FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2021

Return of solutions:
The problem set is available from Friday morning, 22 October.
You may submit handwritten solutions, but they have to be scanned and included in one single file, which is submitted in Inspera before Friday, 29 October, at 12:00.

Language:

Solutions may be written in Norwegian or English depending on your preference.

Questions concerning the problems:

Please ask Joakim Bergli (room V405).

The problem set consists of 1 problem written on 4 pages.

Problem 1: Superradiance

In this problem we will study the phenomenon of superradiance. This is the modifiaction of the emission form an atom in the presence of other identical atoms. To aid you in solving the problems, you may consult any material that you can find on the topic, but as always you should cite the sources you use.

As introduction, we will first consider the emission from a single atom. The interaction beteen the atom and the electromagnetic field is given by the Hamiltonian

$$H_{int} = -\frac{e}{m}\mathbf{A}(\mathbf{r}) \cdot \mathbf{p}.$$

In the following, we will only consider transitions between two atomic states, the ground state $|0\rangle$ and one excited state $|1\rangle$.

a) Show that in the subspace spanned by the atomic states $\{|0\rangle, |1\rangle\}$ and for the purpose of calculating transition rates of spontaneous emission, we can replace the interaction Hamiltonian by

$$H_{int} = \sum_{\mathbf{k}a} g_{\mathbf{k}a}(\hat{a}_{\mathbf{k}a}\sigma^+ + \hat{a}^\dagger_{\mathbf{k}a}\sigma^-)$$

and determine the coupling constants $g_{\mathbf{k}a}$. Here, $\sigma^\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

b) Find the rate of spontaneous emission w_1 from the state $|1\rangle$ to the state $|0\rangle$, summing over all possible final photon states.

We will next study the curious way in which the emission of a photon from an excited atom is modified by the presence of other identical atoms at nearby points in space. If the distance between the atoms is much less than the wavelenght of the emitted light, there is no way to determine which atom the light is emitted from, and we can consider the coupling of each atom to the radiation field to be the same. The interaction Hamiltonian is then

$$H_{int} = \sum_{\mathbf{k}a} g_{\mathbf{k}a}(\hat{a}_{\mathbf{k}a}D^+ + \hat{a}^\dagger_{\mathbf{k}a}D^-)$$

with $D^\pm = \sum_i \sigma_i^\pm$ where the sum is over all the atoms and $\sigma_i^\pm$ is the $\sigma^\pm$ acting on atom i.

c) Consider first two atoms in the initial state $|10\rangle$ where one atom is excited and the other in the ground state. Initially, one would expect the presence of the second atom not to affect the first, but this is not correct. Show that the state $|10\rangle$ will not always decay to the ground state $|00\rangle$, but sometimes only to the state $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$.

   Hint: write the inital state as

$$|10\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) + \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)\right]$$

   and show that only one of the components decay, while the other is not changed by the interaction.

d) On average, how many photons are emitted every time the experiment is repeated?

We consider now a large number N of atoms (we omly consider even N), all so close in space that the interaction with the field is the same. We study the evolution from the initial state $|11\cdots 1\rangle$ where all the atoms are excited. The permutation symmetry of the inial state as well as the interaction Hamiltonian means that only states symmetric under permutation of the atoms will be populated. These take the form

$$|JM\rangle = \sqrt{\frac{(J+M)!}{N!(J-M)!}}(D^-)^{J-M}|11\cdots 1\rangle$$

with $J = N/2$ and $-J \leq M \leq J$ (M is an integer for even N and half integer for odd N).

e) Show that these states are orthogonal and normalized.

f) Show that the decay rate from the state $|JM\rangle$ is

$$w_{JM} = (J+M)(J-M+1)w_1$$

   where w_1 is the decay rate of a single atom that we studied in question b).

g) For a given N, which M will give the largest decay rate (and consequently the largest photon emission rate)? Show that in some cases the rate of photon emission is much larger than what you would expect from N independent atoms, hence the term superradiance.

The fact that the collection of N atoms radiates at a rate much larger than N independent atoms indicates that there are correlations between the atoms. We will study these correlations using two different approaches.

First, we consider the correlation of operators on pairs of atoms. The decay rate w_{JM} that you calculated in question f) is proportional to the matrix element squared,

$$|\langle J, M-1|D^-|JM\rangle|^2$$

h) Show that this is equal to the expectation value

$$\langle JM|D^+D^-|JM\rangle$$

i) Show that

$$\langle JM|\sum_i \sigma_i^+\sigma_i^-|JM\rangle = J+M.$$

j) The permutation symmetry of the state implies that the expectation value $\langle JM|\sigma_i^+\sigma_j^-|JM\rangle$ (for $i \neq j$) is independent of i and j. Show that

$$\langle JM|\sigma_i^+\sigma_j^-|JM\rangle = \frac{J^2-M^2}{N(N-1)}.$$

k) The physical interpretation of this correlation is not so easy to see. To make it more concrete, we can ask what is the probability of measuring some property of atom j given that we know the result of some measurement on atom i. Imagine that we measure σ_x on atom i (if it was a real spin, we know how to do that. Never mind how to do it on an atom, just assume that it can be done). What is the probability that we will get the same result if we then measure σ_x on atom j? Give the answer as a function of J and M. What is the maximal value for having the same result on measuring both atoms and for what J and M does it occur?

A second approach to study the correlations between atoms in the state $|JM\rangle$ is to study the entanglement beween different subsystems. In this part we will limit ourselves to the case $N=4$.

l) Find the reduced density matrix for the first atom for all possible values of M. Calculate the entanglement entropy between the first atom and the rest of the system.

m) For $M=0$, find the reduced density matrix of the first two atoms, and the entanglement entropy with the rest of the system.

The above argument follows closely the original ideas of Dicke (Phys. Rev. **93**, 99 (1954)) and is dependent on the fact that the coupling to the field is perfectly identical for all the atoms, and that there is no interaction between them. In reality, this may be difficult to achieve. The presence of interactions between the atoms means that the atomic eigenstates are no longer eigenstates of the Hamiltonian with interactions included. This implies that we have to take into account the internal dynamis of the atoms during the emission process. One approach to this is to use the master equation in the Lindblad form. To get any real benefit from this method requires more work than we have time for during the exam, so we will limit ourselves to a simple situation without interactions.

We study two identical atoms coupled identically to the electromagnetic field as in question c). The Hamiltonian is then

$$H = -\frac{1}{2}\omega_0(\sigma_1^z+\sigma_2^z).$$

The interaction with the field gives rise to a Lindblad equation of the usual form

$$\frac{d\rho}{dt} = -i[H,\rho] - \frac{\gamma}{2}(D^+D^-\rho + \rho D^+D^- - 2D^-\rho D^+)$$

n) Determine all stationary states (states with $\frac{d\rho}{dt}=0$) of the system.

o) Use the result of the preceeding question to confirm the result in question c) that the initial state $|10\rangle$ will not decay to the ground state and find the final state at $t \to \infty$.

p) The conclusion that not all states will decay to the ground state is a consequence of the identity of the atomic couplings to the electromagnetic field. This means that it is impossible to determine which atom has emitted a given photon. A different situation arises if there is a way to discriminate the photons coming from the two atoms. One way to achieve this is to have independent environments for each atom. Write the Lindblad equation for two two-level atoms coupled to independent environments. A full derivation is not needed, only a reasonable justification. Use this equation to show that any initial state will decay to the ground state.