Solutions to problem set 10

10.1 Uncle Charlie's gift

a) If $|\phi\rangle$ is any of the states $|Aa\rangle, |Ab\rangle, |Ba\rangle, |Bb\rangle$ then the probabilities for the possible measurement outcomes in the $\{|0\rangle, |1\rangle\}$ basis are

$$|\langle 00|\phi\rangle|^2 = |\langle 01|\phi\rangle|^2 = |\langle 10|\phi\rangle|^2 = |\langle 11|\phi\rangle|^2 = \frac{1}{4}$$

independent of which of the four states $|\phi\rangle$ is. Therefore it is not possible to distinguish between these states in the $\{|0\rangle, |1\rangle\}$ basis.

b)

$$\sqrt{2}|u\rangle = |0\rangle + |1\rangle, \quad \sqrt{2}|v\rangle = |0\rangle - |1\rangle$$

In the new basis for Alice, the states take the form:

$$\begin{aligned} |Aa\rangle &= \frac{1}{\sqrt{2}}(|u0\rangle + |v1\rangle) \\ |Ab\rangle &= \frac{1}{\sqrt{2}}(|v0\rangle + |u1\rangle) \\ |Ba\rangle &= \frac{1}{\sqrt{2}}(|u0\rangle - |v1\rangle) \\ |Bb\rangle &= \frac{1}{\sqrt{2}}(-|v0\rangle + |u1\rangle) \end{aligned}$$

We see that if Alice measures u, then there are two states that corresponds to Bob measuring 0 $|Aa\rangle$ and $|Ba\rangle$. If Bob measures 1 and Alice u, we get $|Ab\rangle$ or $|Bb\rangle$. If Alice measures v and Bob 0, we get $|Ab\rangle$ or $|Bb\rangle$, and if Bob instead masured 1, we have the states $|Aa\rangle$ or $|Ba\rangle$. In this setup, we can distinguish between which gift is given, but not who recieves it.

c) In this new configuration of bases, the states take the following form:

$$\begin{aligned} |Aa\rangle &= \frac{1}{\sqrt{2}}(|0u\rangle + |1v\rangle) \\ |Ab\rangle &= \frac{1}{\sqrt{2}}(|0u\rangle - |1v\rangle) \\ |Ba\rangle &= \frac{1}{\sqrt{2}}(|0v\rangle + |1u\rangle) \\ |Bb\rangle &= \frac{1}{\sqrt{2}}(-|0v\rangle + |1u\rangle) \end{aligned}$$

In this current setup, we have the pairs $|Aa\rangle$ or $|Ab\rangle$ and $|Ba\rangle$ or $|Bb\rangle$. Now we can distinguish between who recieves the gift, but not what gift.

If both decides on using the $\{u, v\}$ basis, then the states take the following form:

$$\begin{aligned} |0\rangle &= \frac{\sqrt{2}|u\rangle - |1\rangle + \sqrt{2}|v\rangle + |1\rangle}{2} = \frac{1}{\sqrt{2}}\left(|u\rangle + |v\rangle\right) \\ |1\rangle &= \frac{\sqrt{2}|u\rangle - |0\rangle - \sqrt{2}|v\rangle + |0\rangle}{2} = \frac{1}{\sqrt{2}}\left(|u\rangle - |v\rangle\right) \end{aligned}$$

$$\begin{aligned} |00\rangle &= |0\rangle \otimes |0\rangle = \frac{1}{2}\left(|uu\rangle + |uv\rangle + |vu\rangle + |vv\rangle\right) \\ |01\rangle &= |0\rangle \otimes |1\rangle = \frac{1}{2}\left(|uu\rangle - |uv\rangle + |vu\rangle - |vv\rangle\right) \\ |10\rangle &= |1\rangle \otimes |0\rangle = \frac{1}{2}\left(|uu\rangle + |uv\rangle - |vu\rangle - |vv\rangle\right) \\ |11\rangle &= |1\rangle \otimes |1\rangle = \frac{1}{2}\left(|uu\rangle - |uv\rangle - |vu\rangle + |vv\rangle\right) \end{aligned}$$

