FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2020. Solution

Problem 1: Superradiance

a) From the lecture notes we have

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k}a} \sqrt{\frac{\hbar}{2V\omega_0\epsilon_0}} \left[\hat{a}_{\mathbf{k}a} e^{i\mathbf{kr}} + \hat{a}^\dagger_{\mathbf{k}a} e^{-i\mathbf{kr}}\right] \epsilon_{\mathbf{k}a}.$$

Restricted to the $\{|0\rangle, |1\rangle\}$ subspace we can write

$$\mathbf{p} = \langle 0|\mathbf{p}|1\rangle |0\rangle\langle 1| + \langle 1|\mathbf{p}|0\rangle |1\rangle\langle 0| = \langle 0|\mathbf{p}|1\rangle \sigma^- + \langle 1|\mathbf{p}|0\rangle \sigma^+$$

When calculating transition rates, there will appear a δ-function ensuring energy conservation. This means that terms of the form $\hat{a}\sigma^-$ or $\hat{a}^\dagger\sigma^+$ never will contribute. We choose the position of the atom to be $\mathbf{r} = 0$ and in the dipole approximation it means that $e^{-i\mathbf{kr}} \approx 1$ and we get

$$H_{int} = -\frac{e}{m} \sum_{\mathbf{k}a} \sqrt{\frac{\hbar}{2V\omega_0\epsilon_0}} \left[\hat{a}_{\mathbf{k}a}\sigma^+ + \hat{a}^\dagger_{\mathbf{k}a}\sigma^-\right] \langle 0|\mathbf{p}|1\rangle \cdot \epsilon_{\mathbf{k}a} = \sum_{\mathbf{k}a} g_{\mathbf{k}a} (\hat{a}_{\mathbf{k}a}\sigma^+ + \hat{a}^\dagger_{\mathbf{k}a}\sigma^-)$$

with

$$g_{\mathbf{k}a} = -\frac{e}{m} \sqrt{\frac{\hbar}{2V\omega_0\epsilon_0}} \langle 0|\mathbf{p}|1\rangle \cdot \epsilon_{\mathbf{k}a}$$

The relative phase of $|0\rangle$ and $|1\rangle$ can always be choosen so that $\langle 1|\mathbf{p}|0\rangle = \langle 0|\mathbf{p}|1\rangle$ is real.

b) The rate of spontaneous emission is

$$w_1 = \sum_{\mathbf{k}a} \frac{2\pi}{\hbar} |\langle 0, 1_{\mathbf{k}a}|H_{int}|1, 0\rangle|^2 \delta(E_0 + \hbar\omega_k - E_1)$$

where E_0 and E_1 are the energies of $|0\rangle$ and $|1\rangle$ and $|1, 0\rangle$ refers to the atom in state $|1\rangle$ and field in vacuum state. As in the lecture notes, eq (4.101) we get

$$w_1 = \frac{e^2\omega}{3\pi c^3 \hbar m^2 \epsilon_0} |\langle 0|\mathbf{p}|1\rangle|^2$$

where $\hbar\omega = E_1 - E_0$. To compare with (4.101) recall (4.80): $\langle 0|\mathbf{p}|1\rangle = im\omega\langle 0|\mathbf{r}|1\rangle$.

c) As indicated in the problem, we write $|10\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle + |\psi^-\rangle)$ with $|\psi^\pm\rangle \frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle)$. The state $|\psi^-\rangle$ is an eigenstate of the Hamiltonian (both the Hamiltonian of the atom, and the interaction) and this part of the initial state will not decay. The remaining $|\psi^+\rangle$ has a nonzero matrix element $\langle 00|D^-|\psi^+\rangle$ and will decay to the ground state $|00\rangle$.

d) There is a probability $\frac{1}{2}$ to be in the state $|\psi^-\rangle$ and therefore not decay. Otherwise, one photon is emitted. On average, $\frac{1}{2}$ photon is emitted for each repetition of the experiment.

e) We have

$$(D^-)^{J-M}|11\cdots 1\rangle \sim |\underbrace{00\cdots 0}_{J-M}\underbrace{11\cdots 1}_{J+M}\rangle + \text{All permutations with J+M atoms in } |1\rangle \text{ and J-M atoms in } |0\rangle$$

