} | A, 0\rangle|^2$$

$$= N' \sum_a |\vec{\varepsilon}^{\,*}_{\vec{k}a} \cdot \vec{e}_z|^2 \qquad (\vec{r}_{BA} = z_{BA}\,\vec{e}_z)$$

N, N' normalization factors

$$\sum_a |\vec{\varepsilon}^{\,*}_{\vec{k}a} \cdot \vec{e}_z|^2 = \vec{e}_z^{\,2} - \frac{(\vec{k}\cdot\vec{e}_z)^2}{k^2} = 1 - \cos^2\theta = \sin^2\theta$$

Normalization of probability

$$\iint p(\theta,\varphi) \sin\theta \, d\theta \, d\varphi = 1$$

$$\Rightarrow (N')^{-1} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, (1-\cos^2\theta)\sin\theta$$

$$= 2\pi \int_{-1}^{1} du \, (1-u^2) \qquad (u = \cos\theta)$$

$$= \frac{8\pi}{3} \qquad \Rightarrow \quad \underline{p(\theta,\varphi) = \frac{3}{8\pi}\sin^2\theta}$$

c) $2s \to 1s$ is electric dipole "forbidden". Electric dipole matrix element vanishes due to selection rule for parity. Other interaction matrix elements are much smaller. Implies slower transition and longer life time.

Problem 3

a) Density operators, general properties

1) $\hat{\rho} = \hat{\rho}^+$ hermiticity
2) $\hat{\rho} \geq 0$ positivity
3 $\mathrm{Tr}\,\hat{\rho} = 1$ normalization

Spectral decomposition (eigenvector expansion):

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k| \qquad p_k \geq 0 \quad \sum_k p_k = 1$$

Pure state: $\hat{\rho} = |\psi\rangle\langle\psi|$, only one term

Mixed state: several terms with $0 < p_k < 1$

b) Composite system, Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \quad \text{tensor product}$$

Density operator $\hat{\rho}$, acts on $\mathcal{H}$

1) Uncorrelated states, $\hat{\rho}$ factorizes

$$\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B \Rightarrow \langle \hat{A}\hat{B} \rangle = \langle \hat{A} \rangle \langle \hat{B} \rangle$$

for operator $\hat{A}$ acting on $\mathcal{H}_A$ and $\hat{B}$ acting on $\mathcal{H}_B$

2) Classical correlations (separable states)

$\hat{\rho}$ expressed as a probability distribution over uncorrelated states $\hat{\rho} = \sum_{kl} \hat{\rho}^A_k \otimes \hat{\rho}^B_l \, p_{kl}$; $p_{kl} > 0$ $\sum_{kl} p_{kl} = 1$

3) Entangled states:

$\hat{\rho}$ cannot be expressed in the form 2)

Correlations in the wave functions, not simply in a probability distribution over product states.

c) Schmidt decomposition of a pure state in a composite system

$$\underline{|\psi\rangle = \sum_k c_k |k\rangle_A \otimes |k\rangle_B} \quad \text{with } \langle k|k'\rangle_A = \langle k|k'\rangle_B = \delta_{kk'}$$

any $|\psi\rangle$ can be brought into this form

Density operators $\hat{\rho} = \sum_{kk'} c_k c_{k'}^{*} |k\rangle\langle k'|_A \otimes |k\rangle\langle k'|_B$

$$\hat{\rho}_A = Tr_B\, \hat{\rho} = \sum_k |c_k|^2 |k\rangle\langle k|_A$$

$$\hat{\rho}_B = Tr_A\, \hat{\rho} = \sum_k |c_k|^2 |k\rangle\langle k|_B$$

Entropies $\underline{S_A = S_B = -\sum_k |c_k|^2 \log |c_k|^2}$

FYS4110, Eksamen 2009

Løsninger

Oppgave 1

a) $\hat{H}|\psi(t)\rangle = -i\hbar\lambda(\sin\lambda t\,|+-\rangle - \cos\lambda t\,|-+\rangle)$

$$= i\hbar\frac{d}{dt}|\psi(t)\rangle$$

Tetthetsoperator

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| = \cos^2\lambda t\,|+-\rangle\langle+-| + \sin^2\lambda t\,|-+\rangle\langle-+|$$
$$+\cos\lambda t\sin\lambda t\,(|+-\rangle\langle-+| + |-+\rangle\langle+-|)$$

b) Benytter:

$$|+\rangle\langle+| = \tfrac{1}{2}(\mathbb{1}+\sigma_z),\quad |-\rangle\langle-| = \tfrac{1}{2}(\mathbb{1}-\sigma_z)$$
$$|+\rangle\langle-| = \sigma_+ \quad , \quad |-\rangle\langle+| = \sigma_-$$

$$\Rightarrow |+-\rangle\langle+-| = \tfrac{1}{4}(\mathbb{1}+\sigma_z)\otimes(\mathbb{1}-\sigma_z) = \tfrac{1}{4}(\mathbb{1}+\sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z)$$
$$|-+\rangle\langle-+| = \tfrac{1}{4}(\mathbb{1}-\sigma_z)\otimes(\mathbb{1}+\sigma_z) = \tfrac{1}{4}(\mathbb{1}-\sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z)$$
$$|+-\rangle\langle-+| = \sigma_+\otimes\sigma_- \quad ; \quad |-+\rangle\langle+-| = \sigma_-\otimes\sigma_+$$

$$\Rightarrow \hat{\rho}(t) = \tfrac{1}{4}\mathbb{1} + \tfrac{1}{4}(\cos^2\lambda t - \sin^2\lambda t)(\sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z) - \tfrac{1}{4}\sigma_z\otimes\sigma_z$$
$$+\cos\lambda t\sin\lambda t\,(\sigma_+\otimes\sigma_- + \sigma_-\otimes\sigma_+)$$
$$= \tfrac{1}{4}\mathbb{1} + \tfrac{1}{4}\cos 2\lambda t\,(\sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z) - \tfrac{1}{4}\sigma_z\otimes\sigma_z + \tfrac{1}{2}\sin 2\lambda t\,(\sigma_+\otimes\sigma_- + \sigma_-\otimes\sigma_+)$$

Reduserte tetthetsoperatorer, benytter $\mathrm{Tr}\,\sigma_z = \mathrm{Tr}\,\sigma_\pm = 0$

$$\hat{\rho}_A(t) = \mathrm{Tr}_B\,\hat{\rho}(t) = \tfrac{1}{2}(\mathbb{1} + \cos 2\lambda t\,\sigma_z)$$
$$\hat{\rho}_B(t) = \mathrm{Tr}_A\,\hat{\rho}(t) = \tfrac{1}{2}(\mathbb{1} - \cos 2\lambda t\,\sigma_z)$$

c) Graden av sammenfiltring = von Neumann entropien til delsystemene:

$$S = -\mathrm{Tr}_A(\hat{\rho}_A \log \rho_A) = -\mathrm{Tr}_B(\hat{\rho}_B \log \hat{\rho}_B)$$

$$\hat{\rho}_A = \tfrac{1}{2}(1+\cos 2\lambda t)|+\rangle\langle+| + \tfrac{1}{2}(1-\cos 2\lambda t)|-\rangle\langle-|$$

$$= \cos^2\lambda t\, |+\rangle\langle+| + \sin^2\lambda t\, |-\rangle\langle-|$$

$$\Rightarrow \log\hat{\rho}_A = \log[\cos^2\lambda t]\,|+\rangle\langle+| + \log[\sin^2\lambda t]\,|-\rangle\langle-|$$

$$\underline{S = -(\cos^2\lambda t \log[\cos^2\lambda t] + \sin^2\lambda t \log[\sin^2\lambda t])}$$

<u>Oppgave 2</u>

a) $c^\dagger c = \mu^2 a^\dagger a + \nu^2 b^\dagger b + \mu\nu(a^\dagger b + b^\dagger a)$

$d^\dagger d = \nu^2 a^\dagger a + \mu^2 b^\dagger b - \mu\nu(a^\dagger b + b^\dagger a)$

$$\Rightarrow \omega_c c^\dagger c + \omega_d d^\dagger d = (\mu^2\omega_c + \nu^2\omega_d)a^\dagger a + (\nu^2\omega_c + \mu^2\omega_d)b^\dagger b + \mu\nu(\omega_c - \omega_d)(a^\dagger b + b^\dagger a)$$

Setter: $\omega = \mu^2\omega_c + \nu^2\omega_d = \nu^2\omega_c + \mu^2\omega_d$ I

og $\mu\nu(\omega_c - \omega_d) = \lambda$ II

I $\Rightarrow$ $\omega = \frac{1}{2}(\mu^2+\nu^2)(\omega_c+\omega_d) = \frac{1}{2}(\omega_c+\omega_d)$ (1)

$\Rightarrow \mu^2 = \nu^2 = \frac{1}{2}$

$\underline{\mu = \nu = \frac{1}{\sqrt{2}}}$ $\Rightarrow$ $\frac{1}{2}(\omega_c - \omega_d) = \lambda$ (2)

(1) & (2) $\Rightarrow$ $\underline{\omega_c = \omega + \lambda}$, $\underline{\omega_d = \omega - \lambda}$

Kommutasjonsrelasjoner

$[c, c^\dagger] = \mu^2[a,a^\dagger] + \nu^2[b,b^\dagger] = (\mu^2+\nu^2)\mathbb{1} = \mathbb{1}$

$[d, d^\dagger] = \nu^2[a,a^\dagger] + \mu^2[b,b^\dagger] = (\mu^2+\nu^2)\mathbf{1} = \mathbf{1}$

$[c, d^\dagger] = -\mu\nu([a,a^\dagger] - [b,b^\dagger]) = 0 \Rightarrow [c^\dagger, d] = 0$

andre kommutatorer $= 0$

$\Rightarrow$ To uavh. sett med harm. osc. operatorer

b) Tidsutvikling av koherent tilstand

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle \quad ; \quad \hat{U}(t) = \exp[-i(\omega_c c^\dagger c + \omega_d d^\dagger d + \omega \mathbb{1})]$$

$$\hat{c}|\psi(t)\rangle = \hat{U}(t)\hat{U}(t)^{-1}\hat{c}\,\hat{U}(t)|\psi(0)\rangle$$

$$\hat{U}(t)^{-1}\hat{c}\,\hat{U}(t) = e^{i\omega_c t\, c^\dagger c}\,\hat{c}\,e^{-i\omega t\, c^\dagger c}$$

$$= c + i\omega_c t\,[c^\dagger c, c] + \tfrac{1}{2}(i\omega_c t)^2\,[c^\dagger c,[c^\dagger c, c]] + \cdots$$

$$= (1 - i\omega_c t + \tfrac{1}{2}(-i\omega_c t)^2 + \cdots)\,c = e^{-i\omega_c t}\,c$$

$$\Rightarrow \quad \hat{c}|\psi(t)\rangle = e^{-i\omega_c t}\,\hat{U}(t)\,\hat{c}\,|\psi(0)\rangle = \underline{e^{-i\omega_c t} z_{c0}|\psi(t)\rangle}$$

c) $\hat{c} = \frac{1}{\sqrt{2}}(\hat{a}+\hat{b})$, $\hat{d} = -\frac{1}{\sqrt{2}}(\hat{a}-\hat{b})$

$\Rightarrow \hat{a} = \frac{1}{\sqrt{2}}(\hat{c}-\hat{d})$, $\hat{b} = \frac{1}{\sqrt{2}}(\hat{c}+\hat{d})$

Operatorene har felles egentilstander med egenverdier

$$z_a(t) = \tfrac{1}{\sqrt{2}}(z_c(t) - z_d(t)) = \tfrac{1}{\sqrt{2}}(e^{-i\omega_c t} z_{c0} - e^{-i\omega_d t} z_{d0})$$

$$= \tfrac{1}{2} e^{-i\omega t}(e^{-i\lambda t}(+z_{a0} + z_{b0}) + e^{i\lambda t}(z_{a0} - z_{b0}))$$

$$= \underline{e^{-i\omega t}(\cos\lambda t\, z_{a0} - i\sin\lambda t\, z_{b0})}$$

$$z_b(t) = -\tfrac{1}{2} e^{-i\omega t}(e^{-i\lambda t}(z_{a0} + z_{b0}) - e^{i\lambda t}(z_{a0} - z_{b0}))$$

$$= \underline{e^{-i\omega t}(i\sin\lambda t\, z_{a0} + \cos\lambda t\, z_{b0})}$$

Oppgave 3

a) Krav til tetthetsmatrise

1) Hermitisitet: $\hat{\rho}^\dagger = e^{-\beta \hat{H}^\dagger} = e^{-\beta \hat{H}} = \hat{\rho}$ (β reell)

2) Positivitet: Egenverdier $\hat{\rho}|n\rangle = e^{-\beta E_n}|n\rangle$

$e^{-\beta E_n} > 0$ for alle n

3) Normering $\mathrm{Tr}\,\hat{\rho} = 1 \Leftrightarrow N^{-1} = \mathrm{Tr}\, e^{-\beta \hat{H}}$

bestemmer N

Normeringskonstant

$$N^{-1} = \sum_n e^{-\beta E_n} = e^{-\frac{1}{2}\beta\hbar\omega} \sum_{n=0}^{\infty} (e^{-\beta\hbar\omega})^n = \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{1\cancel{2}}{2\sinh(\frac{1}{2}\beta\hbar\omega)}$$

$$\underline{N = 2\cancel{\tfrac{1}{2}} \sinh(\tfrac{1}{2}\beta\hbar\omega)}$$

b) Forventningsverdi for energien

$$E = Tr(N e^{-\beta\hat{H}}\hat{H}) = -N\frac{d}{d\beta} Tr(e^{-\beta\hat{H}})$$

$$= -N\frac{d}{d\beta}(N^{-1}) = \underline{\frac{1}{N}\frac{dN}{d\beta}}$$

$$\frac{dN}{d\beta} = \frac{1}{4}\hbar\omega\cosh(\tfrac{1}{2}\beta\hbar\omega) \Rightarrow E = \tfrac{1}{2}\hbar\omega\coth(\tfrac{1}{2}\beta\hbar\omega)$$

$\beta \to \infty$: $\coth(\frac{1}{2}\beta\hbar\omega) \to 1 \Rightarrow \underline{E \to \frac{1}{2}\hbar\omega}$ grunntilst. energien

c) $$\hat{\rho} = \int \frac{d^2z}{\pi} p(|z|)\,|z\rangle\langle z| = \sum_{n,n'} \underbrace{\int \frac{d^2z}{\pi} p(|z|)\,\langle n|z\rangle\langle z|n'\rangle}_{\equiv I_{nn'}}\,|n\rangle\langle n'|$$

$$I_{nn'} = \int \frac{d^2z}{\pi} p(|z|)\frac{z^n z^{*n'}}{\sqrt{n!\,n'!}} e^{-|z|^2}$$

$$= \frac{1}{\pi}\int_0^{2\pi} d\varphi \int_0^{\infty} dr\, r\, p(r)\frac{r^{n+n'}e^{i\varphi(n-n')}}{\sqrt{n!\,n'!}} e^{-r^2} \;; \quad \int_0^{2\pi} e^{i\varphi(n-n')}d\varphi = 2\pi\delta_{nn'}$$

$$= 2\int_0^{r} dr\, r^{2n+1} e^{-r^2} p(r)\frac{1}{n!}\delta_{nn'}$$

$$\Rightarrow \hat{\rho} = \sum_n p_n |n\rangle\langle n| \quad \text{med} \quad \underline{p_n = \frac{2}{n!}\int_0^{\infty} dr\, r^{2n+1} e^{-r^2} p(r)}$$

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Løsninger

Oppgave 1

a) En tilstandsvektor eller tetthetsoperator som ikke er på tensor-produktform inneholder korrelasjoner mellom delsystemene.
Her er det en ren tilstand som ikke er på produktform,
$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$.
Korrelasjonene ligger i tilstandsvektoren, ikke i tetthetsoperatoren, dvs $\hat{\rho} = |\psi\rangle\langle\psi| \neq \sum_k p_k \hat{\rho}_k^A \otimes \hat{\rho}_k^B \otimes \hat{\rho}_k^C$; tilstanden er ikke separabel, men sammenfiltret.

b) Tetthetsoperator

$$\hat{\rho} = \tfrac{1}{2}\left(|uuu\rangle\langle uuu| + |ddd\rangle\langle ddd| - |uuu\rangle\langle ddd| - |ddd\rangle\langle uuu|\right)$$

Reduserte tetthetsoperatorer

$$\hat{\rho}_A = \mathrm{Tr}_{BC}\,\hat{\rho} = \tfrac{1}{2}\left(|u\rangle\langle u| + |d\rangle\langle d|\right)_A = \tfrac{1}{2}\mathbb{1}_A$$

$$\hat{\rho}_{BC} = \mathrm{Tr}_A\,\hat{\rho} = \tfrac{1}{2}\left(|uu\rangle\langle uu| + |dd\rangle\langle dd|\right)_{BC}$$

Sammenfiltringsentropien til todelt system er lik von Neumann-entropien til hvert av delsystemene (som er like)

Her $S = S_A = S_{BC} = -\sum_k p_k \log p_k = -2\left(\tfrac{1}{2}\log\tfrac{1}{2}\right) = \underline{\log 2}$

$\hat{\rho}_A$ er maksimalt blandet, dvs S_A har maksimal verdi
$\Rightarrow$ S maksimal, de to delsystemene er maksimalt sammenfiltret.

Delsystem BC: $\hat{\rho}_{BC} = \tfrac{1}{2}\left(\hat{\rho}_u^B \otimes \hat{\rho}_u^C + \hat{\rho}_d^B \otimes \hat{\rho}_d^C\right)$; $\quad \hat{\rho}_u = |u\rangle\langle u|$, $\hat{\rho}_d = |d\rangle\langle d|$

$\hat{\rho}_{BC}$ separabel $\Rightarrow$ B og C <u>ikke</u> sammenfiltret.

c) Uttrykker $|\psi\rangle$ ved $|f\rangle$ og $|b\rangle$ for delsystem A

$$|u\rangle = \frac{1}{\sqrt{2}}(|f\rangle + |b\rangle) \; ; \; |d\rangle = \frac{1}{\sqrt{2}}(|f\rangle - |b\rangle) \Rightarrow$$

$$|\psi\rangle = \tfrac{1}{2}(|f\rangle \otimes (|uu\rangle + |dd\rangle) + |b\rangle \otimes (|uu\rangle - |dd\rangle))$$

Måling med f som resultat $\Rightarrow$ ny tilstand proporsjonal med $|f\rangle_A$ $\Rightarrow$ $|\psi'\rangle = \frac{1}{\sqrt{2}}(|f\rangle \otimes (|uu\rangle + |dd\rangle))$ etter måling

$$\equiv |\psi_A'\rangle \otimes |\psi_{BC}'\rangle$$

Tetthetsoperator

$$\hat{\rho}' = |\psi'\rangle\langle\psi'| = |\psi_A'\rangle\langle\psi_A'| \otimes |\psi_{BC}'\rangle\langle\psi_{BC}'| \equiv \hat{\rho}_A' \otimes \hat{\rho}_{BC}'$$

Delsystemene A og BC ikke lenger korrelerte

$$\hat{\rho}_{BC}' = \tfrac{1}{2}(|uu\rangle\langle uu| + |dd\rangle\langle dd| + |uu\rangle\langle dd| + |dd\rangle\langle uu|)$$

$$\Rightarrow \hat{\rho}_B' = \mathrm{Tr}_C\, \hat{\rho}_{BC}' = \tfrac{1}{2}\mathbb{1}_B \; ; \; \hat{\rho}_C' = \tfrac{1}{2}\mathbb{1}_C$$

Spinnene B og C er nå maksimalt sammenfiltret!