$$\begin{aligned} |Aa\rangle &= \frac{1}{2}\left(|uu\rangle + |uv\rangle + |vu\rangle - |vu\rangle\right) \\ |Ab\rangle &= \frac{1}{2}\left(|uu\rangle - |uv\rangle + |vu\rangle + |vu\rangle\right) \\ |Ba\rangle &= \frac{1}{2}\left(|uu\rangle + |uv\rangle - |vu\rangle + |vv\rangle\right) \\ |Bb\rangle &= \frac{1}{2}\left(|uu\rangle - |uv\rangle - |vu\rangle - |vv\rangle\right) \end{aligned}$$

From this, we get the same argument as from problem a). A measurement in the $\{|u\rangle, |v\rangle\}$ basis yields no information as the information lies in the superposition of states.

d) We solve the given equations to get the original basis vectors in terms of the new. For Alice:

$$|0\rangle = \alpha^*|w\rangle - \beta|x\rangle \qquad |1\rangle = \beta^*|w\rangle + \alpha|x\rangle$$

For Bob:

$$|0\rangle = \gamma^*|y\rangle - \delta|z\rangle \qquad |1\rangle = \delta^*|y\rangle + \gamma|z\rangle$$

Then we get

$$
\begin{aligned}
|Aa\rangle = \frac{1}{2}\{&[\alpha^*(\gamma^* + \delta^*) + \beta^*(\gamma^* - \delta^*)]\,|wy\rangle + [-\alpha^*(\delta - \gamma) - \beta^*(\gamma + \delta)]\,|wz\rangle \\
&+ [-\beta(\gamma^* + \delta^*) + \alpha(\gamma^* - \delta^*)]\,|xy\rangle + [\beta(\delta - \gamma) - \alpha(\gamma + \delta)]\,|xz\rangle\} \\
|Ab\rangle = \frac{1}{2}\{&[\alpha^*(\gamma^* + \delta^*) - \beta^*(\gamma^* - \delta^*)]\,|wy\rangle + [-\alpha^*(\delta - \gamma) + \beta^*(\gamma + \delta)]\,|wz\rangle \\
&+ [-\beta(\gamma^* + \delta^*) - \alpha(\gamma^* - \delta^*)]\,|xy\rangle + [\beta(\delta - \gamma) + \alpha(\gamma + \delta)]\,|xz\rangle\} \\
|Ba\rangle = \frac{1}{2}\{&[\alpha^*(\gamma^* - \delta^*) + \beta^*(\gamma^* + \delta^*)]\,|wy\rangle + [-\alpha^*(\delta + \gamma) + \beta^*(\gamma - \delta)]\,|wz\rangle \\
&+ [-\beta(\gamma^* - \delta^*) + \alpha(\gamma^* + \delta^*)]\,|xy\rangle + [\beta(\delta + \gamma) + \alpha(\gamma - \delta)]\,|xz\rangle\} \\
|Bb\rangle = \frac{1}{2}\{&[-\alpha^*(\gamma^* - \delta^*) + \beta^*(\gamma^* + \delta^*)]\,|wy\rangle + [\alpha^*(\delta + \gamma) + \beta^*(\gamma - \delta)]\,|wz\rangle \\
&+ [\beta(\gamma^* - \delta^*) + \alpha(\gamma^* + \delta^*)]\,|xy\rangle + [-\beta(\delta + \gamma) + \alpha(\gamma - \delta)]\,|xz\rangle\}
\end{aligned}
$$

We observe that the coefficients in front of $|wz\rangle$ and $|xy\rangle$ are complex conjugates of each other (or the negative of the complex conjugate), and therefore 0 at the same time. We want $|Aa\rangle$ and $|Bb\rangle$ to give the same results, and only have projsections on two of the basis states. We want to eliminate $|wz\rangle$ and $|xy\rangle$ from both of them, which requires

$$
\begin{aligned}
-\alpha^*(\delta - \gamma) - \beta^*(\gamma + \delta) = 0 \\
\alpha^*(\delta + \gamma) + \beta^*(\gamma - \delta) = 0
\end{aligned}
\tag{1}
$$