Therefore $\langle JM|JM'\rangle = 0$ if $M \neq M'$ since the number of excited atoms are different. To check normalization we note that there are $\binom{N}{J-M} = \frac{N!}{(J-M)!(J+M)!}$ different terms in $(D^-)^{J-M}|11\cdots 1\rangle$. But the operator generates each term seveal times. For a given set of $J-M$ atoms to be de-excited, the order in which they are de-excited does not matter, which means that

$$|JM\rangle = A(D^-)^{J-M}|11\cdots 1\rangle = A(J-M)!(|\underbrace{00\cdots 0}_{J-M}\underbrace{11\cdots 1}_{J+M}\rangle + \text{permutations})$$

where A is the normalization to be determined. We then have

$$\langle JM|JM\rangle = |A|^2[(J-M)!]^2(\langle 00\cdots 011\cdots 1| + \text{permutations})(|00\cdots 011\cdots 1\rangle + \text{permutations}).$$

Each permutation has inner product 1 with itself and 0 with all other permutations, so

$$\langle JM|JM\rangle = |A|^2[(J-M)!]^2\frac{N!}{(J-M)!(J+M)!}.$$

Requiring $\langle JM|JM\rangle = 1$ gives

$$A = \sqrt{\frac{(J+M)!}{N!(J-M)!}}.$$

f) The decay rate from the state $|JM\rangle$ is

$$w_{JM} = \sum_{\mathbf{k}a}\frac{2\pi}{\hbar}|\langle J, M-1, 1_{\mathbf{k}a}|H_{int}|JM, 0\rangle|^2\delta(E_{J,M-1} + \hbar\omega_k - E_{JM}).$$

The difference from the one atom case is that $\langle 0|\sigma^-|1\rangle$ is replaced by

$$\langle J, M-1|D^-|JM\rangle = \sqrt{\frac{(J+M)!}{N!(J-M)!}}\langle J, M-1|(D^-)^{J-M+1}|11\cdots 1\rangle = \sqrt{(J+M)(J-M+1)}$$

where we used that

$$(D^-)^{J-M+1}|11\cdots 1\rangle = \sqrt{\frac{N!(J-M+1)!}{(J+M-1)!}}|J, M-1\rangle.$$

This gives

$$w_{JM} = (J+M)(J-M+1)w_1.$$

g) The decy rate is maximal for $M = 0$ and $M = 1$.

$$w_{J0} = W_{J1} = J(J+1)w_1 = \frac{N}{2}(\frac{N}{2}+1)w_1 \approx \frac{N^2}{4}w_1.$$

One atom emits a photon at the rate w_1, so N independent atoms will emit at the rate $N\omega_1$. For $N \gg 1$ we see that $w_{J0} \gg Nw_1$ so the emission rate is much larger than for N independent atoms.

h)

$$|\langle J,M-1|D^-|JM\rangle|^2 = \langle JM|D^+|J,M-1\rangle\langle J,M-1|D^-|JM\rangle = \langle JM|D^+\sum_{M'}|JM'\rangle\langle JM'|D^-|JM\rangle$$

since $\langle JM'|D^-|JM\rangle = 0$ for all $M' \neq M-1$. Since the states $|JM\rangle$ constitute a complete set, the sum of projectors is the identity and we get

$$|\langle J,M-1|D^-|JM\rangle|^2 = \langle JM|D^+D^-|JM\rangle.$$

i) If $|a_1\cdots a_N\rangle$ with $a_k = 0$ or 1 is some state, we have

$$\sigma_i^+\sigma_i^-|a_1\cdots a_N\rangle = a_i|a_1\cdots a_N\rangle.$$

This means that if $a_k = 0$ for $J-M$ atoms and $a_k = 1$ for $J+M$ atoms

$$\sum_i \sigma_i^+\sigma_i^-|a_1\cdots a_N\rangle = (J+M)|a_1\cdots a_N\rangle.$$

This applies to all permutations and depends only on the number of excited atoms, so $\sum_i \sigma_i^+\sigma_i^-|JM\rangle = (J+M)|JM\rangle$, which menas that

$$\langle JM|\sum_i \sigma_i^+\sigma_i^-|JM\rangle = J+M.$$

j) We have

$$\langle JM|D^+D^-|JM\rangle = \langle JM|\sum_{ij}\sigma_i^+\sigma_j^-|JM\rangle = \langle JM|\sum_i \sigma_i^+\sigma_i^-|JM\rangle + \langle JM|\sum_{i\neq j}\sigma_i^+\sigma_j^-|JM\rangle.$$