Oppgave 2

a) Vinkelavhengigheten til matriseelementet sitter i faktoren $(\vec{k} \times \vec{\varepsilon}_{ka}) \cdot \vec{\sigma}_{BA} = \vec{\varepsilon}_{ka} \cdot (\vec{\sigma}_{BA} \times \vec{k})$. Sannsynlighetsfordelingen $p(\theta,\varphi)$ er uavhengig av polarisasjonen, då vi summerer over a,

$$p(\theta,\varphi) = N \sum_a |\vec{\varepsilon}_{ka} \cdot (\vec{\sigma}_{BA} \times \vec{k})|^2$$

$$= N\, |\vec{\sigma}_{BA} \times \vec{k}|^2 \qquad \frac{\vec{k}}{k} \cdot (\vec{\sigma}_{BA} \times \vec{k}) = 0$$

N: normeringsfaktor bestemt av $\int d\varphi \int d\theta \sin\theta \; p(\theta,\varphi) = 1$

$$\vec{\sigma}_{BA} = (0\;1)\begin{pmatrix} \vec{e}_z & \vec{e}_x - i\vec{e}_y \\ \vec{e}_x + i\vec{e}_y & -\vec{e}_z \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \vec{e}_x + i\vec{e}_y$$

$$\vec{k} = k(\sin\theta\cos\varphi\, \vec{e}_x + \sin\theta\sin\varphi\, \vec{e}_y + \cos\theta\, \vec{e}_z)$$

$$\Rightarrow \vec{\sigma}_{BA} \times \vec{k} = k(i\cos\theta\, \vec{e}_x - \cos\theta\, e_y - i\sin\theta\, e^{i\varphi}\vec{e}_z)$$

$$\Rightarrow |\vec{\sigma}_{BA} \times \vec{k}|^2 = k^2(2\cos^2\theta \cancel{\tfrac{1}{2}} + \sin^2\theta) = k^2(1 + \cos^2\theta) \quad \text{uavh. av } \varphi$$

$$p(\theta,\varphi) = N k^2 (1+\cos^2\theta)$$

$$\Rightarrow \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \, p(\theta,\varphi) = 2\pi N k^2 \int_{-1}^{1} du \, (1+u^2) \qquad u = -\cos\theta$$

$$= 2\pi N k^2 \left[u + \tfrac{1}{3}u^3 \right]_{-1}^{1} = \tfrac{16}{3}\pi N k^2$$

normering: $N = \frac{3}{16\pi} \frac{1}{k^2}$

$$\Rightarrow \underline{p(\theta,\varphi) = \frac{3}{16\pi}(1+\cos^2\theta)}$$

b) $\vec{k} = k\vec{e}_x \Rightarrow$

$$|\vec{\varepsilon}(\alpha)\cdot(\vec{\sigma}_{BA}\times\vec{k})|^2 = k^2 |(\cos\alpha\,\vec{e}_y + \sin\alpha\,\vec{e}_z)\cdot(-i\vec{e}_z)|^2$$

$$= k^2 \sin^2\alpha$$

Sannsynlighetsfordeling

$$p(\alpha) = N' \sin^2\alpha$$

$$\int_0^{\pi} p(\alpha)\,d\alpha = N' \int_0^{\pi} \sin^2\alpha \, d\alpha = N' \frac{\pi}{2}$$

(definerer $0 \le \alpha < \pi$, siden α og $\alpha+\pi$ def. samme polarisasjonstilstand)

Normering $\Rightarrow N' = \frac{2}{\pi} \Rightarrow \underline{p(\alpha) = \frac{2}{\pi}\sin^2\alpha}$

c) $P_A(t) = e^{-t/\tau_A} = 1 - \frac{t}{\tau_A} + \cdots$

for små t $(t \ll \tau_A)$: $P_A \simeq 1 - \left(\frac{1}{\tau_A}\right)t$

Overgangssannslighet pr. tid for $A \to B$: $w_{BA} = \frac{1}{\tau_A}$

$$w_{BA} = \frac{V}{(2\pi\hbar)^2} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk\, k^2 \frac{e^2\hbar^3}{8Vm^2\omega\varepsilon_0} \sum_\alpha |(\vec{k}\times\vec{\varepsilon}_{k\alpha})\cdot\vec{\sigma}_{BA}|^2 \delta(\omega-\omega_0)$$

$k = \omega/c$

$$= \frac{e^2\hbar\,\omega_0}{32\pi^2 m^2 \varepsilon_0 c^2} \frac{\omega_0^2}{c^3} \frac{16\pi}{3} \int d\varphi \underbrace{\int d\theta \sin\theta \, p(\theta,\varphi)}_{=1}$$

$$= \frac{1}{6\pi^2} \frac{e^2\hbar\,\omega_0^3}{m^2\varepsilon_0 c^5}$$

$$\Rightarrow \underline{\tau_A = 6\pi^2 \frac{m^2\varepsilon_0 c^5}{e^2\hbar\,\omega_0^3}}$$

Oppgave 3

a) $\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}] = -i\omega_0 \hat{a} - i\lambda e^{-i\omega t}\mathbb{1} \quad = \dot{\hat{a}}$

$$\frac{d^2\hat{a}}{dt^2} = \frac{i}{\hbar}[\hat{H}, \dot{\hat{a}}] + \frac{\partial}{\partial t}\dot{\hat{a}} = -i\omega_0(-i\omega_0\hat{a} - i\lambda e^{-i\omega t}\mathbb{1}) - i\lambda(-i\omega)e^{-i\omega t}\mathbb{1}$$

$$= -\omega_0^2 \hat{a} - \lambda(\omega_0 + \omega)e^{-i\omega t}\mathbb{1}$$

$\hat{x} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \Rightarrow$

$$\underline{\frac{d^2\hat{x}}{dt^2} + \omega_0^2 \hat{x} = -\lambda(\omega_0 + \omega)\cos\omega t} \qquad \underline{C = -\lambda(\omega_0 + \omega)}$$

b) $i\hbar \frac{d}{dt}|\psi_T(t)\rangle = \hat{T}(t)\hat{H}(t)|\psi(t)\rangle + i\hbar\frac{d\hat{T}}{dt}|\psi(t)\rangle$

$$= \hat{H}_T(t)|\psi_T(t)\rangle$$

hvor $\hat{H}_T(t) = \hat{T}(t)\hat{H}(t)\hat{T}^\dagger(t) + i\hbar\frac{d\hat{T}}{dt}\hat{T}^\dagger(t)$

$$\hat{T}\hat{a}\hat{T}^\dagger = e^{i\omega t \hat{a}^\dagger\hat{a}}\,\hat{a}\,e^{-i\omega t\hat{a}^\dagger\hat{a}} = \hat{a}e^{-i\omega t} \qquad \hat{T}\hat{a}^\dagger\hat{T}^\dagger = \hat{a}^\dagger e^{i\omega t}$$

$$\Rightarrow \hat{T}\hat{H}\hat{T} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}) + \hbar\lambda(\hat{a}^\dagger + \hat{a})$$

$$i\hbar\frac{d\hat{T}}{dt}\hat{T}^\dagger = -\hbar\omega\hat{a}^\dagger\hat{a}$$

$$\Rightarrow \underline{\hat{H}_T = \hbar(\omega_0 - \omega)\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{a} + \hat{a}^\dagger) + \tfrac{1}{2}\hbar\omega_0\mathbb{1}}$$

c) $|\psi_T(t)\rangle = \hat{U}_T(t)|\psi_T(0)\rangle$, $\hat{U}_T(t) = e^{-\frac{i}{\hbar}\hat{H}_T t}$

$\Rightarrow |\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$, $\hat{U}(t) = \hat{T}^\dagger(t)\hat{U}_T(t) = e^{-i\omega t\hat{a}^\dagger\hat{a}}e^{-\frac{i}{\hbar}\hat{H}_T t}$

Antar $|\psi(0)\rangle = |0\rangle$, $\hat{a}|0\rangle = 0$

Sjekker om $|\psi(t)\rangle$ er en koherent tilstand ved å anvende $\hat{a}$,

$$\hat{a}|\psi(t)\rangle = \hat{U}(t)\hat{U}^\dagger(t)\hat{a}\,\hat{U}(t)|\psi(0)\rangle$$

$$\hat{U}^\dagger(t)\,\hat{a}\,\hat{U}(t) = e^{\frac{i}{\hbar}\hat{H}_T t}e^{+i\omega t\hat{a}^\dagger\hat{a}}\,\hat{a}\,e^{-i\omega t\hat{a}^\dagger\hat{a}}e^{-\frac{i}{\hbar}\hat{H}_T t}$$

$$= e^{\frac{i}{\hbar}\hat{H}t}e^{-i\omega t}\hat{a}\,e^{-\frac{i}{\hbar}\hat{H}t}$$

$$[\hat{H}_T, \hat{a}] = \hbar(\omega-\omega_0)\hat{a} - \hbar\lambda \mathbb{1}$$

$$[\hat{H}_T, [\hat{H}_T, \hat{a}]] = \hbar(\omega-\omega_0)(\hbar(\omega-\omega_0)\hat{a} - \hbar\lambda \mathbb{1})$$

....

$$\Rightarrow e^{\frac{i}{\hbar}\hat{H}_T t}\, \hat{a}\, e^{-\frac{i}{\hbar}\hat{H}_T t} = \hat{a} + \frac{i}{\hbar}[\hat{H}_T, \hat{a}] + \frac{1}{2!}\left(\frac{i}{\hbar}\right)^2 [\hat{H}_T, [\hat{H}_T, \hat{a}]] + \cdots$$

$$= \left(1 + i(\omega-\omega_0)t + \frac{1}{2!}[i(\omega-\omega_0)t]^2 + \cdots\right)\hat{a}$$

$$-i\lambda\left(t + \frac{1}{2!}(i(\omega-\omega_0))\,t^2 + \cdots \right)\mathbb{1}$$

$$= e^{i(\omega-\omega_0)t}\,\hat{a} - \frac{\lambda}{\omega-\omega_0}\left(e^{i(\omega-\omega_0)t} - 1\right)\mathbb{1}$$

$$\Rightarrow \hat{a}\,\hat{U}(t) = \hat{U}(t)\left(e^{-i\omega_0 t}\,\hat{a} - \frac{\lambda}{\omega-\omega_0}\left(e^{-i\omega_0 t} - e^{-i\omega t}\right)\mathbb{1}\right)$$

$$\Rightarrow \hat{a}\,|\psi(t)\rangle = \underline{-\frac{\lambda}{\omega-\omega_0}\left(e^{-i\omega_0 t} - e^{-i\omega t}\right)|\psi(t)\rangle}$$

egentilstand for $\hat{a}$, med egenverdi

$$\underline{z(t) = -\frac{\lambda}{\omega-\omega_0}\left(e^{-i\omega_0 t} - e^{-i\omega t}\right)}$$

Bevegelsesligning

$$\ddot{z} = -\frac{\lambda}{\omega-\omega_0}\left(-\omega_0^2 e^{-i\omega_0 t} + \omega^2 e^{-i\omega t}\right)$$

$$= -\omega_0^2 z - \frac{\lambda}{\omega-\omega_0}\left(\omega^2 - \omega_0^2\right)e^{-i\omega t}$$

$$\ddot{z} + \omega_0^2 z = -\lambda(\omega+\omega_0)\,e^{-i\omega t}$$

Realdel $\quad \underline{\ddot{x} + \omega_0^2 z = -\lambda(\omega+\omega_0)\cos\omega t}$ som for $\hat{x}$

Bevegelse i z-planet: Spiralerende bane med $|z| = 0$ når $e^{-i\omega_0 t}$ og $e^{-i\omega t}$ er i motfase og $|z| = \frac{2\lambda}{|\omega-\omega_0|}$ (maksimal) når — " — er i fase.

FYS4110/9110, Exam 2011

Solutions

Problem 1

a) Matrix elements of the Hamiltonian

$$\hat{H}|-,1\rangle = (-\tfrac{1}{2}\hbar\omega_0 + \hbar\omega)|-,1\rangle - i\hbar\lambda|+,0\rangle$$

$$\hat{H}|+,0\rangle = \tfrac{1}{2}\hbar\omega_0|+,0\rangle + i\hbar\lambda|-,1\rangle$$

$$\Rightarrow \langle -,1|\hat{H}|-,1\rangle = \tfrac{1}{2}\hbar(2\omega-\omega_0)$$

$$\langle +,0|\hat{H}|+,0\rangle = \tfrac{1}{2}\hbar\omega_0$$

$$\langle -,1|\hat{H}|+,0\rangle = i\hbar\lambda$$

$$\langle +,0|\hat{H}|-,1\rangle = -i\hbar\lambda$$

in matrix form

$$H = \tfrac{1}{2}\hbar\begin{pmatrix}\omega_0 & -2i\lambda\\ 2i\lambda & 2\omega-\omega_0\end{pmatrix} = \tfrac{1}{2}\hbar\begin{pmatrix}\omega_0-\omega & -2i\lambda\\ 2i\lambda & \omega-\omega_0\end{pmatrix} + \tfrac{1}{2}\hbar\omega\,\mathbb{1}$$

$$= \tfrac{1}{2}\hbar\Delta\begin{pmatrix}\cos\varphi & -i\sin\varphi\\ i\sin\varphi & -\cos\varphi\end{pmatrix} + \varepsilon\mathbb{1}$$

with $\Delta\cos\varphi = \omega_0-\omega$, $\Delta\sin\varphi = 2\lambda$, $\underline{\varepsilon = \tfrac{1}{2}\hbar\omega}$

$$\Rightarrow \underline{\Delta = \sqrt{(\omega-\omega_0)^2+4\lambda^2}}\,,\quad \underline{\cos\varphi = \frac{\omega_0-\omega}{\Delta}}\,,\quad \underline{\sin\varphi = \frac{2\lambda}{\Delta}}$$

b) Eigenvectors determined by

$$\begin{pmatrix}\cos\varphi & -i\sin\varphi\\ i\sin\varphi & -\cos\varphi\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \mu\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\begin{vmatrix}\cos\varphi-\mu & -i\sin\varphi\\ i\sin\varphi & -\cos\varphi-\mu\end{vmatrix} = 0 \quad\Rightarrow\quad \mu = \pm 1$$

Energies $E_\pm = \tfrac{1}{2}\hbar\omega \pm \tfrac{1}{2}\hbar\Delta = \underline{\tfrac{1}{2}\hbar\left(\omega \pm \sqrt{(\omega-\omega_0)^2+4\lambda^2}\right)}$

Eigenvectors

$$\cos\varphi\, \alpha_\pm - i\sin\varphi\, \beta_\pm = \pm\alpha_\pm$$

$$(\cos\varphi \mp 1)\alpha_\pm - i\sin\varphi\, \beta_\pm = 0$$

$$\Rightarrow \quad \alpha_\pm = N i \sin\varphi\,, \quad \beta_\pm = N(\cos\varphi \mp 1)$$

normalization $N^2(\sin^2\varphi + (\cos\varphi \mp 1)^2) = 1$

$$\Rightarrow \quad N = \frac{1}{\sqrt{2(1\mp\cos\varphi)}}$$

$$\psi_\pm(\varphi) = \frac{1}{\sqrt{2(1\mp\cos\varphi)}}\begin{pmatrix} i\sin\varphi \\ \cos\varphi \mp 1\end{pmatrix}$$

$$\sin\varphi = 2\sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2}\;;\quad \cos\varphi = 2\cos^2\tfrac{\varphi}{2} - 1 = 1 - 2\sin^2\tfrac{\varphi}{2}$$

$$\Rightarrow \quad |\psi_+(\varphi)\rangle = -\sin\tfrac{\varphi}{2}|-,1\rangle + i\cos\tfrac{\varphi}{2}|+,0\rangle$$

$$|\psi_-(\varphi)\rangle = \cos\tfrac{\varphi}{2}|-,1\rangle + i\sin\tfrac{\varphi}{2}|+,0\rangle$$

$$\cos\left(\frac{\varphi+\pi}{2}\right) = -\sin\frac{\varphi}{2}\,, \quad \sin\left(\frac{\varphi+\pi}{2}\right) = \cos\frac{\varphi}{2}$$

$$\Rightarrow \quad |\psi_-(\varphi+\pi)\rangle = |\psi_+(\varphi)\rangle$$

c) Density operator of the $|\psi_-(\varphi)\rangle$ state

$$\rho(\varphi) = |\psi_-(\varphi)\rangle\langle\psi_-(\varphi)|$$

$$= \cos^2\tfrac{\varphi}{2}|-,1\rangle\langle -,1| + \sin^2\tfrac{\varphi}{2}|+,0\rangle\langle +,0| + i\cos\tfrac{\varphi}{2}\sin\tfrac{\varphi}{2}\left(|+,0\rangle\langle -,1| - |-,1\rangle\langle +,0|\right)$$

$$\rho_{ph}(\varphi) = \langle -|\rho(\varphi)|-\rangle + \langle +|\rho(\varphi)|+\rangle = \sin^2\tfrac{\varphi}{2}|0\rangle\langle 0| + \cos^2\tfrac{\varphi}{2}|1\rangle\langle 1|$$

$$\rho_{atom}(\varphi) = \langle 0|\rho(\varphi)|0\rangle + \langle 1|\rho(\varphi)|1\rangle = \cos^2\tfrac{\varphi}{2}|-\rangle\langle -| + \sin^2\tfrac{\varphi}{2}|+\rangle\langle +|$$

$\cos^2\frac{\varphi}{2} > \sin^2\frac{\varphi}{2}$ $\left(-\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right)$: the state is mainly a one-photon state

$\cos^2\frac{\varphi}{2} < \sin^2\frac{\varphi}{2}$ $\left(\frac{\pi}{2} < \varphi < 3\frac{\pi}{2}\right)$: the state is mainly an excited atomic state

d) Entanglement entropy

$$S = -\mathrm{Tr}_{ph}(\rho_{ph}\log\rho_{ph}) = -\mathrm{Tr}_{atom}(\rho_{atom}\log\rho_{atom})$$

$$= -\left(\cos^2\tfrac{\varphi}{2}\log(\cos^2\tfrac{\varphi}{2}) + \sin^2\tfrac{\varphi}{2}\log(\sin^2\tfrac{\varphi}{2})\right)$$

Min. value when $|\psi_-(\varphi)\rangle$ is a product state:

$\cos\frac{\varphi}{2} = 0$ or $\sin\frac{\varphi}{2} = 0 \Rightarrow \varphi = 0, \pi$

gives $\underline{S = 0}$

Max. value, when ρ_{ph} (ρ_{atom}) is maximally mixed:

$\cos^2\frac{\varphi}{2} = \sin^2\frac{\varphi}{2} = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{2}, 3\frac{\pi}{2}$

$\Rightarrow \rho_{ph} = \frac{1}{2}\mathbb{1} \Rightarrow \underline{S = \log 2}$ max. entangled

e) Time evolution: expand in energy eigenstates

$$|\psi(0)\rangle = |-,1\rangle = \cos\frac{\varphi}{2}|\psi_-(\varphi)\rangle - \sin\frac{\varphi}{2}|\psi_+(\varphi)\rangle$$

$$\Rightarrow |\psi(t)\rangle = \cos\frac{\varphi}{2} e^{-\frac{i}{\hbar}E_-t}|\psi_-(\varphi)\rangle - \sin\frac{\varphi}{2} e^{-\frac{i}{\hbar}E_+t}|\psi_+(\varphi)\rangle$$

$$= \left(\cos^2\frac{\varphi}{2} e^{-\frac{i}{\hbar}E_-t} + \sin^2\frac{\varphi}{2} e^{-\frac{i}{\hbar}E_+t}\right)|-,1\rangle$$

$$+ i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\left(e^{-\frac{i}{\hbar}E_-t} - e^{-\frac{i}{\hbar}E_+t}\right)|+,0\rangle$$

Probability for a photon present

$$p(t) = |\langle -,1|\psi(t)\rangle|^2 = \cos^4\frac{\varphi}{2} + \sin^4\frac{\varphi}{2}$$

$$+ \cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}\left(e^{-\frac{i}{\hbar}(E_--E_+)t} + e^{+\frac{i}{\hbar}(E_--E_+)t}\right)$$

$$= \frac{1}{4}(1+\cos\varphi)^2 + \frac{1}{4}(1-\cos\varphi)^2 + \frac{1}{2}\sin^2\varphi\cos\left(\frac{E_--E_+}{\hbar}t\right)$$

$$= \underline{\frac{1}{2}\left(1+\cos^2\varphi + \sin^2\varphi\cos\Delta t\right)} \qquad \Delta = \sqrt{(\omega-\omega_0)^2 + 4\lambda^2}$$

Oscillations due to time dependent mixing of the one-photon state with the excited atom state. Frequency Δ, amplitude $\frac{1}{2}\sin^2\varphi = \underline{\frac{2\lambda^2}{(\omega-\omega_0)^2 + 4\lambda^2}}$

Problem 2

a) Time evolution of the two-level system, $\kappa = 0$:

$$\mathcal{U}(t) = e^{-\frac{i}{2}\omega_A t \sigma_z} = \begin{pmatrix} e^{-\frac{i}{2}\omega_A t} & 0 \\ 0 & e^{\frac{i}{2}\omega_A t} \end{pmatrix}$$

$$\rho_A(t) = \mathcal{U}(t)\rho_A(0)\mathcal{U}^\dagger(t)$$

$$= \begin{pmatrix} e^{-\frac{i}{2}\omega_A t} & 0 \\ 0 & e^{\frac{i}{2}\omega_A t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \begin{pmatrix} e^{\frac{i}{2}\omega_A t} & 0 \\ 0 & e^{-\frac{i}{2}\omega_A t} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1+z & e^{-i\omega_A t}(x-iy) \\ e^{i\omega_A t}(x+iy) & 1-z \end{pmatrix} \Rightarrow x(t) + iy(t) = e^{i\omega_A t}(x+iy)$$

$$\Rightarrow x(t) = x\cos\omega_A t - y\sin\omega_A t$$
$$y(t) = x\sin\omega_A t + y\cos\omega_A t$$
$$z(t) = z$$

Precession of $\vec{r}$ around the z-axis, with ang. freq. ω_A

b) Interaction matrix element

$$\langle -, 1_k | \hat{H}_{int} | +, 0\rangle = \kappa\sqrt{\frac{\hbar}{2L\omega_k}}$$

decay rate:

$$\gamma = \frac{L}{(2\pi\hbar)^2}\int dk\, \frac{\kappa\hbar^2}{2L\omega_k}\,\delta(\omega_k - \omega_A) \qquad k = \frac{\omega_k}{c}$$

$$= \frac{L}{4\pi^2\hbar^2}\,\frac{\kappa^2\hbar}{2Lc\omega_A} = \underline{\frac{\kappa^2}{8\pi^2\hbar c\,\omega_A}}$$

c) $|\psi(t)\rangle = |\phi(t)\rangle \otimes |0\rangle + \sum_k c_k(t)|-, 1_k\rangle$

with $|\phi(t)\rangle = e^{-\frac{i}{2}\omega_A t - \gamma t/2}\alpha|+\rangle + e^{\frac{i}{2}\omega_A t}\beta|-\rangle$