Dividing the two equations we get

$$
\frac{\delta - \gamma}{\delta + \gamma} = \frac{\gamma + \delta}{\gamma - \delta}
$$

which gives $\delta = i\gamma$. Since $|\delta|^2 + |\gamma|^2 = 1$ this implies that $|\gamma| = \frac{1}{\sqrt{2}}$, and we can choose $\gamma = \frac{1}{\sqrt{2}}$ to be real. Inserting this into (1) we get $\beta = i\alpha$, and we have that $\alpha = \gamma = \frac{1}{\sqrt{2}}$ and $\beta = \delta = \frac{i}{\sqrt{2}}$ is a solution. We then have

$$
\begin{aligned}
|Aa\rangle &= \frac{1}{\sqrt{2}}\left(e^{i\pi/4}|wy\rangle - e^{i\pi/4}|xz\rangle\right) \\
|Ab\rangle &= \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|wz\rangle - e^{i\pi/4}|xy\rangle\right) \\
|Ba\rangle &= \frac{1}{\sqrt{2}}\left(-e^{i\pi/4}|wz\rangle + e^{-i\pi/4}|xy\rangle\right) \\
|Bb\rangle &= \frac{1}{\sqrt{2}}\left(-e^{i\pi/4}|wy\rangle + e^{i\pi/4}|xz\rangle\right)
\end{aligned}
$$

which gives the measurement outcomes $|wy\rangle$ or $|xz\rangle$ for the states $|Aa\rangle$ or $|Bb\rangle$ and the outcomes $|wz\rangle$ or $|xy\rangle$ for the states $|Ab\rangle$ or $|Ba\rangle$ as we wanted.

e) When performing a measurement, we destroy the entanglement due to the wavefunction "collapsing" on determinate states. The result is a separable product state, which doesn't contain any entanglement.

10.2 Distributed information (Exam 2012)

a) If we rewrite our state as as:

$$|\psi_n\rangle = \frac{1}{\sqrt{3}}\left(|+\rangle \otimes |--\rangle + \eta^n|-\rangle \otimes |+-\rangle + (\eta^*)^n\,|-\rangle \otimes |-+\rangle\right)$$

We see that the density operator $\hat{\rho}_n = |\psi_n\rangle\langle\psi_n|$ will contain terms of the form $|a\rangle\langle b| \otimes |cd\rangle\langle ef|$, $a, b, c, d, e, f \in \{+, -\}$. To find the reduced density for A, we need to take the trace over the second factor in the tensor product:

$$\mathrm{Tr}\left(|cd\rangle\langle ef|\right) = \langle ef|cd\rangle = \delta_{ec}\delta_{fd}$$

Thus, the only contributions will be:

$$\begin{aligned}\hat{\rho}_A &= \frac{1}{3}\left(|+\rangle\langle+| \otimes \mathrm{Tr}|--\rangle\langle--| + \eta^n\,(\eta^n)^*\,|-\rangle\langle-| \otimes \mathrm{Tr}|+-\rangle\langle+-| + (\eta^*)^n\,\eta^n|-\rangle\langle-| \otimes \mathrm{Tr}|-+\rangle\langle-+|\right) \\ &= \frac{1}{3}\left(|+\rangle\langle+| + 2|-\rangle\langle-|\right)\end{aligned}$$

We see that the density operator doesn't depend on n, and since all information was contained in n, A has no information of the distributed spin. Let's now look at what information A, B and C together can get from measuring their qbit. When all measure in the basis I, the probability of measuring different states are given by $\langle abc|\hat{\rho}_n|def\rangle = \langle abc|\psi_n\rangle\langle\psi_n|def\rangle$. The only terms that survive are diagonal terms, i.e terms $|def\rangle \subset |\psi_n\rangle$. From the calculation of the reduced density matrix $\hat{\rho}_A$, we see that these terms will be independent of n, and thus, A, B, C won't get any information about n from these measurements.