Due to the permutation symmetry of the state, the last sum consists of $N(N-1)$ identical terms. From f) and h) we have that

$$\langle JM|D^+D^-|JM\rangle = |\langle J,M-1|D^-|JM\rangle|^2 = (J+M)(J-M+1)$$

which gives

$$\langle JM|\sigma_i^+\sigma_j^-|JM\rangle = \frac{J^2-M^2}{N(N-1)}.$$

k) We have

$$\sigma_i^+\sigma_j^- = \frac{1}{4}(\sigma_x^i + i\sigma_y^i)(\sigma_x^j - i\sigma_y^j) = \frac{1}{4}(\sigma_x^i\sigma_x^j + \sigma_y^i\sigma_y^j - i\sigma_x^i\sigma_y^j + i\sigma_y^i\sigma_x^j).$$

From the permutation symmetry of $|JM\rangle$ we get

$$\langle JM|\sigma_x^i\sigma_y^j|JM\rangle = \langle JM|\sigma_y^i\sigma_x^j|JM\rangle.$$

There is also symmetry with respect to x and y, so

$$\langle JM|\sigma_x^i\sigma_x^j|JM\rangle = \langle JM|\sigma_y^i\sigma_y^j|JM\rangle$$

which means that

$$\langle JM|\sigma_x^i\sigma_x^j|JM\rangle = 2\langle JM|\sigma_i^+\sigma_j^-|JM\rangle = 2\frac{J^2 - M^2}{N(N-1)}.$$

We denote the probability that the measurements of σ_x^i and σ_y^j gives the same result as P_+ and the probability to get opposite results as $P_- = 1 - P_+$. Then $\langle JM|\sigma_x^i\sigma_x^j|JM\rangle = P_+ - P_- = 2P_+ - 1$ which gives that

$$P_+ = \frac{1}{2} + \frac{J^2 - M^2}{N(N-1)}.$$

For $N = 2$ and $M = 0$ we get $P_+ = 1$. For large N and $M = 0$ we get $P_+ \approx \frac{3}{4}$.

l) We have $N = 4$, $J = 2$, $M = -2, -1, 0, 1, 2$.

$$M = 2|22\rangle = |1111\rangle$$
$$\rho_1 = |1\rangle\langle 1|$$
$$S = 0 \text{ (no entanglement)}$$
$$M = 1|21\rangle = \frac{1}{2}(|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle)$$
$$\rho_1 = \frac{1}{4}(|0\rangle\langle 0| + 3|1\rangle\langle 1|)$$
$$S = -\frac{1}{4}\ln\frac{1}{4} - \frac{3}{4}\ln\frac{3}{4}$$
$$M = 0|20\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle)$$
$$\rho_1 = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$
$$S = -2\frac{1}{2}\ln\frac{1}{2} = \ln 2.$$

Negative M gives the same with 0 and 1 interchanged.

m)

$$\rho_{12} = \frac{1}{6}(|00\rangle\langle 00| + 2|01\rangle\langle 01| + 2|10\rangle\langle 10| + |11\rangle\langle 11| + 2|01\rangle\langle 10| + 2|10\rangle\langle 01|) = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where we use the matrix representation $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Two eigenvalues are $p_1 = p_4 = 1/6$. We find the other two eigenvalues

$$\begin{vmatrix} \frac{1}{3} - p & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} - p \end{vmatrix} = p^2 - \frac{2}{3}p + \frac{1}{12} = 0,$$

which gives

$$p_2 = \frac{2}{3} \qquad p_3 = 0.$$

The entropy is

$$S = -\sum_n p_n \ln p_n = \frac{1}{3}\ln 6 - \frac{2}{3}\ln\frac{2}{3} = \ln 3 - \frac{1}{3}\ln 2.$$

n) We have

$$D^- = \sigma^- \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and

$$H = -\frac{\omega_0}{2}(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) = -\omega_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We parametrize the density matrix

$$\rho = \begin{pmatrix} p & a & b & c \\ a^* & q & d & e \\ b^* & d^* & r & f \\ c^* & e^* & f^* & s \end{pmatrix}. \tag{1}$$

with $p, q, r, s \in \mathbb{R}$ and $p + q + r + s = 1$. Using the Lindblad equation we find (after some calculations)