Normalization

$$\langle\psi(t)|\psi(t)\rangle = \langle\phi(t)|\phi(t)\rangle + \sum_k |c_k(t)|^2$$

$$= e^{-\gamma t}|\alpha|^2 + |\beta|^2 + \sum_k |c_k(t)|^2 \overset{!}{=} 1$$

$$\Rightarrow \sum_k |c_k(t)|^2 = \underline{|\alpha|^2(1-e^{-\gamma t})}$$

Reduced density operator of the two-level system

$$\rho_A(t) = \mathrm{Tr}_B(|\psi(t)\rangle\langle\psi(t)|) = |\phi(t)\rangle\langle\phi(t)| + \sum_k |c_k(t)|^2 |-\rangle\langle-|$$

$$= e^{-\gamma t}|\alpha|^2|+\rangle\langle+| + (1-e^{-\gamma t}|\alpha|^2)|-\rangle\langle-|$$

$$\underline{+ e^{-\gamma t/2}(\alpha\beta^* e^{-i\omega_A t}|+\rangle\langle-| + \alpha^*\beta e^{i\omega_A t}|-\rangle\langle+|)}$$

d) $\alpha = 1,\ \beta = 0$:

$$\rho_A(t) = e^{-\gamma t}|+\rangle\langle+| + (1-e^{-\gamma t})|-\rangle\langle-|$$

$$= \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1-e^{-\gamma t} \end{pmatrix}$$

$$\Rightarrow z(t) = \underline{2e^{-\gamma t}-1}, \quad \underline{x(t) = y(t) = 0}$$

The excited state decays exponentially into the ground state, as expected

$t = 0$ and $t \to \infty$ ($z = \pm 1$) pure product state, $S_A = 0$

Intermediate time: $e^{-\gamma t} = \frac{1}{2} \Rightarrow \rho_A = \frac{1}{2}\mathbb{1}$, maximally entangled

e) $\alpha = \beta = \frac{1}{\sqrt{2}}$:

$$\rho_A(t) = \tfrac{1}{2} e^{-\gamma t} |+\rangle\langle+| + \left(1 - \tfrac{1}{2} e^{-\gamma t}\right) |-\rangle\langle-|$$
$$+ \tfrac{1}{2} e^{-\gamma t/2} \left(e^{-i\omega_A t} |+\rangle\langle-| + e^{i\omega_A t} |-\rangle\langle+| \right)$$
$$= \frac{1}{2} \begin{pmatrix} e^{-\gamma t} & e^{-\gamma t/2} e^{-i\omega_A t} \\ e^{-\gamma t/2} e^{i\omega_A t} & 2 - e^{-\gamma t} \end{pmatrix} \Rightarrow x(t) + i y(t) = e^{-\gamma t/2} e^{i\omega_A t}$$

$\underline{x(t) = e^{-\gamma t/2} \cos\omega_A t}$, $\underline{y(t) = e^{-\gamma t/2} \sin\omega_A t}$; $\underline{z(t) = e^{-\gamma t} - 1}$

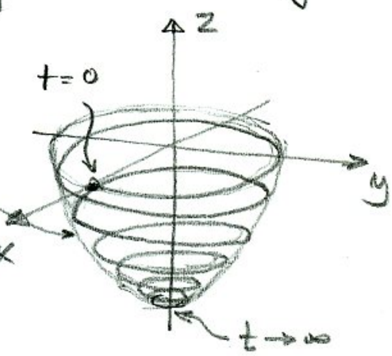
Combination of motions in a) and d) :

$\gamma \ll \omega_A \Rightarrow$ rapid precession of $\vec{r}$ around the z-axis, combined with slow decay towards the ground state

Sketch of the motion

$x^2 + y^2 = z + 1$

$\Rightarrow$ parabolic surface



$r^2 = e^{-\gamma t} + (e^{-\gamma t} + 1)^2$

$= \underline{1 - e^{-\gamma t} + e^{-2\gamma t}}$

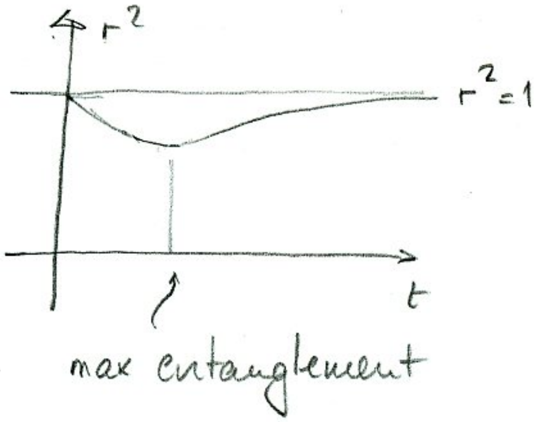
$t = 0 : r^2 = 1$, $t \to \infty : r^2 \to 1$ ent. entropy $S_A = 0$

Intermediate times $0 < r^2 < 1$

min value for $e^{-\gamma t} = \frac{1}{2}$

$\Rightarrow r^2 = \frac{3}{4}$

gives max value for S_A



FYS4110 Eksamensoppgaver 2012

Løsninger

Problem 1

a) Hamiltonian applied to the product states

$$\hat{H}|++\rangle = \tfrac{1}{2}\hbar(\omega_1+\omega_2)|++\rangle$$

$$\hat{H}|--\rangle = -\tfrac{1}{2}\hbar(\omega_1+\omega_2)|--\rangle$$

$$\hat{H}|+-\rangle = \tfrac{1}{2}\hbar\Delta|+-\rangle + \tfrac{1}{2}\hbar\lambda|-+\rangle$$

$$\hat{H}|-+\rangle = -\tfrac{1}{2}\hbar\Delta|-+\rangle + \tfrac{1}{2}\hbar\lambda|+-\rangle$$

In the subspace spanned by $|+-\rangle$ and $|-+\rangle$,

$$H = \tfrac{1}{2}\hbar\begin{pmatrix}\Delta & \lambda \\ \lambda & -\Delta\end{pmatrix} = \tfrac{1}{2}\hbar\mu\begin{pmatrix}\cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi\end{pmatrix}$$

The matrix is determined by φ, with μ as a scale factor. This implies that the eigenstates are determined by φ.

b) Eigenvalues in subspace

$$\begin{vmatrix}\cos\varphi-\varepsilon & \sin\varphi \\ \sin\varphi & -\cos\varphi-\varepsilon\end{vmatrix} = 0 \Rightarrow \varepsilon_\pm = \pm 1 .$$

energies $E_\pm = \pm\tfrac{1}{2}\hbar\mu = \underline{\pm\tfrac{1}{2}\hbar\sqrt{\Delta^2+\lambda^2}}$

Eigenstates

$$(\cos\varphi \mp 1)\alpha_\pm + \sin\varphi\,\beta_\pm = 0$$

$$(\cos\varphi \pm 1)\beta_\pm - \sin\varphi\,\alpha_\pm = 0$$

$$\Rightarrow (\cos\varphi \mp 1)\beta_\mp - \sin\varphi\,\alpha_\mp = 0$$

$$\frac{\beta_+}{\alpha_+} = -\frac{\alpha_-}{\beta_-} = -\frac{\cos\varphi - 1}{\sin\varphi} = \frac{2\sin^2\varphi/2}{2\cos\varphi/2\,\sin\varphi/2} = \tan\varphi/2$$

Normalized solutions

$\alpha_+ = \cos\frac{\varphi}{2} \quad \beta_+ = \sin\frac{\varphi}{2} \qquad |\psi_+\rangle = \cos\frac{\varphi}{2}|+-\rangle + \sin\frac{\varphi}{2}|-+\rangle$

$\alpha_- = \sin\frac{\varphi}{2} \quad \beta_- = -\cos\frac{\varphi}{2} \qquad |\psi_-\rangle = \sin\frac{\varphi}{2}|+-\rangle - \cos\frac{\varphi}{2}|-+\rangle$

c) $\Delta = 0 \Rightarrow \cos\varphi = 0 \Rightarrow \varphi = \pi/2 \Rightarrow \cos\frac{\varphi}{2} = \sin\frac{\varphi}{2} = \frac{1}{\sqrt{2}}$

$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|+-\rangle \pm |-+\rangle)$

$|\pm\mp\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle \pm |\psi_-\rangle) = |\psi(0)\rangle$

Time evolution

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{2}\mu t}|\psi_+\rangle + e^{\frac{i}{2}\mu t}|\psi_-\rangle\right) \qquad \mu = \lambda$$

$$= \frac{1}{2}\left(e^{-\frac{i}{2}\mu t}(|+-\rangle + |-+\rangle) + e^{\frac{i}{2}\mu t}(|+-\rangle - |-+\rangle)\right)$$

$$= \underline{\cos\left(\tfrac{\mu t}{2}\right)|+-\rangle - i\sin\left(\tfrac{\mu t}{2}\right)|-+\rangle} \equiv c(t)|+-\rangle + i s(t)|-+\rangle$$

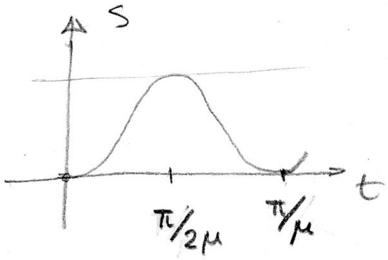
Density operator

$$\rho(t) = c(t)^2|+-\rangle\langle+-| + s(t)^2|-+\rangle\langle-+|$$
$$+ c(t)s(t)\left(|+-\rangle\langle-+| + |-+\rangle\langle+-|\right)$$

Reduced density operators

$\rho_1(t) = c(t)^2|+\rangle\langle+| + s(t)^2|-\rangle\langle-|$

$\rho_2(t) = c(t)^2|-\rangle\langle-| + s(t)^2|+\rangle\langle+|$



Entanglement entropy

$S_1 = S_2 = -c^2\log c^2 - s^2\log s^2$

max value : $c^2 = s^2 = \frac{1}{2} \Rightarrow S_A = \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = \log 2$

min value : $c^2 = 1 \vee s^2 = 1 \quad S = 0$ for $c = 0 \vee s = 0$, $t = 0, \frac{\pi}{\mu}, \frac{\pi}{2\mu}, \ldots$

period $\underline{T = \frac{\pi}{\mu}}$

Problem 2

a) Hamiltonian applied to the product states

$$\hat{H}|g,1\rangle = \hbar(\tfrac{1}{2}\omega - i\gamma)|g,1\rangle + \tfrac{1}{2}\hbar\lambda|e,0\rangle$$

$$\hat{H}|e,0\rangle = \tfrac{1}{2}\hbar\omega|e,0\rangle + \tfrac{1}{2}\hbar\lambda|g,1\rangle$$

$$\hat{H}|g,0\rangle = -\tfrac{1}{2}\hbar\omega|g,0\rangle$$

In the space spanned by $|g,1\rangle$ and $|e,0\rangle$

$$H = \tfrac{1}{2}\hbar(\omega - i\gamma)\mathbb{1} + \tfrac{1}{2}\hbar\begin{pmatrix} -i\gamma & \lambda \\ \lambda & i\gamma \end{pmatrix} \equiv H_0 + H_1$$

b) Define $|\psi(t)\rangle = e^{-\frac{i}{2}\omega t - \frac{1}{2}\gamma t}|\phi(t)\rangle$

$$|\phi(t)\rangle = (\cos(\Omega t) + a\sin(\Omega t))|e,0\rangle + ib\sin(\Omega t)|g,1\rangle$$

$\Rightarrow$ $|\psi(0)\rangle = |\phi(0)\rangle = |e,0\rangle$

satisfies the initial condition

need to show that $|\psi(t)\rangle$ satisfies the Schrödinger eq.

Note $i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$ $\quad$ I

$\Leftrightarrow$ $i\hbar\frac{d}{dt}|\phi(t)\rangle = \hat{H}_1|\phi(t)\rangle$ II

Need to show that II is satisfied

$$i\hbar\frac{d}{dt}|\phi(t)\rangle = i\hbar\Omega[ib\cos(\Omega t)|g,1\rangle + (-\sin\Omega t + a\cos(\Omega t))|e,0\rangle]$$

$$\hat{H}_1|\phi(t)\rangle = \tfrac{1}{2}\hbar\{\gamma b\sin(\Omega t) + \lambda(\cos(\Omega t) + a\sin(\Omega t))\}|g,1\rangle$$

$$\tfrac{1}{2}\hbar(i\lambda b\sin\Omega t + i\gamma(\cos(\Omega t) + a\sin(\Omega t))|e,0\rangle$$

$$= \tfrac{1}{2}\hbar[\{\lambda\cos(\Omega t) + (a\lambda + \gamma b)\sin(\Omega t)\}|g,1\rangle$$

$$+ i\{\gamma\cos\Omega t + (\lambda b + \gamma a)\sin(\Omega t)\}|e,0\rangle]$$

Conditions for equality

$$-\Omega b = \tfrac{1}{2}\lambda \qquad \mathrm{I}$$

$$a\lambda + \gamma b = 0 \qquad \mathrm{II}$$

$$\Omega a = \tfrac{1}{2}\gamma \qquad \mathrm{III}$$

$$-\Omega = \tfrac{1}{2}(\lambda b + \gamma a) \qquad \mathrm{IV}$$

$$\mathrm{I} \Rightarrow \quad \underline{b = -\frac{\lambda}{2\Omega}} \qquad \mathrm{III} \quad \underline{a = \frac{\gamma}{2\Omega}}$$

$$\Rightarrow a\lambda + \gamma b = \frac{\gamma\lambda - \gamma\lambda}{2\Omega} = 0$$ consistent with II

$$\mathrm{IV} \Rightarrow \quad \Omega = \frac{1}{4\Omega}(\lambda^2 - \gamma^2)$$

$$\Omega^2 = \tfrac{1}{4}(\lambda^2 - \gamma^2) \Rightarrow \underline{\Omega = \tfrac{1}{2}\sqrt{\gamma^2 - \lambda^2}}$$

c) Assume $\mathrm{Tr}\, \rho_{tot} = 1$

$$\Rightarrow \mathrm{Tr}\rho(t) + f(t) = 1 \qquad f(t) = 1 - \mathrm{Tr}\rho(t)$$

$$\mathrm{Tr}\rho(t) = \langle \psi(t) | \psi(t) \rangle = e^{-\gamma t} \langle \phi(t) | \phi(t) \rangle$$

$$\langle \phi(t) | \phi(t) \rangle = \cos^2(\Omega t) + a^2 \sin^2(\Omega t) + 2a \cos\Omega t \sin\Omega t + b^2 \sin^2\Omega t$$

$$= 1 + (a^2 + b^2 - 1)\sin^2\Omega t + 2a\cos\Omega t \sin\Omega t$$

$$= 1 + \tfrac{1}{2}(a^2 + b^2 - 1) - \tfrac{1}{2}(a^2 + b^2 - 1)\cos(2\Omega t) + a\sin(2\Omega t)$$

$$a^2 + b^2 - 1 = \frac{\lambda^2 + \gamma^2}{\lambda^2 - \gamma^2} - 1 = \frac{2\gamma^2}{\lambda^2 - \gamma^2}$$

$$1 + \tfrac{1}{2}(a^2 + b^2 - 1) = 1 + \frac{\gamma^2}{\lambda^2 - \gamma^2} = \frac{\lambda^2}{\lambda^2 - \gamma^2}$$

$$= \mathrm{Tr}\rho = \underline{e^{-\gamma t}\left(\frac{\lambda^2}{\lambda^2 - \gamma^2} - \frac{\gamma^2}{\lambda^2 - \gamma^2}\cos(\sqrt{\lambda^2 - \gamma^2}\, t) + \frac{\gamma}{\sqrt{\lambda^2 - \gamma^2}}\sin(\sqrt{\lambda^2 - \gamma^2}\, t)\right)}$$

$$\underline{f(t) = 1 - \mathrm{Tr}\rho(t)}$$?

The decay of $\mathrm{Tr}\rho$ is due to the leakage of the cavity photon out of the system. For the cavity states this corresponds to the transition $|g,1\rangle \to |g,0\rangle$. The second term in Eq. (5) gives the build up of probability in $|g,0\rangle$ due to this process.

With $\gamma = 0$, there are oscillations between $|g,1\rangle$ and $|e,0\rangle$ due to the coupling between the atom and the cavity photon. The time evolution of the state (4) shows, for $\gamma \neq 0$, decay of the probabilities due to the leakage $|g,1\rangle \to |g,0\rangle$, superimposed on these oscillations

Problem 3

a) The full density operator

$$\rho_n = \tfrac{1}{3}\{ |{+}{-}{-}\rangle\langle{+}{-}{-}| + |{-}{+}{-}\rangle\langle{-}{+}{-}| + |{-}{-}{+}\rangle\langle{-}{-}{+}|$$
$$+ \eta^n (|{-}{+}{-}\rangle\langle{+}{-}{-}| + |{+}{-}{-}\rangle\langle{-}{-}{+}|) + (\eta^*)^n (|{+}{-}{-}\rangle\langle{-}{+}{-}| + |{-}{-}{+}\rangle\langle{+}{-}{-}|)$$
$$+ \eta^{2n} |{-}{+}{-}\rangle\langle{-}{-}{+}| + (\eta^*)^{2n} |{-}{-}{+}\rangle\langle{-}{+}{-}|$$

Reduced density operator

$$\rho_n^A = \mathrm{Tr}_{BC}\,\rho_n = \tfrac{1}{3}(|+\rangle\langle+| + 2|-\rangle\langle-|) \quad \text{independent of } n,$$

information about n can therefore not be detected by A

Measurement by A, B, C in basis I, gives result determined by probabilities of the form $\langle abc|\rho_n|abc\rangle$ with $|abc\rangle$ as a product of states $|\pm\rangle$. Only the diagonal terms in ρ_n give contributions, and these are independent of n.

Again there are no measurable differences between different n.

b) Reduced density operator

$$\rho_n^{AB} = Tr_C \rho_n = \frac{1}{3}\{|+-\rangle\langle+-| + |-+\rangle\langle-+| + |--\rangle\langle--|$$
$$+ \eta^n |-+\rangle\langle+-| + (\eta^*)^n |+-\rangle\langle-+|\}$$

probabilities $p(k|n) = \langle\phi_k|\rho_n^{AB}|\phi_k\rangle$

Need overlap between vectors of basis I and II:

$$\langle 0|+\rangle = \langle 0|-\rangle = \langle 1|+\rangle = \frac{1}{\sqrt{2}} \quad \langle 1|-\rangle = -\frac{1}{\sqrt{2}}$$

note: only sign change for $\langle 1|-\rangle$

$$p(1|0) = \langle 00|\rho_0^{AB}|00\rangle = \frac{1}{3}\,\frac{5}{4} = \frac{5}{12}$$
$$p(2|0) = \langle 01|\rho_0^{AB}|01\rangle = \frac{1}{3}\left(\frac{3}{4} - \frac{2}{4}\right) = \frac{1}{12}$$
$$p(1|1) = \langle 00|\rho_1^{AB}|00\rangle = \frac{1}{3}\left(\frac{3}{4} + \frac{\eta+\eta^*}{4}\right) = \frac{1}{6}$$
$$p(2|1) = \langle 01|\rho_1^{AB}|01\rangle = \frac{1}{3}\left(\frac{3}{4} - \frac{\eta+\eta^*}{4}\right) = \frac{1}{3}$$

Have used $\eta + \eta^* = -1$

The change $n=1 \rightarrow n=2$ corresponds to $\eta \rightarrow \eta^*$ since $\eta^2 = \eta^*$
no change since the probabilities are real

c) Normalization of probabilities

$$\sum_n \bar{p}(n|k) = 1 \Rightarrow p(k) = \sum_n p(k|n)$$

$$p(1) = p(1|0) + p(1|1) + p(1|2) = \frac{5}{12} + 2\cdot\frac{1}{6} = \frac{9}{12} = \frac{3}{4}$$

Probabilities for $k=1$, $n = 0, 1, 2$

$$\bar{p}(0|1) = \frac{p(1|0)}{p(1)} = \frac{5}{12}\cdot\frac{12}{9} = \frac{5}{9}$$
$$\bar{p}(1|1) = \frac{p(1|1)}{p(1)} = \frac{1}{6}\cdot\frac{12}{9} = \frac{2}{9}$$
$$\bar{p}(2|1) = \text{—} \quad \text{''} \quad \text{—} = \frac{2}{9}$$

The message $n=0$ is most probable, with probability $\frac{5}{9}$, while $n=1$ and 2 have probability $\frac{2}{9}$.