b) Let's write our state as

$$|\psi_n\rangle = \frac{1}{\sqrt{3}}\left(|+-\rangle \otimes |-\rangle + \eta^n|-+\rangle \otimes |-\rangle + (\eta^*)^n\,|--\rangle \otimes |+\rangle\right)$$

Then by taking the trace over C, the only contributing elements are the diagonal terms in C.

$$\begin{aligned}\hat{\rho}_n^{AB} &= \mathrm{Tr}_C\hat{\rho}_n \\ &= \frac{1}{3}\{|+-\rangle\langle+-| \otimes \mathrm{Tr}\left(|-\rangle\langle-|\right) + (\eta^n)^*\,|+-\rangle\langle-+| \otimes \mathrm{Tr}\,|-\rangle\langle-| \\ &\quad + \eta^n|-+\rangle\langle+-| \otimes \mathrm{Tr}\,|-\rangle\langle-| + |-+\rangle\langle-+| \otimes \mathrm{Tr}\,|-\rangle\langle-| + |--\rangle\langle--| \otimes \mathrm{Tr}\,|+\rangle\langle+|\} \\ &= \frac{1}{3}\{|+-\rangle\langle+-| + |-+\rangle\langle-+| + |--\rangle\langle--| + \eta^n|-+\rangle\langle+-| + (\eta^n)^*\,|+-\rangle\langle-+|\}\end{aligned}$$

The probabilities for measuring different $|\phi_k\rangle$ given an n, are:

$$\begin{aligned}p\,(k|n) &= \langle\phi_k|\hat{\rho}_n^{AB}|\phi_k\rangle \\ &= \frac{1}{3}\{\langle\phi_k|+-\rangle\langle+-|\phi_k\rangle + \langle\phi_k|-+\rangle\langle-+|\phi_k\rangle + \langle\phi_k|--\rangle\langle--|\phi_k\rangle \\ &\quad + \eta^n\langle\phi_k|-+\rangle\langle+-|\phi_k\rangle + (\eta^n)^*\,\langle\phi_k|+-\rangle\langle-+|\phi_k\rangle\}\end{aligned}$$

The overlaps are given by:

$$\langle 0|+\rangle = \langle 0|-\rangle = \langle 1|+\rangle = \frac{1}{\sqrt{2}}, \quad \langle 1|-\rangle = -\frac{1}{\sqrt{2}}$$

Then we're ready to calculate the probabilities:

$$\begin{aligned}
p\,(1|0) &= \frac{1}{3}\{\langle 00|+-\rangle\langle +-|00\rangle + \langle 00|-+\rangle\langle -+|00\rangle + \langle 00|--\rangle\langle --|00\rangle \\
&\quad +\eta^0\langle 00|-+\rangle\langle +-|00\rangle + \left(\eta^0\right)^*\langle 00|+-\rangle\langle -+|00\rangle\Big\} \\
&= \frac{1}{3}\cdot 5\frac{1}{4} = \frac{5}{12} \\
p\,(2|0) &= \frac{1}{3}\{\langle 01|+-\rangle\langle +-|01\rangle + \langle 01|-+\rangle\langle -+|01\rangle + \langle 01|--\rangle\langle --|01\rangle \\
&\quad +\eta^0\langle 01|-+\rangle\langle +-|01\rangle + \left(\eta^0\right)^*\langle 01|+-\rangle\langle -+|01\rangle\Big\} \\
&= \frac{1}{3}\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-\frac{1}{4}-\frac{1}{4}\right) = \frac{1}{12} \\
p\,(1|1) &= \frac{1}{3}\{\langle 00|+-\rangle\langle +-|00\rangle + \langle 00|-+\rangle\langle -+|00\rangle + \langle 00|--\rangle\langle --|00\rangle \\
&\quad +\eta\langle 00|-+\rangle\langle +-|00\rangle + \eta^*\langle 00|+-\rangle\langle -+|00\rangle\} \\
&= \frac{1}{3}\left(\frac{3}{4}+\frac{1}{4}\left(\eta+\eta^*\right)\right) \\
&= \frac{2}{12} = \frac{1}{6} \\
p\,(2|1) &= \frac{1}{3}\{\langle 01|+-\rangle\langle +-|01\rangle + \langle 01|-+\rangle\langle -+|01\rangle + \langle 01|--\rangle\langle --|01\rangle \\
&\quad +\eta\langle 01|-+\rangle\langle +-|01\rangle + \eta^*\langle 01|+-\rangle\langle -+|01\rangle\} \\
&= \frac{1}{3}\left(\frac{3}{4}-\frac{1}{4}\left(\eta+\eta^*\right)\right) \\
&= \frac{1}{3}
\end{aligned}$$