$$\frac{d\rho}{dt} = -i\omega_0 \begin{pmatrix} 0 & -a & -b & -2c \\ a^* & 0 & 0 & -e \\ b^* & 0 & 0 & -f \\ 2c^* & e^* & f^* & 0 \end{pmatrix}$$

$$- \frac{\gamma}{2} \begin{pmatrix} 4p & 3a+b & a+3b & 2c \\ 3a^*+b^* & 2q+d+d^*-2p & q+r+2d-2p & e+f-2a-2b \\ a^*+3b^* & q+r+2d^*-2p & 2r+d+d^*-2p & e+f-2a-2b \\ 2c^* & e+f^*-2a^*-2b^* & e+f^*-2a^*-2b^* & -2(q+r+d+d^*) \end{pmatrix}.$$

A stationary state is a state with $\frac{d\rho}{dt} = 0$, which means that all matrix elements of $\frac{d\rho}{dt}$ are 0. The 11 element gives that $p = 0$. The 23 and 32 elements give that $d = d^*$ and then the 22 ad 23 elements give that $q = r = -d$. The condition $p + q + r + s = 1$ then implies $q = \frac{1}{2}(1 - s)$. The 12 and 13 elements together imply that $a = b = 0$ and if we know that, the elements 42 and 43 give $e = f = 0$. The 14 element gives $c = 0$. The only remaining free parameter is s, and the density matrix has the form

$$\rho = s|00\rangle\langle 00| + (1 - s)|\psi^-\rangle\langle\psi^-|.$$

o) IF the initial state is $|10\rangle$, the initial density matrix has $q = 1$ and all other elements are =0. From the expression for $\frac{d\rho}{dt}$ we see that only the elements q, r, s and d will ever be nonzero. They satisfy the equations

$$\begin{aligned} \dot{q} &= -\gamma(q + d) \\ \dot{r} &= -\gamma(r + d) \\ \dot{d} &= -\frac{\gamma}{2}(q + r + 2d) \\ \dot{s} &= \gamma(q + r + 2d) \end{aligned}$$

Summing the first two equations and subtracting twice the third we get

$$\frac{d}{dt}(q + r - 2d) = 0$$

which implies that $q + r - 2d = 1$ since this is the value at $t = 0$. In the final stationary state we have $q + r + 2d = 0$, so we have $d = -\frac{1}{4}$. Then $q + r = \frac{1}{2}$ and $s = \frac{1}{2}$. The final stationary state is then

$$\rho = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|\psi^-\rangle\langle\psi^-|$$

in accordance with what we found in c).

p) With independent environments for each atom (e.g. distiguishable photon modes) we have one Lindblad operator for each process (atom 1 emits and atom 2 emits).

$$\frac{d\rho}{dt} = -i[H, \rho] - \frac{\gamma_1}{2}(\sigma_1^+\sigma_1^-\rho + \rho\sigma_1^+\sigma_1^- - 2\sigma_1^-\rho\sigma_1^+) - \frac{\gamma_2}{2}(\sigma_2^+\sigma_2^-\rho + \rho\sigma_2^+\sigma_2^- - 2\sigma_2^-\rho\sigma_2^+)$$

where $\sigma_1^{\pm} = \sigma^{\pm} \otimes \mathbb{1}$ and $\sigma_2^{\pm} = \mathbb{1} \otimes \sigma^{\pm}$. With the density matrix as in Eq. (1) we get

$$\begin{aligned}\frac{d\rho}{dt} = -i\omega_0 &\begin{pmatrix} 0 & -a & -b & -2c \\ a^* & 0 & 0 & -e \\ b^* & 0 & 0 & -f \\ 2c^* & e^* & f^* & 0 \end{pmatrix} \\ &- \frac{\gamma_1}{2} \begin{pmatrix} 2p & 2a & b & c \\ 2a^* & 2q & d & e \\ b^* & d^* & -2p & -2a \\ c^* & e^* & -2a^* & -2q \end{pmatrix} - \frac{\gamma_2}{2} \begin{pmatrix} 2p & a & 2b & c \\ a^* & -2p & d & -2b \\ 2b^* & d^* & 2r & f \\ c^* & -2b^* & f^* & -2r \end{pmatrix}.\end{aligned}$$

In a stationary state we have $\frac{d\rho}{dt} = 0$ which gives $p = a = b = c = d = e = f = r = q = 0$ and $s = 1$, so the only stationary state is $|00\rangle\langle 00|$ which means that any initial state will decay to the ground state.