FYS4110/9110 Eksamen 2013

Løsninger

Oppgave 1

a) Utnytter $\hat{\alpha}^\dagger\hat{\alpha} = |e\rangle\langle g|g\rangle\langle e| = |e\rangle\langle e|$

Lindbladligning

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_0,\hat{\rho}] - \frac{1}{2}\gamma\{|e\rangle\langle e|\hat{\rho} + \hat{\rho}|e\rangle\langle e| - 2|g\rangle\langle e|\hat{\rho}|e\rangle\langle g|\}$$

for matriseelementer, utnytter

$$\langle e|[\hat{H}_0,\hat{\rho}]|e\rangle = \langle g|[\hat{H}_0,\hat{\rho}]|g\rangle = 0$$

$$\langle e|[\hat{H}_0,\hat{\rho}]|g\rangle = (E_e - E_g)\langle e|\hat{\rho}|g\rangle = \hbar\omega\langle e|\hat{\rho}|g\rangle$$

$$\Rightarrow \quad \frac{dp_e}{dt} = -\gamma p_e \qquad p_e(t) = e^{-\gamma t}p_e(0)$$

$$\frac{dp_g}{dt} = \gamma p_e \quad \Rightarrow \quad p_g(t) = 1 - p_e(t)$$

$$\frac{db}{dt} = (-i\omega - \tfrac{1}{2}\gamma)\, b \quad \Rightarrow \quad b(t) = e^{-i\omega t - \frac{1}{2}\gamma t}\, b(0)$$

Initialbetingelser

$$p_e(0) = 1, \quad p_g(0) = 0, \quad b(0) = 0$$

$$\Rightarrow \quad \underline{p_e(t) = e^{-\gamma t}}, \quad \underline{p_g(t) = 1 - e^{-\gamma t}}, \quad \underline{b(t) = 0}$$

b) Nye initialbetingelser

$$p_e(0) = |\langle e|\psi\rangle|^2 = \tfrac{1}{2}$$

$$p_g(0) = |\langle g|\psi\rangle|^2 = \tfrac{1}{2}$$

$$b(0) = \langle e|\psi\rangle\langle\psi|g\rangle = \tfrac{1}{2}$$

Tidsutvikling

$$p_e(t) = \frac{1}{2} e^{-\gamma t}, \quad p_g(t) = 1 - \frac{1}{2} e^{-\gamma t}, \quad b(t) = \frac{1}{2} e^{-i\omega t - \frac{1}{2}\gamma t}$$

$$\Rightarrow \quad \underline{\rho(t) = \frac{1}{2} \begin{pmatrix} e^{-\gamma t} & e^{-i\omega t - \frac{1}{2}\gamma t} \\ e^{i\omega t - \frac{1}{2}\gamma t} & 2 - e^{-\gamma t} \end{pmatrix}}$$

c)

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

$$\Rightarrow z = p_e - p_g, \quad x = 2\,\mathrm{Re}\, b, \quad y = -2\,\mathrm{Im}\, b$$

$$\Rightarrow r^2 = (p_e - p_g)^2 + 4|b|^2$$

Tilfelle a):

$$r^2 = (2e^{-\gamma t} - 1)^2$$

minimum for $e^{-\gamma t} = \frac{1}{2}$, $\underline{t = \frac{1}{\gamma}\ln 2}$ $\quad \underline{r_{min} = 0}$

$\Rightarrow \hat{\rho} = \frac{1}{2}\mathbb{1}$, maksimalt blandet $\Rightarrow$ A+B er maksimalt sammenfiltret.

Tilfelle b)

$$r^2 = (e^{-\gamma t} - 1)^2 + e^{-\gamma t} = e^{-2\gamma t} - e^{-\gamma t} + 1$$

$$\frac{d}{dt} r^2 = 0 \Rightarrow -2e^{-2\gamma t} + e^{-\gamma t} = 0 \Rightarrow e^{-\gamma t} = \frac{1}{2}, \quad \underline{t = \frac{1}{\gamma}\ln 2}$$

$$\Rightarrow r^2_{min} = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}, \quad \underline{r_{min} = \frac{1}{2}\sqrt{3}}$$

Siden $r_{min} < 1$ er $\hat{\rho}$ en blandet tilstand,

$\Rightarrow$ A+B er sammenfiltret, men mindre enn i tilfellet a)

I begge tilfeller er $r = 1$ både for $t = 0$ og $t \to \infty$, dvs. sammenfiltringen er bare midlertidig under henfallet $|\psi\rangle_{init} \to |g\rangle$.

Oppgave 2

a) Reduserte tetthetsoperatorer

$$\hat{\rho}_A = \mathrm{Tr}_{BC}(|\psi\rangle\langle\psi|) = \tfrac{1}{2}(|u\rangle\langle u| + |d\rangle\langle d|) = \underline{\tfrac{1}{2}\mathbb{1}_A}$$

$$\hat{\rho}_{BC} = \mathrm{Tr}_A(|\psi\rangle\langle\psi|) = \underline{\tfrac{1}{2}(|uu\rangle\langle uu| + |dd\rangle\langle dd|)}$$

$\hat{\rho}_A$ er maksimalt blandet $\Rightarrow$ sammenfiltringsentropien er maksimal: $S = -\mathrm{Tr}_A(\hat{\rho}_A \log \rho_A) = \log 2$

$\hat{\rho}_{BC}$ er separabel, dvs en sum av produkttilstander, $|u\rangle \otimes |u\rangle$ og $|d\rangle \otimes |d\rangle$. Ingen sammenfiltring

b) Uttrykker A-tilstanden i $|\frac{\pi}{2},+\rangle \equiv |f\rangle$ og $|\frac{\pi}{2},-\rangle \equiv |b\rangle$

$$|u\rangle = \tfrac{1}{\sqrt{2}}(|f\rangle - |b\rangle), \quad |d\rangle = \tfrac{1}{\sqrt{2}}(|f\rangle + |b\rangle)$$

$$\Rightarrow |\psi\rangle = \tfrac{1}{2}|f\rangle \otimes (|uu\rangle + |dd\rangle) + \tfrac{1}{2}|b\rangle \otimes (|uu\rangle - |dd\rangle)$$

Målingen gir f (spinn opp) $\Rightarrow$

$$|\psi\rangle \to |\psi'\rangle = \tfrac{1}{\sqrt{2}}|f\rangle \otimes (|uu\rangle + |dd\rangle) \quad \text{normert}$$

$$\hat{\rho}_{BC} \to \hat{\rho}'_{BC} = \tfrac{1}{2}(|uu\rangle\langle uu| + |dd\rangle\langle dd| + |uu\rangle\langle dd| + |dd\rangle\langle uu|)$$

Dette er en ren tilstand

$$\hat{\rho}_B = \mathrm{Tr}_C\, \hat{\rho}_{BC} = \tfrac{1}{2}(|u\rangle\langle u| + |d\rangle\langle d|) = \tfrac{1}{2}\mathbb{1}_B$$

Denne er maksimalt blandet $\Rightarrow$ B+C er maks. sammenfiltret

Målingen på A gjør B+C sammenfiltret!

c) Roterte tilstander

$$|u\rangle = \cos\frac{\theta}{2}\,|\theta,+\rangle - \sin\frac{\theta}{2}\,|\theta,-\rangle$$

$$|d\rangle = \sin\frac{\theta}{2}\,|\theta,+\rangle + \cos\frac{\theta}{2}\,|\theta,-\rangle$$

$\Longrightarrow$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big\{|\theta,+\rangle \otimes \left(\cos\frac{\theta}{2}\,|uu\rangle + \sin\frac{\theta}{2}\,|dd\rangle\right)$$

$$+ \; |\theta,-\rangle \otimes \left(-\sin\frac{\theta}{2}\,|uu\rangle + \cos\frac{\theta}{2}\,|dd\rangle\right)\Big\}$$

Måleresultat $(\theta,+)$ $\Rightarrow$

$$|\psi\rangle \to |\theta,+\rangle \otimes \left(\cos\frac{\theta}{2}\,|uu\rangle + \sin\frac{\theta}{2}\,|dd\rangle\right)$$

$$\equiv |\theta,+\rangle \otimes |\psi'_{BC}(\theta)\rangle$$

$$\hat{\rho}_{BC} \to \hat{\rho}'_{BC} = |\psi'_{BC}\rangle\langle\psi'_{BC}| \qquad \text{ren tilstand}$$

$$= \underline{\cos^2\frac{\theta}{2}\,|uu\rangle\langle uu| + \sin^2\frac{\theta}{2}\,|dd\rangle\langle dd|}$$

$$\underline{+ \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(|uu\rangle\langle dd| + |dd\rangle\langle uu|\right)}$$

Redusert tetthetsoperator

$$\hat{\rho}_B = \mathrm{Tr}_C\,\hat{\rho}_{BC} = \cos^2\frac{\theta}{2}\,|u\rangle\langle u| + \sin^2\frac{\theta}{2}\,|d\rangle\langle d|$$

$\langle u|d\rangle = 0 \Rightarrow \cos^2\frac{\theta}{2}$ og $\sin^2\frac{\theta}{2}$ er egenverdier til $\hat{\rho}_B$

Entropi $S = \underline{-\cos^2\frac{\theta}{2}\ln(\cos^2\frac{\theta}{2}) - \sin^2\frac{\theta}{2}\ln\sin^2\frac{\theta}{2}}$

= sammenfiltringsentropien mellom B og C

Oppgave 3

a) $\vec{\sigma} = \sigma_x \vec{e}_x + \sigma_y \vec{e}_y + \sigma_z \vec{e}_z$

$$= \begin{pmatrix} \vec{e}_z & \vec{e}_x - i\vec{e}_y \\ \vec{e}_x + i\vec{e}_y & -\vec{e}_z \end{pmatrix}$$

$$\vec{\sigma}_{BA} = (0\ 1)\left(- - - \right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{e}_x + i\vec{e}_y \equiv \vec{e}_+$$

$$(\vec{k} \times \vec{\varepsilon}_{\vec{k}a}) \cdot \vec{e}_+ = (\vec{e}_+ \times \vec{k}) \cdot \vec{\varepsilon}_{\vec{k}a}$$

$$\vec{k} = k(\cos\varphi \sin\theta\, \vec{e}_x + \sin\varphi \sin\theta\, \vec{e}_y + \cos\theta\, \vec{e}_z)$$

$$\Rightarrow \vec{e}_+ \times \vec{k} = ik(\cos\theta\, \vec{e}_+ - e^{i\varphi} \sin\theta\, \vec{e}_z)$$

Vinkelavhengighet til $|\langle B\, 1_{\vec{k}a} | \hat{H}_1 | A, 0 \rangle|^2$:

$$p(\theta, \varphi) = N \sum_a |(\vec{e}_+ \times \vec{k}) \cdot \vec{\varepsilon}_{\vec{k}a}|^2$$

$= 0$ $\quad$ N norm. faktor

$$= N\left(|\vec{e}_+ \times \vec{k}|^2 - |(\vec{e}_+ \times \vec{k}) \cdot \tfrac{\vec{k}}{k}|^2 \right)$$

$$= Nk^2 (2\cos^2\theta + \sin^2\theta) \qquad |\vec{e}_+|^2 = 2$$

$$= Nk^2 (1 + \cos^2\theta) \quad \text{uavh av } \varphi$$

Normering $\int d\varphi \int d\theta \sin\theta\,(1 + \cos^2\theta) = 2\pi \int_{-1}^{1} (1 + u^2)\, du = 2\pi \left[u + \tfrac{1}{3}u^3 \right]_{-1}^{1}$

$$= \frac{16}{3}\pi$$

$$\Rightarrow \underline{p(\theta, \varphi) = \frac{3}{16\pi}(1 + \cos^2\theta)}$$

b) $\vec{k} = k\vec{e}_x$

Sannsynlighet for deteksjon av foton med polarisasjon i retning $\vec{\varepsilon}(\alpha)$, $\qquad \vec{e}_+ \times \vec{e}_x = -i\vec{e}_z$

$$p(\alpha) = N' |(\vec{e}_+ \times \vec{e}_x) \cdot \vec{\varepsilon}(\alpha)|^2$$

$$= N' |\vec{e}_z \cdot \vec{\varepsilon}(\vec{\alpha})|^2$$

$$= N' \sin^2\alpha$$

$$p(\alpha) + p(\alpha + \tfrac{\pi}{2}) = N' = 1 \quad \Rightarrow \quad \underline{p(\alpha) = \sin^2\alpha}$$

Sannsynlighet for deteksjon:

$p(0) = 0 \qquad \alpha = 0 \Rightarrow \vec{\varepsilon} = \vec{e}_y$

$p(\frac{\pi}{2}) = 1 \qquad \alpha = \frac{\pi}{2} \Rightarrow \vec{\varepsilon} = \vec{e}_z$

viser at fotoner utsendt langs x-aksen
er polarisert langs z-aksen

FYS4110, Exam 2014

Solutions

Problem 1

a) $\hat{\rho}_I = \cos^2 x\, |1\rangle\langle 1| + \sin^2 x\, |2\rangle\langle 2| + \cos x \sin x\, (|1\rangle\langle 2| + |2\rangle\langle 1|)$

$= \frac{1}{2}\cos^2 x\, (|+-\rangle\langle +-| + |-+\rangle\langle -+| + |+-\rangle\langle -+| + |-+\rangle\langle +-|)$

$+ \frac{1}{2}\sin^2 x\, (|+-\rangle\langle +-| + |-+\rangle\langle -+| - |+-\rangle\langle -+| - |-+\rangle\langle +-|)$

$+ \frac{1}{2}\cos x \sin x\, (|+-\rangle\langle +-| - |-+\rangle\langle -+| - |+-\rangle\langle -+| + |-+\rangle\langle +-|)$

$+ \frac{1}{2}\cos x \sin x\, (|+-\rangle\langle +-| - |-+\rangle\langle -+| + |+-\rangle\langle -+| - |-+\rangle\langle +-|)$

$= \frac{1}{2}(1 + \sin(2x))|+-\rangle\langle +-| + \frac{1}{2}(1 - \sin(2x))|-+\rangle\langle -+|$

$+ \frac{1}{2}\cos 2x\, (|+-\rangle\langle -+| + |-+\rangle\langle +-|)$

Reduced density operators

$\hat{\rho}_{IA} = \mathrm{Tr}_B\, \hat{\rho}_I = \frac{1}{2}(1+\sin(2x))\,|+\rangle\langle +| + \frac{1}{2}(1-\sin(2x))\,|-\rangle\langle -|$

$= \frac{1}{2}(\mathbb{1} + \sin(2x)\,\sigma_z)$

$\hat{\rho}_{IB} = \mathrm{Tr}_A\, \hat{\rho}_I = \frac{1}{2}(1-\sin(2x))\,|+\rangle\langle +| + \frac{1}{2}(1+\sin(2x))\,|-\rangle\langle -|$

$= \frac{1}{2}(\mathbb{1} - \sin(2x)\,\sigma_z)$

Entropies: $S_I = 0$ (pure state)

$S_{IA} = S_{IB} = -\frac{1}{2}(1+\sin(2x))\log(\frac{1}{2}(1+\sin(2x))) - \frac{1}{2}(1-\sin(2x))\log(\frac{1}{2}(1-\sin(2x)))$

$x = 0, \frac{\pi}{2}$ $\quad S_{IA} = S_{IB} = \log 2$; maximally entangled states

$x = \frac{\pi}{4}$ $\quad S_{IA} = S_{IB} = 0$, non-entangled, product state $|\psi\rangle = |+\rangle \otimes |-\rangle$

b) Case II

$$\hat{\rho}_{II} = \cos^2 x\, |1\rangle\langle 1| + \sin^2 x\, |2\rangle\langle 2|$$

$$\Rightarrow S_{II} = \underline{-\cos^2 x \log(\cos^2 x) - \sin^2 x \log(\sin^2 x)}$$

$\hat{\rho}_{II}$ obtained from $\hat{\rho}_I$ by deleting terms proportional to $\cos x \sin x = \frac{1}{2}\sin(2x)$:

$$\hat{\rho}_{II} = \tfrac{1}{2}(|+-\rangle\langle +-| + |-+\rangle\langle -+|) + \tfrac{1}{2}\cos(2x)(|+-\rangle\langle -+| + |-+\rangle\langle +-|)$$

$$\Rightarrow \underline{\hat{\rho}_{IIA} = \hat{\rho}_{IIB} = \tfrac{1}{2}\mathbb{1}} \Rightarrow \underline{S_{IIA} = S_{IIB} = \log 2}$$

$x = 0, \pi/2$ Same as in case I

$x = \pi/4$, $S_{II} = \log 2$; maximally mixed

$$\hat{\rho}_{II} = \tfrac{1}{2}(|+-\rangle\langle +-| + |-+\rangle\langle -+|)$$

separable (sum of product states) $\Rightarrow$ non-entangled

c) $\Delta_I = -S_{IA} = -S_{IB}$

is negative, unless $S_{IA} = S_{IB} = 0$,

which happens for $x = \pi/4$.

$$\Delta_{II} = S_{II} - \log 2$$

$S_{II} \leq \log 2$ since the Hilbert space is two-dimensional

$\Rightarrow \Delta_{II} \leq 0$, $\Delta_{II} = 0$ only when $S_{II} = \log 2$,

this happens only when $\underline{x = \pi/4}$ $\Rightarrow \cos^2 x = \sin^2 x = \frac{1}{2}$

Problem 2

a) Matrix elements of $\hat{x}$

$$X_{mn} = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle m|\hat{a}^{+}|n\rangle + \langle m|\hat{a}|n\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right)$$

Non-vanishing: $X_{n-1,n} = X_{n,n-1} = \sqrt{\frac{\hbar n}{2m\omega}}$

Photon emission: $|n\rangle \rightarrow |n-1\rangle$ $(E_n \rightarrow E_{n-1} + \hbar\omega)$

$$\Rightarrow W_{n-1,n} = \frac{2\alpha\hbar}{3mc^2}\omega^2 n = \gamma n$$

b) $$\frac{dp_n}{dt} = \langle n|\left(-\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}] - \frac{1}{2}\gamma(\hat{a}^{+}\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^{+}\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^{+})\right)|n\rangle$$

$$= -\gamma\left(np_n - (n+1)p_{n+1}\right)$$

$W_{n-1,n}$ = transition rate when state $|n\rangle$ occupied

$\Rightarrow p_n = 1$, $p_m = 0$ $m \neq n$

With this assumption, conservation of probability gives

$$\frac{dp_n}{dt} = -W_{n-1,n}$$

$$= -\gamma n \quad \text{(from eq. (9))}$$

consistent with eq. (8).

c) Excitation energy

$$E = Tr(\hat{H}_0 \hat{\rho}) - \tfrac{1}{2}\hbar\omega$$

$$= \sum_n \hbar\omega(n+\tfrac{1}{2})\langle n|\hat{\rho}|n\rangle - \tfrac{1}{2}\hbar\omega$$

$$= \sum_n \hbar\omega\, n p_n$$

$$\Rightarrow \frac{dE}{dt} = \hbar\omega \sum_n n \frac{dp_n}{dt}$$

$$= -\gamma\hbar\omega \sum_n (n^2 p_n - n(n+1)p_{n+1})$$

$$= -\gamma\hbar\omega \sum_n (n^2 - n(n-1))\, p_n$$

$$= -\gamma\hbar\omega \sum_n n p_n$$

$$= \underline{-\gamma E}$$

Integrated

$$\frac{dE}{E} = -\gamma dt \Rightarrow \ln E = -\gamma t + \text{const}$$

$$\Rightarrow \underline{E(t) = E(0)e^{-\gamma t}}$$ exponential decay

Problem 3

a) $\mathrm{Tr}\hat{\rho} = 1 \Rightarrow N(\beta)^{-1} = \mathrm{Tr}(e^{-\beta H})$

$$= \sum_n e^{-\beta E_n}$$

$$E(\beta) = \mathrm{Tr}(\hat{H}\hat{\rho}) = N\,\mathrm{Tr}(\hat{H}e^{-\beta\hat{H}})$$

$$= -N\frac{\partial}{\partial\beta}\mathrm{Tr}(e^{-\beta\hat{H}}) = -N\frac{\partial}{\partial\beta}N^{-1}$$

$$= \frac{1}{N}\frac{\partial}{\partial\beta}\ln N = \underline{\frac{\partial}{\partial\beta}\ln N(\beta)}$$

Entropy: $S(\beta) = -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$

$$= -\mathrm{Tr}(Ne^{-\beta\hat{H}}(\ln N - \beta\hat{H}))$$

$$= -\ln N\,\mathrm{Tr}\hat{\rho} + \beta\,\mathrm{Tr}(\hat{H}\hat{\rho})$$

$$= -\ln N + \beta E(\beta)$$

$$= \underline{\beta\frac{\partial}{\partial\beta}\ln N(\beta) - \ln N(\beta)}$$

b) $\hat{H} = \frac{1}{2}\varepsilon\sigma_z \Rightarrow E_\pm = \pm\frac{1}{2}\varepsilon$

$$\Rightarrow N^{-1} = e^{\frac{1}{2}\varepsilon\beta} + e^{-\frac{1}{2}\varepsilon\beta} = 2\cosh(\tfrac{1}{2}\varepsilon\beta)$$

$$N(\beta) = \underline{\frac{1}{2\cosh(\frac{1}{2}\varepsilon\beta)}}$$

$$E(\beta) = -2\cosh(\tfrac{1}{2}\varepsilon\beta)\frac{1}{2\cosh^2(\frac{1}{2}\varepsilon\beta)}\sinh(\tfrac{1}{2}\varepsilon\beta)\cdot\tfrac{1}{2}\varepsilon$$

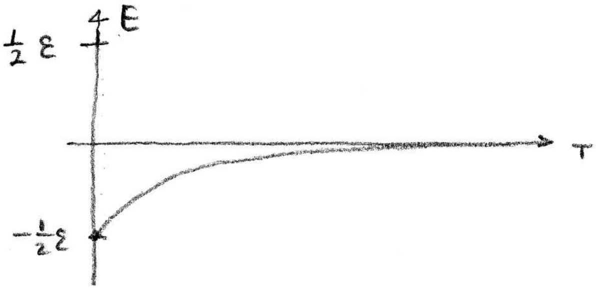
$$= \underline{-\tfrac{1}{2}\varepsilon\tanh(\tfrac{1}{2}\varepsilon\beta)}$$

$$\underline{S(\beta) = -\tfrac{1}{2}\varepsilon\beta\tanh(\tfrac{1}{2}\varepsilon\beta) + \ln(2\cosh(\tfrac{1}{2}\varepsilon\beta))}$$

$$E(\beta) = -\tfrac{1}{2}\varepsilon \tanh(\tfrac{1}{2}\varepsilon\beta)$$
$$= -\frac{1}{2}\varepsilon \; \frac{e^{\frac{1}{2}\varepsilon\beta} - e^{-\frac{1}{2}\varepsilon\beta}}{e^{\frac{1}{2}\varepsilon\beta} + e^{-\frac{1}{2}\varepsilon\beta}}$$

$$T \to 0 \Rightarrow \beta \to \infty \Rightarrow E(\beta) \simeq -\frac{1}{2}\varepsilon(1 - e^{-\varepsilon\beta}) \to -\frac{1}{2}\varepsilon$$

$$T \to \infty \Rightarrow \beta \to 0 \Rightarrow E(\beta) \simeq -\frac{1}{4}\varepsilon^2\beta = -\frac{1}{4}\frac{\varepsilon^2}{k_B T} \to 0$$



c) $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}) \Rightarrow \vec{r} = Tr(\vec{\sigma}\hat{\rho})$

since $Tr\,\sigma_i = 0$ and $Tr(\sigma_i\sigma_j) = 2\delta_{ij}$

$$\vec{r} = N\, Tr(\vec{\sigma}\, e^{-\frac{1}{2}\varepsilon\beta\sigma_z})$$
$$= N\, Tr(\sigma_z e^{-\frac{1}{2}\varepsilon\beta\sigma_z})\,\vec{k}$$
$$= -\frac{2}{\varepsilon}N\frac{\partial}{\partial\beta}(Tr\, e^{-\frac{1}{2}\varepsilon\beta\sigma_z})\,\vec{k}$$
$$= -\frac{2}{\varepsilon}N\frac{\partial}{\partial\beta}N^{-1}\vec{k}$$
$$= -\frac{2}{\varepsilon}E(\beta)\,\vec{k}$$
$$= \underline{\tanh(\tfrac{1}{2}\varepsilon\beta)\vec{k}}$$

$\vec{r} = r\vec{k}$ with $r = -\frac{2}{\varepsilon}E(\beta)$

$T = 0\ (\beta = \infty)$: $r = 1$ pure state

$T \to \infty\ (\beta \to 0)$: $r \to 0$ maximally mixed

EXAM in FYS 4110/9110 Modern Quantum Mechanics 2015
Solutions

PROBLEM 1

a) Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega(\,\sigma_z \otimes \mathbb{1} - \,\mathbb{1} \otimes \sigma_z) + \hbar\lambda(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \tag{1}$$

Action on the basis states

$$\begin{aligned} \hat{H}|++\rangle &= \hat{H}|--\rangle = 0 \\ \hat{H}|+-\rangle &= \hbar\omega|+-\rangle + \hbar\lambda|-+\rangle \\ \hat{H}|-+\rangle &= -\hbar\omega|-+\rangle + \hbar\lambda|+-\rangle \end{aligned} \tag{2}$$

Matrix form of $\hat{H}$

$$H = \hbar\begin{pmatrix} \omega & \lambda \\ \lambda & -\omega \end{pmatrix} = \hbar a \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \tag{3}$$

b) Eigenvalue equation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \epsilon \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{4}$$

Secular equation

$$\epsilon^2 - \cos^2\theta - \sin^2\theta = 0 \quad \Rightarrow \quad \epsilon = \pm 1 \equiv \epsilon_\pm \tag{5}$$

Energy eigenvalues

$$E_\pm = \pm\hbar a = \pm\hbar\sqrt{\omega^2 + \lambda^2} \tag{6}$$

Eigenvectors

$$\begin{aligned} \cos\theta\alpha_\pm + \sin\theta\beta_\pm &= \pm\alpha_\pm \\ \Rightarrow \quad \alpha_+/\beta_+ = (1+\cos\theta)/\sin\theta &= \cot\frac{\theta}{2} \\ \alpha_-/\beta_- = (-1+\cos\theta)/\sin\theta &= -\tan\frac{\theta}{2} \end{aligned} \tag{7}$$

$$\begin{aligned} \Rightarrow \quad |\psi_+\rangle &= \cos\frac{\theta}{2}|+-\rangle + \sin\frac{\theta}{2}|-+\rangle \\ |\psi_-\rangle &= \sin\frac{\theta}{2}|+-\rangle - \cos\frac{\theta}{2}|-+\rangle \end{aligned} \tag{8}$$

The states $|++\rangle$ and $|--\rangle$ are energy eigenstates with eigenvalues $E = 0$.

c) Product states

$$\hat{\rho}_1 = |++\rangle\langle++|\,, \quad \hat{\rho}_2 = |--\rangle\langle--| \tag{9}$$

have no entanglement. Reduced density operators

$$\hat{\rho}_1^A = \hat{\rho}_1^B = |+\rangle\langle+|\,, \quad \hat{\rho}_2^A = \hat{\rho}_2^B = |-\rangle\langle-| \tag{10}$$

Non-product states

$$\begin{aligned}\hat{\rho}_\pm = |\psi_\pm\rangle\langle\psi_\pm| &= \cos^2\frac{\theta}{2}|+-\rangle\langle+-| + \sin^2\frac{\theta}{2}|-+\rangle\langle+-| \\ &\pm \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}(|+-\rangle\langle-+| + |-+\rangle\langle+-|)\end{aligned} \tag{11}$$

Reduced density operators

$$\begin{aligned}\hat{\rho}_+^A = \hat{\rho}_-^B &= \cos^2\frac{\theta}{2}|+\rangle\langle+| + \sin^2\frac{\theta}{2}|-\rangle\langle-| \\ \hat{\rho}_-^A = \hat{\rho}_+^B &= \sin^2\frac{\theta}{2}|+\rangle\langle+| + \cos^2\frac{\theta}{2}|-\rangle\langle-|\end{aligned} \tag{12}$$

Entanglement entropies

$$S_\pm(\theta) = \cos^2\frac{\theta}{2}\log(\cos^2\frac{\theta}{2}) + \sin^2\frac{\theta}{2}\log(\sin^2\frac{\theta}{2}) \tag{13}$$

Minimum entanglement for $\theta = 0$ ($\lambda/\omega = 0$), with $S_\pm(0) = 0$, maximum entanglement for $\theta = \pm\pi/2$ ($\omega/\lambda = 0$), with $S_\pm(0) = \log 2$. This is identical to the maximum possible entanglement entropy in the two-spin system.

PROBLEM 2

a) Hamiltonian

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar\lambda(\hat{a}^\dagger e^{-i\omega t} + \hat{a}e^{i\omega t}) \tag{14}$$

In the Heisenberg picture

$$\dot{\hat{a}}_H = \frac{i}{\hbar}\left[\hat{H}, \hat{a}\right]_H = -i\omega_0\hat{a}_H - i\lambda e^{-i\omega t}\mathbb{1} \tag{15}$$

gives

$$\ddot{\hat{a}}_H = \frac{i}{\hbar}\left[\hat{H}, \dot{\hat{a}}_H\right] + \frac{\partial\dot{a}_H}{\partial t} = -\omega_0^2\hat{a}_H - \lambda(\omega_0 + \omega)e^{-i\omega t}\mathbb{1} \tag{16}$$

which gives $C = -\lambda(\omega_0 + \omega)$.

b) Assume

$$\hat{a}_H = \hat{a}e^{-i\omega_0 t} + D(e^{-i\omega t} - e^{-i\omega_0 t})\mathbb{1} \tag{17}$$

Differentiation gives

$$\begin{aligned}\ddot{\hat{a}}_H &= -\omega_0^2 \hat{a} e^{-i\omega_0 t} - D(\omega^2 e^{-i\omega t} - \omega_0^2 e^{-i\omega_0 t}) \\ &= -\omega_0^2 \hat{a}_H - (\omega^2 - \omega_0^2) D e^{-i\omega t}\end{aligned} \tag{18}$$

which is of the form (16) with

$$D = \frac{\lambda}{\omega - \omega_0} \tag{19}$$

c) Time evolution

$$\begin{aligned}|\psi(0)\rangle &= |0\rangle\,, \quad \hat{a}|0\rangle = 0 \\ |\psi(t)\rangle &= \hat{\mathcal{U}}(t)|\psi(0)\rangle\end{aligned} \tag{20}$$

gives

$$\begin{aligned}\hat{a}|\psi(t)\rangle &= \hat{\mathcal{U}}(t)\hat{\mathcal{U}}^\dagger(t)\,\hat{a}\,\hat{\mathcal{U}}(t)|\psi(0)\rangle \\ &= \hat{\mathcal{U}}(t)\;\hat{a}_H(t)\,|\psi(0)\rangle \\ &= \hat{\mathcal{U}}(t)\;(\hat{a}e^{-i\omega_0 t} + D(e^{-i\omega t} - e^{-i\omega_0 t})\,|\psi(0)\rangle \\ &= \frac{\lambda}{\omega - \omega_0}\,(e^{-i\omega t} - e^{-i\omega_0 t})\,|\psi(t)\rangle\end{aligned} \tag{21}$$

This shows that $|\psi(t)\rangle$ is a coherent state with time dependent complex parameter $z(t)$, and with real part $x(t)$, given by

$$z(t) = \frac{\lambda}{\omega - \omega_0}(e^{-i\omega t} - e^{-i\omega_0 t})\,, \quad x(t) = \frac{\lambda}{\omega - \omega_0}(\cos\omega t - \cos\omega_0 t) \tag{22}$$

The time evolution of the coordinate $x(t)$ is the same as for the classical driven harmonic oscillator,

$$\ddot{x} + \omega_0^2 x = -\lambda(\omega_0 + \omega)\cos\omega t \tag{23}$$

PROBLEM 3

a) Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\lambda(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) \tag{24}$$

Action on the states $|-,1\rangle$ and $|+,0\rangle$,

$$\begin{aligned}\hat{H}|-,1\rangle &= \frac{1}{2}\hbar\,(\omega|-,1\rangle + \lambda|+,0\rangle) \\ \hat{H}|+,0\rangle &= \frac{1}{2}\hbar\,(\omega|+,0\rangle + \lambda|-,1\rangle)\end{aligned} \tag{25}$$

Matrix form

$$\hat{H} = \frac{1}{2}\hbar\omega\mathbb{1} + \frac{1}{2}\hbar\lambda\sigma_x \tag{26}$$

Eigenvalues for σ_x are ± 1, gives energy eigenvalues

$$E_\pm = \frac{1}{2}\hbar(\omega \pm \lambda) \tag{27}$$

Energy eigenstates

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|-,1\rangle \pm |+,0\rangle)\,, \quad \hat{H}|\psi_\pm\rangle = E_\pm|\psi_\pm\rangle \tag{28}$$

Time dependent state

$$|\psi(t)\rangle = c_+ e^{-\frac{i}{\hbar}E_+ t}|\psi_+\rangle + c_- e^{-\frac{i}{\hbar}E_- t}|\psi_-\rangle \tag{29}$$

Initial condition $|\psi(0)\rangle = |-,1\rangle$ implies $c_+ = c_- = \frac{1}{\sqrt{2}}$,

$$|\psi(t)\rangle = e^{-\frac{i}{2}t}(\cos(\frac{\lambda}{2}t)|-,1\rangle - i(\sin(\frac{\lambda}{2}t)|+,0\rangle) \tag{30}$$

which gives $\epsilon = -\omega/2$ and $\Omega = \lambda/2$.

b) The Lindblad equation gives for the occupation probability of the ground state

$$\frac{dp_g}{dt} = -\frac{i}{\hbar}\langle -,0|\left[\hat{H},\hat{\rho}\right]|-,0\rangle + \gamma\langle -,0|\hat{a}\hat{\rho}\hat{a}^\dagger|-,0\rangle = \gamma\langle -,1|\hat{\rho}|-,1\rangle \tag{31}$$

When a photon is present in the cavity, $\langle -,1|\hat{\rho}|-,1\rangle \neq 0$, this gives $\dot{p}_g > 0$, which implies that the occupation probability of the ground state increases until there is no photon in the cavity, $\langle -,1|\hat{\rho}|-,1\rangle = 0$.

c) Evaluation of the matrix elements of the Lindblad equation in the subspace spanned by $|-,1\rangle$ and $|+,0\rangle$ gives

$$\begin{aligned}
\dot{p}_1 &= -\frac{i}{2}\lambda(\langle +,0|\hat{\rho}|-,1\rangle - \langle -,1|\hat{\rho}|+,0\rangle) - \gamma p_1 \\
\dot{p}_0 &= -\frac{i}{2}\lambda(\langle -,1|\hat{\rho}|+,0\rangle - \langle +,0|\hat{\rho}|-,1\rangle) \\
\dot{b} &= -\frac{i}{2}\lambda(\langle +,0|\hat{\rho}|+,0\rangle - \langle -,1|\hat{\rho}|-,1\rangle) - \frac{1}{2}\gamma b
\end{aligned} \tag{32}$$

which simplifies to

$$\begin{aligned}
\dot{p}_1 &= -\gamma p_1 - \lambda b \\
\dot{p}_0 &= \lambda b \\
\dot{b} &= -\frac{1}{2}\gamma b + \frac{1}{2}\lambda(p_1 - p_0)
\end{aligned} \tag{33}$$

Expected time evolution: Exponentially damped oscillations between the states $|-,1\rangle$ and $|+,0\rangle$, with the system ending in the photon less ground state $|-,0\rangle$.

Exam FYS4110, fall semester 2016
Solutions

PROBLEM 1
a) Matrix elements of $\hat{H}$ in the two-dimensional subspace

$$\begin{aligned} \hat{H}|0,+1\rangle &= \frac{1}{2}\hbar(\omega_0+\omega_1)|0,+1\rangle + \lambda\hbar|1,-1\rangle \\ \hat{H}|1,-1\rangle &= \frac{1}{2}\hbar(3\omega_0-\omega_1)|0,+1\rangle + \lambda\hbar|0,+1\rangle \end{aligned} \tag{1}$$

In matrix form

$$H = \frac{1}{2}\hbar \begin{pmatrix} \omega_0+\omega_1 & 2\lambda \\ 2\lambda & 3\omega_0-\omega_1 \end{pmatrix} = \frac{1}{2}\hbar\Delta \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} + \epsilon\hbar\mathbb{1} \tag{2}$$

which gives

$$\Delta\cos\theta = \omega_1-\omega_0\,, \quad \Delta\sin\theta = 2\lambda\,, \quad \epsilon = \omega_0 \tag{3}$$

and from this

$$\Delta = \sqrt{(\omega_1-\omega_0)^2+4\lambda^2} \tag{4}$$

and

$$\cos\theta = \frac{\omega_1-\omega_0}{\sqrt{(\omega_1-\omega_0)^2+4\lambda^2}}\,, \quad \sin\theta = \frac{2\lambda}{\sqrt{(\omega_1-\omega_0)^2+4\lambda^2}} \tag{5}$$

b) Eigenvalue problem for the matrix

$$\begin{aligned} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \delta\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \begin{vmatrix} \cos\theta-\delta & \sin\theta \\ \sin\theta & -\cos\theta-\delta \end{vmatrix} &= 0 \\ \Rightarrow \quad \delta^2-\cos^2\theta-\sin^2\theta &= 0 \quad \Rightarrow \quad \delta = \pm 1 \end{aligned} \tag{6}$$

Energy eigenvalues

$$E_\pm = \hbar(\epsilon \pm \frac{1}{2}\Delta) = \hbar\left(\omega_0 \pm \frac{1}{2}\sqrt{(\omega_1-\omega_0)^2+4\lambda^2}\right) \tag{7}$$

Eigenvectors

$$(\cos\theta \mp 1)\,\alpha + \sin\theta\,\beta = 0 \quad \Rightarrow \quad \frac{\beta}{\alpha} = \pm\frac{1\mp\cos\theta}{\sin\theta} \tag{8}$$

This gives

$$\begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} = N_\pm \begin{pmatrix} \pm\sin\theta \\ 1\mp\cos\theta \end{pmatrix} \tag{9}$$

with normalization factor

$$N_\pm^2 = \sin^2\theta + (1\mp\cos\theta)^2 = 2(1\mp\cos\theta) \tag{10}$$

Finally

$$\begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} = \frac{1}{\sqrt{2(1\mp\cos\theta)}} \begin{pmatrix} \pm\sin\theta \\ 1\mp\cos\theta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm\sqrt{1\pm\cos\theta} \\ \sqrt{1\mp\cos\theta} \end{pmatrix} \tag{11}$$

and in bra-ket form

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}} \left(\pm\sqrt{1\pm\cos\theta}\,|0,+1\rangle + \sqrt{1\mp\cos\theta}\,|1,-1\rangle \right) \tag{12}$$

c) Density operator

$$\begin{aligned} \hat{\rho}_\pm &= \frac{1}{2}(1\pm\cos\theta)(|0\rangle\langle 0|\otimes|+1\rangle\langle +1|) + \frac{1}{2}(1\mp\cos\theta)(|1\rangle\langle 1|\otimes|-1\rangle\langle -1|) \\ &\pm\frac{1}{2}\sin\theta(|0\rangle\langle 1|\otimes|+1\rangle\langle -1| + |1\rangle\langle 0|\otimes|-1\rangle\langle +1|) \end{aligned} \tag{13}$$

Reduced density operators

$$\begin{aligned} \text{position}: \quad & \hat{\rho}^p_\pm = \mathrm{Tr}_s\,\hat{\rho}_\pm = \frac{1}{2}(1\pm\cos\theta)|0\rangle\langle 0| + \frac{1}{2}(1\mp\cos\theta)|1\rangle\langle 1| \\ \text{spin}: \quad & \hat{\rho}^s_\pm = \mathrm{Tr}_p\,\hat{\rho}_\pm = \frac{1}{2}(1\pm\cos\theta)|+1\rangle\langle +1| + \frac{1}{2}(1\mp\cos\theta)|-1\rangle\langle -1| \end{aligned} \tag{14}$$

Entanglement entropy

$$\begin{aligned} S^p_\pm = S^s_\pm &= -[\frac{1}{2}(1-\cos\theta)\log(\frac{1}{2}(1-\cos\theta)) + \frac{1}{2}(1+\cos\theta)\log(\frac{1}{2}(1+\cos\theta)] \\ &= -[\cos^2\frac{\theta}{2}\log(\cos^2\frac{\theta}{2}) + \sin^2\frac{\theta}{2}\log(\sin^2\frac{\theta}{2})] \equiv S \end{aligned} \tag{15}$$

Maximum entanglement

$$\theta = \frac{\pi}{2}: \quad \cos^2\frac{\theta}{2} = \sin^2\frac{\theta}{2} = \frac{1}{2} \quad \Rightarrow \quad S = \log 2 \tag{16}$$

Minimum entanglement

$$\begin{aligned} \theta = 0: \quad & \cos^2\frac{\theta}{2} = 1,\ \sin^2\frac{\theta}{2} = 0 \quad \Rightarrow \quad S = 0 \\ \theta = \pi: \quad & \cos^2\frac{\theta}{2} = 0,\ \sin^2\frac{\theta}{2} = 1 \quad \Rightarrow \quad S = 0 \end{aligned} \tag{17}$$

PROBLEM 2

a) Change of variables

$$\begin{aligned} \hat{c}^\dagger\hat{c} &= \mu^2\hat{a}^\dagger\hat{a} + \nu^2\hat{b}^\dagger\hat{b} + \mu\nu(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \\ \hat{d}^\dagger\hat{d} &= \nu^2\hat{a}^\dagger\hat{a} + \mu^2\hat{b}^\dagger\hat{b} - \mu\nu(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \\ \Rightarrow \quad \omega_c\hat{c}^\dagger\hat{c} + \omega_d\hat{d}^\dagger\hat{d} &= (\mu^2\omega_c + \nu^2\omega_d)\hat{a}^\dagger\hat{a} + (\nu^2\omega_c + \mu^2\omega_d)\hat{b}^\dagger\hat{b} \\ &\quad +\mu\nu(\omega_c - \omega_d)(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \end{aligned} \tag{18}$$

To get the correct form for the Hamiltonian, define ω_c, ω_d, μ and ν so that the following equations are satisfied

$$\begin{array}{rrcl} \mathrm{I} & \mu^2+\nu^2 & = & 1 \\ \mathrm{II} & \mu^2\omega_c+\nu^2\omega_d & = & \omega \\ \mathrm{III} & \nu^2\omega_c+\mu^2\omega_d & = & \omega \\ \mathrm{IV} & \mu\nu(\omega_c-\omega_d) & = & \lambda \end{array} \tag{19}$$

From I, II and III follows

$$\begin{array}{rrcl} \mathrm{IIb} & \frac{1}{2}(\omega_c+\omega_d) & = & \omega \\ \mathrm{IIIb} & (\mu^2-\nu^2)(\omega_c-\omega_d) & = & 0 \end{array} \tag{20}$$

Since $\omega_c \neq \omega_d$ from IV, we have $\mu^2 = \nu^2 = 1/2$, and therefore (by convenient choice of sign factors) $\mu = \nu = 1/\sqrt{2}$. Inserted in IV this gives

$$\mathrm{IVb} \qquad \frac{1}{2}(\omega_c-\omega_d) \;=\; \lambda \tag{21}$$

which together with IIb gives

$$\omega_c = \omega+\lambda\,, \quad \omega_d = \omega-\lambda \tag{22}$$

Commutation relations

$$\begin{aligned} \left[\hat{c},\hat{c}^\dagger\right] &= \mu^2\left[\hat{a},\hat{a}^\dagger\right]+\nu^2\left[\hat{b},\hat{b}^\dagger\right] = (\mu^2+\nu^2)\mathbb{1} = \mathbb{1} \\ \left[\hat{c},\hat{d}^\dagger\right] &= -\mu\nu(\left[\hat{a},\hat{a}^\dagger\right]-\left[\hat{b},\hat{b}^\dagger\right]) = 0 \end{aligned} \tag{23}$$

Similar evaluations of other commutators show that the two sets of ladder operators satify the standard commutation rules for two independent harmonic oscillators.

b) Time evolution of a coherent state

$$\begin{aligned} |\psi(t)\rangle &= \hat{\mathcal{U}}(t)|\psi(0)\rangle\,, \quad \hat{\mathcal{U}}(t) = \exp[-i(\omega_c\hat{c}^\dagger\hat{c}+\omega_d\hat{d}^\dagger\hat{d}+\omega\mathbb{1})] \\ \Rightarrow \quad \hat{c}|\psi(t)\rangle &= \hat{\mathcal{U}}(t)\hat{\mathcal{U}}(t)^{-1}\hat{c}\hat{\mathcal{U}}(t)|\psi(0)\rangle \\ &= \hat{\mathcal{U}}(t)\,e^{i\omega_c t\hat{c}^\dagger\hat{c}}\,\hat{c}\,e^{-i\omega_c t\hat{c}^\dagger\hat{c}}\,|\psi(0)\rangle \\ &= e^{-i\omega_c t}\hat{\mathcal{U}}(t)\,\hat{c}\,|\psi(0)\rangle \\ &= e^{-i\omega_c t}z_{c0}\,|\psi(0)\rangle \end{aligned} \tag{24}$$

$|\psi(t)\rangle$ is thus a coherent state of the c-oscillator with eigenvalue $z_c(t) = e^{-i\omega_c t}z_{c0}$. Simlar result is valid for the d- oscillator with $z_d(t) = e^{-i\omega_d t}z_{d0}$.

c) Since all the operators $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ commute, they have a common set of eigenvalues. This implies that a state which is a coherent state of $\hat{c}$, and $\hat{d}$ will also be a coherent state of $\hat{a}$ and $\hat{b}$. As follows from a) we have

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{c}-\hat{d})\,, \quad \hat{b} = \frac{1}{\sqrt{2}}(\hat{c}+\hat{d}) \tag{25}$$

The corresponding relations between the eigenvalues are

$$\begin{aligned} z_a(t) &= \frac{1}{\sqrt{2}}(z_c(t) - z_d(t)) \\ &= \frac{1}{\sqrt{2}}(e^{-i\omega_c t} z_{c0} - e^{-i\omega_d t} z_{d0}) \\ &= \frac{1}{2} e^{-i\omega t}(e^{-i\lambda t}(z_{a0} + z_{b0}) + e^{i\lambda t}(z_{a0} - z_{b0})) \\ &= \frac{1}{2} e^{-i\omega t}(\cos(\lambda t) z_{a0} - i \sin(\lambda t) z_{b0}) \end{aligned} \tag{26}$$

and similarly

$$\begin{aligned} z_b(t) &= \frac{1}{2} e^{-i\omega t}(-e^{-i\lambda t}(z_{a0} + z_{b0}) + e^{i\lambda t}(z_{a0} - z_{b0})) \\ &= \frac{1}{2} e^{-i\omega t}(i \sin(\lambda t) z_{a0} + \cos(\lambda t) z_{b0}) \end{aligned} \tag{27}$$

PROBLEM 3

a) Time derivatives of matrix elements

$$\begin{aligned} \mathrm{I} \qquad \dot{p}_e &= \langle e|\frac{d\hat{\rho}}{dt}|e\rangle = -\gamma p_e + \gamma' p_g \\ \mathrm{II} \qquad \dot{p}_g &= \langle g|\frac{d\hat{\rho}}{dt}|g\rangle = -\gamma' p_g + \gamma p_e \\ \mathrm{III} \qquad \dot{b} &= \langle e|\frac{d\hat{\rho}}{dt}|g\rangle = [\frac{i}{\hbar}\Delta E - \frac{1}{2}(\gamma + \gamma')]b \end{aligned} \tag{28}$$

From I and II follows $\frac{d}{dt}(p_e + p_g = 0)$, the sum of occupation probabilities is constant.

b) Conditions satisfied by the density operator

$$\begin{aligned} 1) \qquad \hat{\rho} &= \hat{\rho}^\dagger \\ 2) \qquad \hat{\rho} &\geq 0 \\ 3) \qquad \mathrm{Tr}\,\hat{\rho} &= 1 \end{aligned} \tag{29}$$

1) implies that p_e and p_g are real, which is consistent with the interpretation of these as probabilities. 3) gives the normalization $p_e + p_g = 1$. 2) means that the eigenvalues of $\hat{\rho}$ are non-negative. To see the implication of this we find the eigenvalues from the secular equation

$$\begin{aligned} & \begin{vmatrix} p_e - \lambda & b \\ b^* & p_g - \lambda \end{vmatrix} = 0 \\ \Rightarrow \qquad & \lambda^2 - \lambda + p_e p_g - |b|^2 = 0 \\ \Rightarrow \qquad & \lambda_\pm = \frac{1}{2}(1 \pm \sqrt{1 + 4(|b|^2 - p_e p_g)} \end{aligned} \tag{30}$$

Positivity of λ_- then requires $|b|^2 \leq p_e p_g$.

c) At thermal equilibrium we have $\dot{p}_e = \dot{p}_g = \dot{b} = 0$. I then implies

$$\gamma p_e = \gamma' p_g \quad \Rightarrow \quad \frac{p_e}{p_g} = \frac{\gamma'}{\gamma} = e^{-\Delta E/kT} \tag{31}$$

Using $p_g = 1 - p_e$ we find

$$\begin{aligned} p_e = \frac{\gamma'/\gamma}{1+\gamma'/\gamma} &= \frac{1}{1+e^{\Delta E/kT}} \\ p_g = \frac{1}{1+\gamma'/\gamma} &= \frac{1}{1+e^{-\Delta E/kT}} \end{aligned} \tag{32}$$

From III follows $\dot{b} = 0 \Rightarrow b = 0$.

d) From the initial values $p_e(0) = 1$, $p_g(0) = 0$, and the constraint on $|b|^2$ follows

$$|b(0)|^2 \leq p_e(0)p_g(0) = 0 \quad \Rightarrow \quad b(0) = 0 \tag{33}$$

We apply in the following the general formula

$$\dot{x} = ax \quad \Rightarrow \quad x(t) = e^{at}x(0) \tag{34}$$

For b this means

$$b(t) = e^{-\frac{i}{b}\Delta E - \frac{1}{2}(\gamma+\gamma')t}\, b(0) = 0 \tag{35}$$

With $p_e = 1 - p_g$ eq. II gives for p_g

$$\dot{p}_g = -(\gamma+\gamma')p_g + \gamma = -(\gamma+\gamma')(p_g - \frac{1}{1+\gamma'/\gamma}) \tag{36}$$

or

$$\frac{d}{dt}(p_g - \frac{1}{1+\gamma'/\gamma}) = -(\gamma+\gamma')p_g + \gamma = -(\gamma+\gamma')(p_g - \frac{1}{1+\gamma'/\gamma}) \tag{37}$$

Integrating the equation gives

$$p_g(t) - \frac{1}{1+\gamma'/\gamma} = e^{-(\gamma+\gamma')t}(p_g(0) - \frac{1}{1+\gamma'/\gamma}) \tag{38}$$

which with $p_g(0) = 1$ is solved to

$$p_g(t) = \frac{1}{1+\gamma'/\gamma}(1 + (\gamma'/\gamma)e^{-(\gamma+\gamma')t}) \tag{39}$$

and for $p_e = 1 - p_g$ gives

$$p_e(t) = \frac{\gamma'/\gamma}{1+\gamma'/\gamma}(1 + e^{-(\gamma+\gamma')t}) \tag{40}$$

We note that the above expressions reproduce correctly, in the limit $t \to \infty$, the values for p_e and p_g at thermal equilibrium.

The limit $T \to 0$ gives $\gamma'/\gamma \to 0$. This gives $p_g(t) \to 1$ and $p_e(t) \to 0$ consistent with the fact that the system remains in the ground state when $T = 0$. In the limit $T \to \infty$ we have $\gamma'/\gamma \to 1$, which gives

$$\begin{aligned} p_g(t) &\to \frac{1}{2}(1 + e^{-2\gamma t}) \\ p_e(t) &\to \frac{1}{2}(1 - e^{-2\gamma t}) \end{aligned} \tag{41}$$

In this case the time evolution gives $\lim_{t\to\infty} p_e = \lim_{t\to\infty} p_g = \frac{1}{2}$.

Fys 4110 exam 2017 Solutions

Problem 1

a) $$H = \frac{\hbar}{2} g\, \sigma_z^A \otimes \sigma_z^B = \frac{\hbar}{2} g \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U = e^{-\frac{i}{\hbar}Ht} = \begin{pmatrix} z^* & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^* \end{pmatrix} \qquad \text{where } z = e^{\frac{igt}{2}}$$

$$|z| = 1.$$

b) Alternative 1 (Brute force)

$$|\psi(0)\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

$$|\psi(t)\rangle = U|\psi(0)\rangle = \begin{pmatrix} z^* ac \\ z\, ad \\ z\, bc \\ z^* bd \end{pmatrix}$$

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} z^* ac \\ z\, ad \\ z\, bc \\ z^* bd \end{pmatrix} \left(z a^* c^*,\ z^* a^* d^*,\ z^* b^* c^*,\ z b^* d^* \right)$$

$$= \begin{pmatrix} |ac|^2 & z^{*2}|a|^2 c d^* & z^{*2} a b^* |c|^2 & a b^* c d^* \\ z^2 |a|^2 c^* d & |ad|^2 & a b^* c^* d & z^2 a b^* |d|^2 \\ z^2 a^* b |c|^2 & a^* b c d^* & |bc|^2 & z^2 |b|^2 c d^* \\ a^* b c^* d & z^{*2} a^* b |d|^2 & z^{*2} |b|^2 c^* d & |bd|^2 \end{pmatrix}$$

$$\rho_A = \mathrm{Tr}_B\, \rho = \begin{pmatrix} |a|^2 & a b^* (z^{*2}|c|^2 + z^2 |d|^2) \\ a^* b (z^2 |c|^2 + z^{*2} |d|^2) & |b|^2 \end{pmatrix}$$

$$\rho_B = \mathrm{Tr}_A\, \rho = \begin{pmatrix} |c|^2 & c d^* (z^{*2}|a|^2 + z^2 |b|^2) \\ c^* d (z^2 |a|^2 + z^{*2} |b|^2 & |d|^2 \end{pmatrix}$$

Alternative 2 (More sophisticated, but not really simpler...)

With $z = x + iy$ we find

$$U = x\, \mathbb{1}^A \otimes \mathbb{1}^B - iy\, \sigma_z^A \otimes \sigma_z^B$$

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = U \underbrace{|\psi(0)\rangle\langle\psi(0)|}_{\rho(0) = \rho^A(0) \otimes \rho^B(0)} U^+$$

Let $\rho^A(0) = \frac{1}{2}(\mathbb{1} + \vec{m}\cdot\vec{\sigma})$ $\quad \rho^B(0) = \frac{1}{2}(\mathbb{1} + \vec{n}\cdot\vec{\sigma})$

$$\rho(t) = (x\, \mathbb{1}^A \otimes \mathbb{1}^B - iy\, \sigma_z^A \otimes \sigma_z^B)\, \rho^A(0) \otimes \rho^B(0) (x\, \mathbb{1}^A \otimes \mathbb{1}^B + iy\, \sigma_z^A \otimes \sigma_z^B)$$

$$= x^2 \rho^A(0) \otimes \rho^B(0) + y^2 \sigma_z^A \otimes \sigma_z^B\, \rho^A(0) \otimes \rho^B(0)\, \sigma_z^A \otimes \sigma_z^B$$
$$+ ixy \left[\rho^A(0) \otimes \rho^B(0)\, \sigma_z^A \otimes \sigma_z^B - \sigma_z^A \otimes \sigma_z^B\, \rho^A(0) \otimes \rho^B(0)\right]$$

$$= x^2 \rho^A(0) \otimes \rho^B(0) + y^2\, \sigma_z^A \rho^A(0) \sigma_z^A \otimes \sigma_z^B \rho^B(0) \sigma_z^B$$
$$+ ixy \left[\rho^A(0) \sigma_z^A \otimes \rho^B(0) \sigma_z^B - \sigma_z^A \rho^A(0) \otimes \sigma_z^B \rho^B(0)\right]$$

We have

$$\mathrm{Tr}\, \rho^A(0) = 1$$

$$\mathrm{Tr}\, \sigma_z^A \rho^A(0) \sigma_z^A = \frac{1}{2} \mathrm{Tr}\, \sigma_z^A (\mathbb{1} + \vec{m}\cdot\vec{\sigma}) \sigma_z^A = 1$$

$$\mathrm{Tr}\, \rho^A(0)\, \sigma_z^A = \frac{1}{2} \mathrm{Tr} (\sigma_z^A + \vec{m}\cdot\vec{\sigma}\, \sigma_z^A) = m_z = \mathrm{Tr}\, \sigma_z^A \rho^A(0)$$

and similar for system B

$$\Rightarrow \rho^A(t) = \mathrm{Tr}_B\, \rho = x^2 \rho^A(0) + y^2 \sigma_z^A \rho^A(0) \sigma_z^A + ixy[\rho_A(0), \sigma_z^A]$$

$$= \frac{1}{2}\Big[\mathbb{1} + (m_x \cos gt - m_y n_z \sin gt)\sigma_x^A$$

$$+ (m_y \cos gt + m_x n_z \sin gt)\sigma_y^A + m_z \sigma_z^A \Big]$$

$$\rho^B(t) = \frac{1}{2}\Big[\mathbb{1} + (n_x \cos gt - n_y m_z \sin gt)\sigma_x^B$$

$$+ (n_y \cos gt + \cancel{m_x m_z}\, \sin gt)\sigma_y^B + n_z \sigma_z^A \Big]$$

c) Alternative 1

Using $z^2 = e^{igt} = \cos gt + i \sin gt$ and $a = b = \frac{1}{\sqrt{2}}$:

$$\rho^A = \frac{1}{2}\begin{pmatrix} 1 & \cos gt\,(\underbrace{|c|^2+|d|^2}_{1}) - i \sin gt\,(\underbrace{|c|^2-|d|^2}_{m_z}) \\ \text{c.c.} & 1 \end{pmatrix}$$

$$= \frac{1}{2}(\mathbb{1} + \cos gt\, \sigma_x^A + n_z \sin gt\, \sigma_y^A)$$

$$\Rightarrow m_x(t) = \cos gt \qquad m_y(t) = n_z \sin gt \qquad m_z(t) = 0$$

$$m_x(t)^2 + \left(\frac{m_y(t)}{n_z}\right)^2 = 1 \qquad \Rightarrow \text{ellipse}$$

Alternative 2.

$$\rho^A(0) = \begin{pmatrix} a \\ b \end{pmatrix}(a^* \; b^*) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1\;1) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \sigma_x)$$

$$\Rightarrow m_x = 1, \quad m_y = m_z = 0$$

$$\rho^A(t) = \frac{1}{2}(\mathbb{1} + \cos gt\, \sigma_x^A + n_z \sin gt\, \sigma_y^A)$$

d) Maximal entanglement when the Bloch-vector is shortest $\Rightarrow$ $gt = \frac{\pi}{2}$ $\cos gt = 0$ $\sin gt = 1$.

$$\rho^A(t) = \frac{1}{2}(\mathbb{1} + n_z \sigma_y^A) = \frac{1}{2}\begin{pmatrix} 1 & -in_z \\ in_z & 1 \end{pmatrix}$$

Eigenvalues: $(\frac{1}{2}-\lambda)^2 - (\frac{n_z}{2})^2 = 0$ $\Rightarrow \lambda_\pm = \frac{1}{2}(1 \pm n_z)$

$$S_{max} = -\frac{1+n_z}{2}\ln\frac{1+n_z}{2} - \frac{1-n_z}{2}\ln\frac{1-n_z}{2}$$

$$= \ln 2 - \frac{1}{2}\left[(1+n_z)\ln(1+n_z) + (1-n_z)\ln(1-n_z)\right] = \begin{cases} 0 & n_z = \pm 1 \\ \ln 2 & n_z = 0 \end{cases}$$

Problem 2

a) $S(\zeta) = e^{-\frac{1}{2}(\zeta a^2 - \zeta^* a^{+2})}$ $\qquad B = \frac{1}{2}(\zeta a^2 - \zeta^* a^{+2})$

$B^+ = -B$

$$S^+ a S = e^B a e^{-B} = a + [B,a] + \frac{1}{2}[B,[B,a]] + \dots$$

$$[B,a] = -\frac{1}{2}\zeta^*[a^{+2},a] = -\frac{1}{2}\zeta^*(a^+[a^+,a] + [a^+,a]a^+) = \zeta^* a^+$$

$$[B,a^+] = \frac{1}{2}\zeta[a^2,a^+] = \frac{1}{2}\zeta(a[a,a^+] + [a,a^+]a) = \zeta a$$

$$S^+ a S = a + \zeta^* a^+ + \frac{1}{2}\zeta^*\zeta a + \frac{1}{3!}\zeta^{*2}\zeta a^+ + \frac{1}{4!}\zeta^{*2}\zeta^2 a + \dots$$

$$= \left[1 + \frac{1}{2!}|\zeta|^2 + \frac{1}{4!}|\zeta|^4 + \dots\right] a + \left[\zeta^* + \frac{1}{3!}\zeta^{*2}\zeta + \frac{1}{5!}\zeta^{*3}\zeta^2 + \dots\right] a^+$$

$$= \left[1 + \frac{1}{2!}r^2 + \frac{1}{4!}r^4 + \dots\right] a + e^{-i\phi}\left[r + \frac{1}{3!}r^3 + \frac{1}{5!}r^5 + \dots\right] a^+$$

$$= \cosh r \; a + e^{-i\phi}\sinh r \; a^+$$

$$S^+ a^+ S = \cosh r \; a^+ + e^{i\phi}\sinh r \; a$$

b) $\langle sq_{\tau_s}|x|sq_{\tau_s}\rangle = \langle 0|S^\dagger x S|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0|S^\dagger(a^\dagger+a)S|0\rangle$

$= \sqrt{\frac{\hbar}{2m\omega}}\langle 0|(\cosh r + e^{-i\phi}\sinh r)a^\dagger + (\cosh r + e^{i\phi}\sinh r)a|0\rangle = 0$

$\langle sq_{\tau_s}|p|sq_{\tau_s}\rangle = \langle 0|S^\dagger p S|0\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle 0|S^\dagger(a^\dagger - a)S|0\rangle$

$= i\sqrt{\frac{\hbar m\omega}{2}}\langle 0|(\cosh r - e^{-i\phi}\sinh r)a^\dagger - (\cosh r - e^{i\phi}\sinh r)a|0\rangle = 0$

$\Delta x^2 = \langle sq_{\tau_s}|x^2|sq_{\tau_s}\rangle = \langle 0|S^\dagger x S S^\dagger x S|0\rangle$

$= \frac{\hbar}{2m\omega}(\cosh r + e^{i\phi}\sinh r)(\cosh r + e^{-i\phi}\sinh r)$

$= \frac{\hbar}{2m\omega}\Big[\underbrace{\cosh^2 r + \sinh^2 r}_{\cosh 2r} + \underbrace{\cosh r \sinh r}_{\frac{1}{2}\sinh 2r}\underbrace{(e^{i\phi}+e^{-i\phi})}_{2\cos\phi}\Big]$

$= \frac{\hbar}{2m\omega}(\cosh 2r + \sinh 2r\cos\phi)$

$\Delta p^2 = \langle sq_{\tau_s}|p^2|sq_{\tau_s}\rangle = \langle 0|S^\dagger p S S^\dagger p S|0\rangle$

$= \frac{\hbar m\omega}{2}(\cosh r - e^{i\phi}\sinh r)(\cosh r - e^{-i\phi}\sinh r)$

$= \frac{\hbar m\omega}{2}\left[\cosh^2 r + \sinh^2 r - \cosh r\sinh r(e^{i\phi}+e^{-i\phi})\right]$

$= \frac{\hbar m\omega}{2}(\cosh 2r - \sinh 2r\cos\phi)$

8) $\Delta x \, \Delta p = \frac{\hbar}{2}\sqrt{\cosh^2 r - \sinh^2 r \cos^2\phi}$

$$= \frac{\hbar}{2}\sqrt{\cosh^2 r - \sinh^2 r\,(1-\sin^2\phi)}$$

$$= \frac{\hbar}{2}\sqrt{1 + \sinh^2 r \sin^2\phi}$$

Minimal uncertainty: $\Delta x \, \Delta p = \frac{\hbar}{2}$

$\rightarrow \sin\phi = 0 \quad \rightarrow \quad \phi = n\pi$

9) For $\phi = n\pi$:

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\cosh 2r + (-1)^n \sinh 2r} = \sqrt{\frac{\hbar}{2m\omega}}\, e^{(-1)^n r}$$

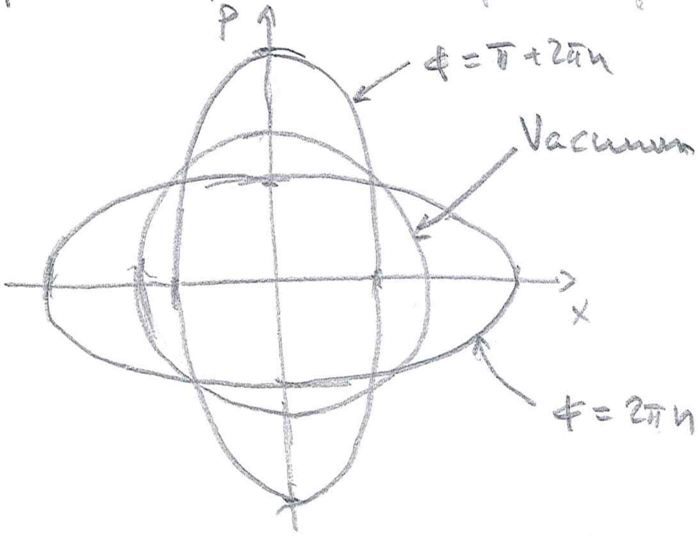
$$\Delta p = \sqrt{\frac{\hbar m\omega}{2}}\sqrt{\cosh 2r - (-1)^n \sinh 2r} = \sqrt{\frac{\hbar m\omega}{2}}\, e^{-(-1)^n r}$$

For n even $\quad \Delta x$ increases by a factor e^r

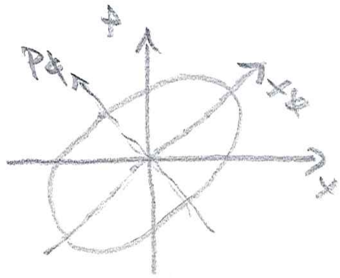
$\quad\quad\quad\quad\quad \Delta p$ decreases by a factor e^r

For n odd $\quad \Delta x$ decreases and Δp increases.

Spread of wave function in phase space (Wigner function)



e) We guess that for other ϕ ~~the~~ the wavefunction is squeezed in a direction not parallel to the axes. Thus we want to define "rotated" operators x_ϕ and p_ϕ. For this to be meaningful we introduce coordinates with same dimension



$$\xi = x\sqrt{m\omega} = \sqrt{\frac{\hbar}{2}}\,(a^\dagger + a)$$

$$\pi = \frac{p}{\sqrt{m\omega}} = i\sqrt{\frac{\hbar}{2}}\,(a^\dagger - a)$$

Coordinates rotated by angle α:

$$\xi_\alpha = \cos\alpha\,\xi - \sin\alpha\,\pi$$

$$\pi_\alpha = \sin\alpha\,\xi + \cos\alpha\,\pi$$

From b): $\langle sq_\xi|\xi^2|sq_\xi\rangle = \frac{\hbar}{2}\left[\cosh 2r + \sinh 2r\cos\phi\right]$

$\langle sq_\xi|\pi^2|sq_\xi\rangle = \frac{\hbar}{2}\left[\cosh 2r - \sinh 2r\cos\phi\right]$

$$\langle sq_\xi|\xi_\alpha|sq_\xi\rangle = \langle sq_\xi|\pi_\alpha|sq_\xi\rangle = 0$$

$$\langle sq_\xi|\xi_\alpha^2|sq_\xi\rangle = \langle sq_\xi|\cos^2\alpha\,\xi^2 - \cos\alpha\sin\alpha\,(\xi\pi + \pi\xi) + \sin^2\alpha\,\pi^2|sq_\xi\rangle$$

We need to find

$$\langle sq_\xi|\xi\pi|sq_\xi\rangle = \langle 0|S^\dagger\xi S S^\dagger \pi S|0\rangle$$

$$= i\frac{\hbar}{2}(\cosh r + e^{i\phi}\sinh r)(\cosh r - e^{-i\phi}\sinh r)$$

$$= i\frac{\hbar}{2}\Big[\underbrace{\cosh^2 r - \sinh^2 r}_{1} + \underbrace{\cosh r\sinh r}_{\frac{1}{2}\sinh 2r}\big(\underbrace{e^{i\phi} - e^{-i\phi}}_{2i\sin\phi}\big)\Big]$$

$$= \frac{\hbar}{2}(i - \sinh 2r\sin\phi) = \langle sq_\xi|\pi\xi|sq_\xi\rangle^*$$

$$\Rightarrow \Delta \xi_\alpha^{\,2} = \frac{\hbar}{2}\Big[\cos^2\alpha(\cosh 2r + \sinh 2r\cos\phi) + \sinh^2\alpha(\cosh 2r - \sinh 2r\cos\phi)$$
$$+ \cos\alpha\sin\alpha\sinh 2r\sin\phi\Big]$$

$$= \frac{\hbar}{2}\left[\cosh 2r + \sinh 2r\cos(2\alpha-\phi)\right]$$

Similarly we find

$$\Delta \pi_\alpha^{\,2} = \frac{\hbar}{2}\left[\cosh 2r - \sinh 2r\cos(2\alpha-\phi)\right]$$

We reproduce the minimal uncertainty expressions from d) if we choose $2\alpha - \phi = 0 \Rightarrow \alpha = \phi/2$

We should check that the commutator is right.

$$[\xi_\alpha, \pi_\alpha] = [\cos\alpha\,\xi - \sin\alpha\,\pi,\ \sin\alpha\,\xi + \cos\alpha\,\pi]$$

$$= \cos^2\alpha[\xi,\pi] - \sin^2\alpha[\pi,\xi] = [\xi,\pi]$$

Problem 1

a) A pure state is the most accurate description possible of a quantum system. It is represented by a state vector $|\psi\rangle$ in Hilbert space. A mixed state is used when we do not know the exact quantum state, but only probabilities p_i for a set of possible states $|\psi_i\rangle$. It is represented by a density matrix $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Mixed states also occur for composite systems in pure states. The reduced density matrix of one component is then a mixed state when there is entanglement between the component and the rest of the system.

b) We measure the spin in the x-direction. $|\rightarrow\rangle$ is an eigenstate of σ_x with eigenvalue $+1$, which means that we will measure spin up in x for all particles in ensemble A. For ensemble B we will measure spin up and spin down randomly with equal probabilities.

c) We consider the density matrices:

$$\rho_B = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow|$$

$$\rho_C = \frac{1}{2}|\rightarrow\rangle\langle\rightarrow| + \frac{1}{2}|\leftarrow\rangle\langle\leftarrow|$$

$$= \frac{1}{4}(|\uparrow\rangle + |\downarrow\rangle)(\langle\uparrow| + \langle\downarrow|) + \frac{1}{4}(|\uparrow\rangle - |\downarrow\rangle)(\langle\uparrow| - \langle\downarrow|)$$

$$= \frac{1}{4}\Big(|\uparrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$$

$$+ |\uparrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| \Big)$$

$$= \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow| = \rho_B$$

Since the density matrices are the same we will get the same statistics for all possible measurements, and we can not distinguish the ensembles.

d) $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

It is clear that if we measure the first particle along the z-axis we have equal probabilities of measuring up or down, and the second particle will collapse to the opposite state, generating ensemble B. Ensemble C is generated by measuring the first particle in the x-direction. To see this we rewrite $|\psi\rangle$ in terms of the states $|\rightarrow\rangle$ and $|\leftarrow\rangle$.

We have $|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\rightarrow\rangle + |\leftarrow\rangle)$

$|\downarrow\rangle = \frac{1}{\sqrt{2}}(|\rightarrow\rangle - |\leftarrow\rangle)$

③

$$|\psi\rangle = \frac{1}{2\sqrt{2}}(|\rightarrow\rangle + |\leftarrow\rangle)\otimes(|\rightarrow\rangle - |\leftarrow\rangle) - \frac{1}{2\sqrt{2}}(|\rightarrow\rangle - |\leftarrow\rangle)(|\rightarrow\rangle + |\leftarrow\rangle)$$

$$= \frac{1}{2\sqrt{2}}\Big(|\rightarrow\rightarrow\rangle - |\rightarrow\leftarrow\rangle + |\leftarrow\rightarrow\rangle - |\leftarrow\leftarrow\rangle$$

$$- |\rightarrow\rightarrow\rangle - |\rightarrow\leftarrow\rangle + |\leftarrow\rightarrow\rangle + |\leftarrow\leftarrow\rangle\Big)$$

$$= \frac{1}{\sqrt{2}}\Big(|\leftarrow\rightarrow\rangle - |\rightarrow\leftarrow\rangle\Big)$$

e) Consider the case where person 1 measures spin along the z-axis and therefore prepares ensemble B. If person 2 also measures along the z-axis, the outcome of the two measurements will always be perfectly anticorrelated. If instead person 1 measures x-spin and prepares ensemble C while person 2 still measures z-spin, the results will be uncorrelated. Nothing changes if person 1 measures after person 2.

Problem 2.

a) $H = \hbar\omega(a^+a + b^+b) + \hbar\lambda(a^+b + b^+a)$

$$H = \hbar\omega_c c^+c + \hbar\omega_d d^+d$$

$$= \hbar\omega_c(\mu a^+ + \nu b^+)(\mu a + \nu b) + \hbar\omega_d(-\nu a^+ + \mu b^+)(-\nu a + \mu b)$$

$$= \hbar\omega_c(\mu^2 a^+a + \mu\nu(a^+b + b^+a) + \nu^2 b^+b)$$

$$+ \hbar\omega_d(\nu^2 a^+a - \mu\nu(a^+b + b^+a) + \mu^2 b^+b)$$

$$= \hbar(\omega_c\mu^2 + \omega_d\nu^2)a^+a + \hbar(\omega_c\nu^2 + \omega_d\mu^2)b^+b$$

$$+ \hbar(\omega_c - \omega_d)\mu\nu(a^+b + b^+a)$$

$$\Rightarrow \left.\begin{array}{l}\omega_c\mu^2 + \omega_d\nu^2 = \omega \\ \omega_c\nu^2 + \omega_d\mu^2 = \omega\end{array}\right\} \mu^2 = \nu^2 \Rightarrow \mu = \nu = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{2}(\omega_c + \omega_d) = \omega$$

$$(\omega_c - \omega_d)\mu\nu = \lambda \qquad \Rightarrow \frac{1}{2}(\omega_c - \omega_d) = \lambda$$

$$\Rightarrow \omega_c = \omega + \lambda \qquad \omega_d = \omega - \lambda$$

$$[c, c^+] = [\mu a + \nu b, \mu a^+ + \nu b^+] = \mu^2\underbrace{[a, a^+]}_{1} + \nu^2\underbrace{[b, b^+]}_{1} = \mu^2 + \nu^2 = \underline{1}$$

$$[d, d^+] = [-\nu a + \mu b, -\nu a^+ + \mu b^+] = \nu^2[a, a^+] + \mu^2[b, b^+] = \underline{1}$$

$$[c, d] = [\mu a + \nu b, -\nu a + \mu b] = 0$$

$$[c, d^+] = [\mu a + \nu b, -\nu a^+ + \mu b^+] = -\mu\nu[a, a^+] + \mu\nu[b, b^+] = 0$$

b) $c = \frac{1}{\sqrt{2}}(a+b)$, $d = \frac{1}{\sqrt{2}}(-a+b)$ $\Rightarrow$ $a = \frac{1}{\sqrt{2}}(c-d)$, $b = \frac{1}{\sqrt{2}}(c+d)$

$$|\psi(0)\rangle = |1_a 0_b\rangle = a^+|0\rangle = \frac{1}{\sqrt{2}}(c^+ - d^+)|0\rangle = \frac{1}{\sqrt{2}}\left(|1_c 0_d\rangle - |0_c 1_d\rangle\right)$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_c t c^+ c - i\omega_d t d^+ d}\left(|1_c 0_d\rangle - |0_c 1_d\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\omega_c t}|1_c 0_d\rangle - e^{-i\omega_d t}|0_c 1_d\rangle\right)$$

$$= \frac{1}{2}\left[e^{-i\omega_c t}(a^+ + b^+)|0\rangle - e^{-i\omega_d t}(-a^+ + b^+)|0\rangle\right]$$

$$= \frac{1}{2}\left[\underbrace{\left(e^{-i\omega_c t} + e^{-i\omega_d t}\right)}_{A}|1_a 0_b\rangle + \underbrace{\left(e^{-i\omega_c t} - e^{-i\omega_d t}\right)}_{B}|0_a 1_b\rangle\right]$$

$$\langle N_A\rangle = \langle\psi(t)|a^+ a|\psi(t)\rangle$$

$$= \frac{1}{4}\left(e^{i\omega_c t} + e^{i\omega_d t}\right)\left(e^{-i\omega_c t} + e^{-i\omega_d t}\right)$$

$$= \frac{1}{4}\Big(2 + \underbrace{e^{-i(\omega_c - \omega_d)t} + e^{i(\omega_c - \omega_d)t}}_{2\cos(\omega_c - \omega_d)t \,=\, 2\cos 2\lambda t}\Big)$$

$$= \frac{1}{2}\left(1 + \cos 2\lambda t\right) = \cos^2 \lambda t$$

$$\langle N_B\rangle = \langle\psi(t)|b^+ b|\psi(t)\rangle$$

$$= \frac{1}{4}\left(e^{i\omega_c t} - e^{i\omega_d t}\right)\left(e^{-i\omega_c t} - e^{-i\omega_d t}\right)$$

$$= \frac{1}{2}\left(1 - \cos 2\lambda t\right) = \sin^2 \lambda t$$

Energy is oscillating between the two oscillators.

c) $\rho_A = \mathrm{Tr}_B |\psi(t)\rangle\langle\psi(t)| = \frac{1}{4}\mathrm{Tr}_B\left(A|1_a 0_b\rangle + B|0_a 1_b\rangle\right)\left(A^*\langle 1_a 0_b| + B^*\langle 0_a 1_b|\right)$

$= \frac{1}{4}\left(|A|^2 |1_a\rangle\langle 1_a| + |B|^2 |0_a\rangle\langle 0_a|\right)$

$= \cos^2\lambda t\, |1_a\rangle\langle 1_a| + \sin^2\lambda t\, |0_a\rangle\langle 0_a|$

$S = -\cos^2\lambda t \ln\cos^2\lambda t - \sin^2\lambda t \ln \sin^2\lambda t$

Maximal value when $\cos^2\lambda t = \sin^2\lambda t = \frac{1}{2}$

$S_{max} = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \ln 2$

$S = 0$ when $\cos^2\lambda t$ or $\sin^2\lambda t = 0$

$\Rightarrow \lambda t = n\frac{\pi}{2} \qquad n = 0, 1, 2 \ldots$

The system is then in state $|1_a 0_b\rangle$ or $|0_a 1_b\rangle$.

Problem 3.

a) $H_0 = \frac{1}{2}\hbar\omega_0\sigma_z$ $\qquad \rho = \begin{pmatrix} P_e & b \\ b^* & P_g \end{pmatrix}$ $\qquad |e\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$|g\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0,\rho] - \frac{\gamma}{2}[\alpha^+\alpha\rho + \rho\alpha^+\alpha - 2\alpha\rho\alpha^+]$$

$\alpha = |g\rangle\langle e| = \begin{pmatrix}0\\1\end{pmatrix}(1\ 0) = \begin{pmatrix}0&0\\1&0\end{pmatrix}$

$\alpha^+\alpha = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}$

$$= -\frac{i\omega_0}{2}\underbrace{\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}P_e&b\\b^*&P_g\end{pmatrix}\right]} - \frac{\gamma}{2}\left[\underbrace{\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}P_e&b\\b^*&P_g\end{pmatrix}}_{\begin{pmatrix}P_e&b\\0&0\end{pmatrix}} + \underbrace{\begin{pmatrix}P_e&b\\b^*&P_g\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}}_{\begin{pmatrix}P_e&0\\b^*&0\end{pmatrix}} - 2\underbrace{\begin{pmatrix}0&0\\1&0\end{pmatrix}\underbrace{\begin{pmatrix}P_e&b\\b^*&P_g\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}}_{\begin{pmatrix}0&P_e\\0&b^*\end{pmatrix}}}_{\begin{pmatrix}0&0\\0&P_e\end{pmatrix}}\right]$$

$$\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}P_e&b\\b^*&P_g\end{pmatrix}}_{\begin{pmatrix}P_e&b\\-b^*&-P_g\end{pmatrix}} - \underbrace{\begin{pmatrix}P_e&b\\b^*&P_g\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{\begin{pmatrix}P_e&-b\\b^*&-P_g\end{pmatrix}}$$

$$= -i\omega_0\begin{pmatrix}0&b\\-b^*&0\end{pmatrix} - \frac{\gamma}{2}\begin{pmatrix}2P_e&b\\b^*&-2P_e\end{pmatrix}$$

$$\Rightarrow \dot{P}_e = -\gamma P_e$$

$$\dot{P}_g = \gamma P_e$$

$$\dot{b} = -\left(\frac{\gamma}{2}+i\omega_0\right)b$$

$$\frac{d}{dt}(P_e+P_g) = \dot{P}_e + \dot{P}_g = -\gamma P_e + \gamma P_e = 0$$

b) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\rho(0) = |\psi(0)\rangle\langle\psi(0)| = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1\ 1) = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$

$\Rightarrow P_e(0) = P_g(0) = b(0) = \frac{1}{2}$

$P_e(t) = e^{-\gamma t}P_e(0) = \frac{1}{2}e^{-\gamma t}$

$P_g(t) = 1 - P_e(t) = 1 - \frac{1}{2}e^{-\gamma t}$

$$b(t) = e^{-(\frac{\gamma}{2}+i\omega_0)t}\, b(0) = \frac{1}{2} e^{-(\frac{\gamma}{2}+i\omega_0)t}$$

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})_{\vec{r}=(x,y,z)} = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

$$\Rightarrow z = P_e - P_g = e^{-\gamma t} - 1$$

$$x = 2\,\mathrm{Re}\, b = e^{-\frac{\gamma}{2}t}\cos\omega_0 t$$

$$y = -2\,\mathrm{Im}\, b = e^{-\frac{\gamma}{2}t}\sin\omega_0 t$$

A spiral in the xy-plane starting on the surface of the Bloch sphere and decaying to the axis and a decay of the z-component to the ground state.

c) $T(t) = e^{\frac{i}{2}\omega t \sigma_z}$

$|\psi'\rangle = T(t)|\psi\rangle$

$\rho' = T\rho T^\dagger$

$$\frac{d\rho'}{dt} = \dot{T}\rho T^\dagger + T\rho\dot{T}^\dagger + T\dot{\rho}T^\dagger$$

$$= \underbrace{\frac{i}{2}\omega\sigma_z\rho' - \frac{i}{2}\omega\rho'\sigma_z}_{\frac{i}{\hbar}[\frac{\hbar}{2}\omega\sigma_z,\rho']} + T\left\{-\frac{i}{\hbar}[H,\rho] - \frac{\gamma}{2}[\alpha^\dagger\alpha\rho + \rho\alpha^\dagger\alpha - 2\alpha\rho\alpha^\dagger]\right\}T^\dagger$$

$$T[H,\rho]T^\dagger = TH\rho T^\dagger - T\rho HT^\dagger = THT^\dagger\rho' - \rho' THT^\dagger$$

$$T = e^{\frac{i}{2}\omega t\sigma_z} = \cos\frac{\omega t}{2} + i\sin\frac{\omega t}{2}\sigma_z$$

$$THT^\dagger = \frac{1}{2}\hbar\omega_0\sigma_z + \frac{1}{2}\hbar\omega_1\left(\cos\omega t\, T\sigma_x T^\dagger + \sin\omega t\, T\sigma_y T^\dagger\right)$$

$$T\sigma_x T^\dagger = \left(\cos\tfrac{\omega t}{2} + i\sin\tfrac{\omega t}{2}\sigma_z\right)\sigma_x\left(\cos\tfrac{\omega t}{2} - i\sin\tfrac{\omega t}{2}\sigma_z\right)$$

$$= \cos^2\tfrac{\omega t}{2}\,\sigma_x + i\sin\tfrac{\omega t}{2}\cos\tfrac{\omega t}{2}\underbrace{[\sigma_z,\sigma_x]}_{2i\sigma_y} + \sin^2\tfrac{\omega t}{2}\underbrace{\sigma_z\sigma_x\sigma_z}_{-\sigma_x}$$

$$= \cos\omega t\,\sigma_x - \sin\omega t\,\sigma_y$$

$$T\sigma_y T^\dagger = \left(\cos\tfrac{\omega t}{2} + i\sin\tfrac{\omega t}{2}\sigma_z\right)\sigma_y\left(\cos\tfrac{\omega t}{2} - i\sin\tfrac{\omega t}{2}\sigma_z\right)$$

$$= \cos^2\tfrac{\omega t}{2}\,\sigma_y + i\sin\tfrac{\omega t}{2}\cos\tfrac{\omega t}{2}\underbrace{[\sigma_z,\sigma_y]}_{-2i\sigma_x} + \sin^2\tfrac{\omega t}{2}\underbrace{\sigma_z\sigma_y\sigma_z}_{-\sigma_y}$$

$$= \cos\omega t\,\sigma_y + \sin\omega t\,\sigma_x$$

$$THT^\dagger = \tfrac{1}{2}\hbar\omega_0\sigma_z + \tfrac{1}{2}\hbar\omega_1\left(\cos^2\omega t\,\sigma_x - \cos\omega t\sin\omega t\,\sigma_y + \cos\omega t\sin\omega t\,\sigma_y + \sin^2\omega t\,\sigma_x\right)$$

$$= \tfrac{1}{2}\hbar\omega_0\sigma_z + \tfrac{1}{2}\hbar\omega_1\sigma_x$$

$$T\alpha T^\dagger = \cos^2\tfrac{\omega t}{2}\,\alpha + i\sin\tfrac{\omega t}{2}\cos\tfrac{\omega t}{2}\underbrace{[\sigma_z,\alpha]}_{\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}}_{-\alpha} - \underbrace{\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{\alpha}} + \sin^2\tfrac{\omega t}{2}\underbrace{\sigma_z\underbrace{\alpha\sigma_z}_{\alpha}}_{-\alpha}$$

$$= (\cos\omega t - i\sin\omega t)\,\alpha = e^{-i\omega t}\alpha$$

$$T\alpha^\dagger T^\dagger = e^{i\omega t}\alpha^\dagger$$

$$\Rightarrow \frac{d\rho'}{dt} = -\frac{i}{\hbar}[H',\rho'] - \frac{\gamma}{2}\left[\alpha^\dagger\alpha\rho' + \rho'\alpha^\dagger\alpha - 2\alpha\rho\alpha^\dagger\right]$$

$$H' = THT^\dagger - \tfrac{1}{2}\hbar\omega\sigma_z = \tfrac{1}{2}\hbar\underbrace{(\omega_0-\omega)}_{\Delta}\sigma_z + \tfrac{1}{2}\hbar\omega_1\sigma_x$$

d) Let $\rho' = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix}$

$$[\sigma_x, \rho'] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix} - \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b^*-b & p_g-p_e \\ p_e-p_g & b-b^* \end{pmatrix}$$

$$\frac{d\rho'}{dt} = -i\Delta\begin{pmatrix} 0 & b \\ -b^* & 0 \end{pmatrix} - \frac{i}{2}\omega_1\begin{pmatrix} b^*-b & p_g-p_e \\ p_e-p_g & b-b^* \end{pmatrix} - \frac{\gamma}{2}\begin{pmatrix} 2p_e & b \\ b^* & -2p_e \end{pmatrix}$$

$$\dot{p}_e = -\frac{i}{2}\omega_1(b^*-b) - \gamma p_e$$

$$\dot{p}_g = \frac{i}{2}\omega_1(b^*-b) + \gamma p_e$$

$$\dot{b} = -i\Delta b - \frac{i}{2}\omega_1(p_g-p_e) - \frac{\gamma}{2} b$$

Stationary state: $\dot{p}_e = \dot{p}_g = \dot{b} = 0$

$$-\frac{i}{2}\omega_1(b^*-b) - \gamma p_e = 0$$

$$-\Delta b - \frac{i}{2}\omega_1(\underbrace{p_g}_{1-p_e} - p_e) - \frac{\gamma}{2} b = 0$$

$$\Rightarrow b = \frac{\omega_1(p_e - \frac{1}{2})}{\Delta - \frac{i\gamma}{2}} \qquad b^* = \frac{\omega_1(p_e-\frac{1}{2})}{\Delta + \frac{i\gamma}{2}}$$

$$p_e = -\frac{i\omega_1}{2\gamma}(b^*-b) = \frac{\frac{1}{4}\omega_1^2}{\Delta^2 + \frac{\gamma^2}{4} + \frac{\omega_1^2}{2}}$$

$$b = -\frac{\omega_1}{2}\,\frac{\Delta + \frac{i\gamma}{2}}{\Delta^2 + \frac{\gamma^2}{4} + \frac{\omega_1^2}{2}}$$

$\omega_1 \ll \sqrt{\Delta^2 + \frac{\gamma^2}{4}}$: $p_e \ll 1$, $|b| \ll 1$ Driving is weak and state is close to ground state.

$\omega_1 \gg \sqrt{\Delta^2 + \frac{\gamma^2}{4}}$: $p_e \approx \frac{1}{2}$ $b \approx 0$ Driving is strong and $p_e \approx p_g$. All relative phases have the same probability and $b \approx 0$.

Fys 4110, 2019 Solutions

Problem 1

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right)$$

a) $\rho = |\psi\rangle\langle\psi| = \frac{1}{3}\left(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right)\left(\langle\uparrow\downarrow\downarrow| + \langle\downarrow\uparrow\downarrow| + \langle\downarrow\downarrow\uparrow|\right)$

$$\rho_A = \mathrm{Tr}_{BC}\,\rho = \sum_{i,j=\uparrow,\downarrow} \langle i_B j_C|\rho|i_B j_C\rangle$$

$$= \frac{1}{3}\left(|\uparrow\rangle\langle\uparrow| + 2|\downarrow\rangle\langle\downarrow|\right)$$

$$\rho_B = \mathrm{Tr}_A\,\rho = \frac{1}{3}\left(|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|\right)$$

$S = -\mathrm{Tr}_A\,\rho_A \ln\rho_A = -\mathrm{Tr}_{BC}\,\rho_{BC}\ln\rho_{BC}$ Easiest to use ρ_A

$$S = -\frac{1}{3}\ln\frac{1}{3} - \frac{2}{3}\ln\frac{2}{3}$$

b) Measure $\uparrow$: $|\psi\rangle \to |\uparrow\downarrow\downarrow\rangle$ $S_{BC} = 0$

Measure $\downarrow$: $|\psi\rangle \to \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$ $S'_{BC} = \ln 2$

c) Eigenstates for σ_x:

$|\rightarrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$ $\sigma_x|\rightarrow\rangle = |\rightarrow\rangle$

$|\leftarrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)$ $\sigma_x|\leftarrow\rangle = -|\leftarrow\rangle$

$|\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle + |\leftarrow\rangle\right)$ $|\downarrow\rangle = \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle - |\leftarrow\rangle\right)$

$$|\psi\rangle = \frac{1}{\sqrt{6}}\left(|\rightarrow\downarrow\downarrow\rangle + |\leftarrow\downarrow\downarrow\rangle + |\rightarrow\uparrow\downarrow\rangle - |\leftarrow\uparrow\downarrow\rangle + |\rightarrow\downarrow\uparrow\rangle - |\leftarrow\downarrow\uparrow\rangle\right)$$

Measure $\rightarrow$: $|\psi\rangle \to |\rightarrow\rangle\frac{1}{\sqrt{3}}\left(|\downarrow\downarrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$

Measure $\leftarrow$: $|\psi\rangle \to |\leftarrow\rangle\frac{1}{\sqrt{3}}\left(|\downarrow\downarrow\rangle - |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

For BC we have

$$|\psi_{BC}\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \pm |\downarrow\downarrow\rangle\right)$$

$$\rho_{BC} = |\psi_{BC}\rangle\langle\psi_{BC}| = \frac{1}{3}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \pm |\downarrow\downarrow\rangle\right)\left(\langle\uparrow\downarrow| + \langle\downarrow\uparrow| \pm \langle\downarrow\downarrow|\right)$$

$$\rho_B = \mathrm{Tr}_C\, \rho_{BC} = \frac{1}{3}\left(2|\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow| \pm |\uparrow\rangle\langle\downarrow| \pm |\downarrow\rangle\langle\uparrow|\right)$$

In matrix form $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$\rho_B = \frac{1}{3}\begin{pmatrix}1 & \pm 1\\ \pm 1 & 2\end{pmatrix}$$

Eigenvalues $\begin{vmatrix}\frac{1}{3}-\lambda & \pm\frac{1}{3}\\ \pm\frac{1}{3} & \frac{2}{3}-\lambda\end{vmatrix} = \left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{2}{3}\right)-\frac{1}{9} = 0$

$\Rightarrow \quad (3\lambda-1)(3\lambda-2)-1 = 9\lambda^2-9\lambda-1 = 0 \quad \Rightarrow \lambda_\pm = \frac{9\pm\sqrt{81+36}}{18} = \frac{1\pm\sqrt{13}}{2}$

Entanglement entropy: $\quad S = -\frac{1+\sqrt{13}}{2}\ln\frac{1+\sqrt{13}}{2} - \frac{1-\sqrt{13}}{2}\ln\frac{1-\sqrt{13}}{2}$

Problem 2.

$$H = \frac{\hbar}{2}\omega_0\sigma_z + \frac{\hbar}{2}A\left(\cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y\right)$$

a) $i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle \qquad |\psi'\rangle = e^{i\frac{\omega t}{2}\sigma_z}|\psi\rangle$

$$i\hbar\frac{d}{dt}|\psi'\rangle = i\hbar\left(i\frac{\omega}{2}\sigma_z|\psi'\rangle + e^{i\frac{\omega t}{2}\sigma_z}\frac{d}{dt}|\psi\rangle\right)$$

$$= \Big(\underbrace{-\frac{\hbar}{2}\omega\sigma_z + e^{i\frac{\omega t}{2}\sigma_z}He^{-i\frac{\omega t}{2}\sigma_z}}_{H'}\Big)|\psi'\rangle$$

$$e^{i\frac{\omega t}{2}\sigma_z}\sigma_x e^{-i\frac{\omega t}{2}\sigma_z} = \left(\cos\frac{\omega t}{2}\mathbb{1} + i\sin\frac{\omega t}{2}\sigma_z\right)\sigma_x\left(\cos\frac{\omega t}{2}\mathbb{1} - i\sin\frac{\omega t}{2}\sigma_z\right)$$

$$= \cos^2\frac{\omega t}{2}\sigma_x + i\cos\frac{\omega t}{2}\sin\frac{\omega t}{2}\underbrace{[\sigma_z,\sigma_x]}_{2i\sigma_y} + \sin^2\frac{\omega t}{2}\underbrace{\underbrace{\sigma_z\sigma_x}_{i\sigma_y}\sigma_z}_{-\sigma_x}$$

$$= \cos\omega t\,\sigma_x - \sin\omega t\,\sigma_y$$

$$e^{i\frac{\omega t}{2}\sigma_z}\sigma_y e^{-i\frac{\omega t}{2}\sigma_z} = \left(\cos\tfrac{\omega t}{2}\mathbb{1} + i\sin\tfrac{\omega t}{2}\sigma_z\right)\sigma_y\left(\cos\tfrac{\omega t}{2}\mathbb{1} - i\sin\tfrac{\omega t}{2}\sigma_z\right)$$

$$= \cos\omega t\,\sigma_y + \sin\omega t\,\sigma_x$$

$$H' = -\frac{\hbar}{2}\omega\sigma_z + \frac{\hbar}{2}\omega_0\sigma_z + \frac{\hbar}{2}A\Big[\cos^2\omega t\,\sigma_x - \cos\omega t\sin\omega t\,\sigma_y$$
$$+ \cos\omega t\sin\omega t\,\sigma_y + \sin^2\omega t\,\sigma_x\Big]$$

$$= \frac{\hbar}{2}(\omega_0-\omega)\sigma_z + \frac{\hbar}{2}A\sigma_x \qquad \text{Time independent}$$

Resonance when $\omega = \omega_0$.

b) $$H' = \frac{\hbar}{2}(\omega_0-\omega)\sigma_z + \frac{\hbar}{2}A\Big[\underbrace{\cos^2\omega t}_{\frac{1}{2}(1+\cos 2\omega t)}\sigma_x - \underbrace{\cos\omega t\sin\omega t}_{\frac{1}{2}\sin 2\omega t}\sigma_y\Big]$$

$$= \frac{\hbar}{2}(\omega_0-\omega)\sigma_z + \frac{\hbar}{4}A\sigma_x + \frac{\hbar A}{4}\underbrace{\left(\cos 2\omega t\,\sigma_x - \sin 2\omega t\,\sigma_y\right)}_{\text{Rotating with frequency } 2\omega}$$

The oscillating field $\cos\omega t\,\sigma_x$ can be thought of as two counterrotating fields

$$\cos\omega t\,\sigma_x = \frac{1}{2}(\cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y) + \frac{1}{2}(\cos\omega t\,\sigma_x - \sin\omega t\,\sigma_y)$$

When transforming to the rotating frame, the first term will appear constant while the second term will appear as rotating at twice the frequency.

We can neglect the term $\frac{\hbar A}{4}(\cos 2\omega t\, \sigma_x - \sin 2\omega t\, \sigma_y)$ when A is sufficiently small because it changes rapidly in time and its effect on the state does not have time to build up before it changes direction. On average it does not have large effect, and the true state will wiggle around the approximate state that we find using the rotating wave approximation.

c) $H' = -\hbar \frac{dS}{dt} + e^{iS} H e^{-iS}$ $\qquad S = \frac{A}{2\omega} \xi \sin\omega t\, \sigma_x = \hat{A}\, \sigma_x$

$$\frac{dS}{dt} = \frac{A}{2} \xi \cos\omega t\, \sigma_x$$

$$e^{iS} \sigma_z e^{-iS} = e^{i\hat{A}\sigma_x} \sigma_z e^{-i\hat{A}\sigma_x} = (\cos\hat{A}\, \mathbb{1} + i\sin\hat{A}\, \sigma_x)\, \sigma_z\, (\cos\hat{A}\, \mathbb{1} - i\sin\hat{A}\, \sigma_x)$$

$$= \cos^2\hat{A}\, \sigma_z + i\cos\hat{A}\sin\hat{A}\, \underbrace{[\sigma_x, \sigma_z]}_{-2i\sigma_y} + \sin^2\hat{A}\, \underbrace{\sigma_x \sigma_z \sigma_x}_{-\sigma_z}$$

$$= \cos 2\hat{A}\, \sigma_z + \sin 2\hat{A}\, \sigma_y$$

$$H' = -\frac{\hbar A}{2} \xi \cos\omega t\, \sigma_x + \frac{\hbar}{2} \omega_0 \cos\left[\frac{A}{\omega} \xi \sin\omega t\right] \sigma_z + \frac{\hbar}{2} \omega_0 \sin\left[\frac{A}{\omega} \xi \sin\omega t\right] \sigma_y$$

$$+ \frac{\hbar}{2} A \cos\omega t\, \sigma_x$$

$$= \frac{\hbar}{2} \omega_0 \left\{ \cos\left[\frac{A}{\omega} \xi \sin\omega t\right] \sigma_z + \sin\left[\frac{A}{\omega} \xi \sin\omega t\right] \sigma_y \right\} + \frac{\hbar}{2} A(1-\xi) \cos\omega t\, \sigma_x$$

d) If $\quad J_1(\frac{A}{\omega}\xi)\omega_0 = \frac{1}{2}A(1-\xi) = \frac{1}{2}A'$ we have

$$H' \approx \frac{\hbar}{2}\omega_0 J_0(\frac{A}{\omega}\xi)\sigma_z + \frac{\hbar}{2}A'(\cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y)$$

With this choice of ξ, the components of the field in the x- and y-directions have the same amplitude, and we have a rotating field similar to that in question a) but with ω_0 rescaled by the Bessel function. The resonance condition is therefore $\quad \omega = \omega_0 J_0(\frac{A}{\omega}\xi)$

e) $$J_1(\frac{A}{\omega}\xi)\,\omega_0 \approx \frac{A}{2\omega}\xi\omega_0 = \frac{1}{2}A(1-\xi)$$

$$\Rightarrow \xi = \frac{1}{1+\frac{\omega_0}{\omega}} = \frac{\omega}{\omega_0+\omega}$$

$$\omega = \omega_0 J_0(\frac{A}{\omega}\xi) = \omega_0 J_0\left(\frac{A}{\omega_0+\omega}\right) \approx \omega_0\left(1 - \frac{A^2}{4(\omega_0+\omega)^2}\right)$$

For $A=0$ we have $\omega=\omega_0$ and in general $\omega = \omega_0 + (\,)A^2$

To lowest order we can then replace $\omega_0+\omega \rightarrow 2\omega_0$ in the denominator to get

$$\omega = \omega_0 - \frac{A^2}{16\omega_0}$$

Problem 3

a) $p(\theta,\phi) = N\sum_a |(\vec{k}\times\vec{\epsilon}_{ka})\cdot\vec{\sigma}_{BA}|^2$ where N is a normalization to be determined at the end.

$$\vec{\sigma}_{BA} = \langle\downarrow|\vec{\sigma}|\uparrow\rangle = \left((0\,1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}, (0\,1)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}, (0\,1)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \right)$$

$$= (1, i, 0)$$

We have $\vec{k}\times\vec{\epsilon}_{k1} = k\,\vec{\epsilon}_{k2}$ $\quad \vec{k}\times\vec{\epsilon}_{k2} = -k\,\vec{\epsilon}_{k1}$

$\uparrow |\vec{k}|$

$$\Rightarrow \sum_a |(\vec{k}\times\vec{\epsilon}_{ka})\cdot\vec{\sigma}_{BA}|^2 = k^2\sum_a |\vec{\epsilon}_{ka}\cdot\vec{\sigma}_{BA}|^2 = k^2\left(|\vec{\sigma}_{BA}|^2 - |\vec{\sigma}_{BA}\cdot\tfrac{\vec{k}}{k}|^2\right)$$

$$\frac{\vec{k}}{k} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{\sigma}_{BA}\cdot\frac{\vec{k}}{k} = \sin\theta\, e^{i\phi} \qquad |\vec{\sigma}_{BA}|^2 = 2$$

$$\Rightarrow p(\theta,\phi) = Nk^2(2-\sin^2\theta) = Nk^2(1+\cos^2\theta)$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, p(\theta,\phi) = Nk^2\cdot 2\pi \int_0^{\pi} d\theta \sin\theta(1+\cos^2\theta) \qquad u = \cos\theta$$

$$= 2\pi N k^2 \underbrace{\int_{-1}^{1} du(1+u^2)}_{2+2/3 = \frac{8}{3}} = \frac{16\pi}{3}Nk^2 = 1 \quad \Rightarrow N = \frac{3}{16\pi k^2}$$

$$\Rightarrow p(\theta,\phi) = \frac{3}{16\pi}(1+\cos^2\theta)$$

b) $\vec{k} = (1,0,0)$ $\quad \vec{\epsilon}_{k1} = (0, \cos\alpha, \sin\alpha)$

$$p(\alpha) = N\,|\underbrace{(\vec{k}\times\vec{\epsilon}_{k1})}_{(0,-\sin\alpha,\cos\alpha)}\cdot\vec{\sigma}_{BA}|^2 = N\sin^2\alpha$$

$$\int_0^{2\pi} p(\alpha)d\alpha = N\int_0^{2\pi}\sin^2\alpha\, d\alpha = N\pi = 1 \quad \Rightarrow \quad N = \frac{1}{\pi}$$

$$\Rightarrow p(\alpha) = \frac{1}{\pi}\sin^2\alpha$$

It is equally reasonable to restrict $0 \le \alpha \le \pi$, since α and $\alpha+\pi$ give the same polarization state, and normalize according to $\int_0^{\pi} d\alpha\, p(\alpha) = 1$

$$\Rightarrow p(\alpha) = \frac{2}{\pi}\sin^2\alpha$$

e) $$w_{BA} = \frac{V}{(2\pi\hbar)^2} \int d^3k \sum_{e} |\langle B, 1_{ke}| H_I |A, 0\rangle|^2 \delta(\omega - \omega_B)$$

$$= \frac{V}{(2\pi\hbar)^2} \frac{e^2\hbar^2}{4m^2} \frac{\hbar}{2V\epsilon_0} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} k^2 dk \, \frac{1}{\omega} \delta(\omega - \omega_B) \cdot \underbrace{\sum_{e} |(\vec{k} \times \vec{\epsilon}_{ke}) \cdot \vec{\sigma}_{BA}|^2}_{P(\theta,\phi)/N = k^2(1+\cos^2\theta)}$$

$$= \frac{e^2\hbar}{32\pi^2 m^2 \epsilon_0 c^5} 2\pi \cdot \underbrace{\int_0^{\pi} d\theta \, (1+\cos^2\theta) \sin\theta}_{8/3} \int_0^{\infty} \omega^3 d\omega \, \delta(\omega - \omega_B)$$

$$= \frac{e^2\hbar\omega_B^3}{6\pi m^2 \epsilon_0 c^5}$$

$$\tau = \frac{1}{w_{BA}} = \frac{6\pi m^2 \epsilon_0 c^5}{e^2\hbar\omega_B^3}$$