The change $n = 1 \to n = 2$ doesn't change the probabilities:

$$\begin{aligned}
\eta^2 &= e^{4\pi i/3} = e^{2\pi i\left(1-\frac{1}{3}\right)} = e^{2\pi i}e^{-2\pi i/3} = \eta^* \\
\left(\eta^*\right)^2 &= e^{-4\pi i/3} = e^{-2\pi i\left(1-\frac{1}{3}\right)} = e^{-2\pi i}e^{2\pi i/3} = \eta
\end{aligned}$$

So $\eta^2 + \left(\eta^*\right)^2 = \eta^* + \eta = -1$. Since the probabilities on the real part of the phase, this doesn't change anything.

c) From normalization:

$$\sum_n \bar{p}(n|k) = 1 \Rightarrow \sum_n \frac{p(k|n)}{p(k)} = 1 \Rightarrow p(k) = \sum_n p(k|n)$$

Thus:

$$\bar{p}(n|k) = \frac{p(k|n)}{\sum_n p(k|n)}$$

For the case $k = 1$, we get:

$$\begin{aligned}\bar{p}\,(0|1) &= \frac{p(1|0)}{p(1|0)+p(1|1)+p(1|2)} = \frac{\frac{5}{12}}{\frac{5}{12}+2\frac{1}{6}} = \frac{\frac{5}{12}}{\frac{9}{12}} = \frac{5}{9} \\ \bar{p}\,(1|1) &= \frac{p(1|1)}{\frac{9}{12}} = \frac{\frac{1}{6}}{\frac{9}{12}} = \frac{2}{9} \\ \bar{p}\,(2|1) &= \frac{p(1|2)}{\frac{9}{12}} = \frac{p(1|1)}{\frac{9}{12}} = \frac{2}{9}\end{aligned}$$

We see that the message $n = 0$ is most probable, while $n = 1, 2$ is equally probable.

10.3 Three-spin entanglement

a)

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{3}\left(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right)\left(\langle\uparrow\downarrow\downarrow| + \langle\downarrow\uparrow\downarrow| + \langle\downarrow\downarrow\uparrow|\right)$$

$$\rho_A = \mathrm{Tr}_{BC}\,\rho = \frac{1}{3}\left(|\uparrow\rangle\langle\uparrow| + 2|\downarrow\rangle\langle\downarrow|\right)$$

$$\rho_{BC} = \mathrm{Tr}_A\,\rho = \frac{1}{3}\left(|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|\right)$$

The von Neumann entropy is

$$S = -\,\mathrm{Tr}\,\rho_A \ln\rho_A = -\frac{1}{3}\ln\frac{1}{3} - \frac{2}{3}\ln\frac{2}{3}$$

We could also use ρ_{BC}, but it is more complicated since it is not diagonal.

b) Measure $\uparrow$: $|\psi\rangle \to |\uparrow\downarrow\downarrow\rangle$. $S_{BC} = 0$.
Measure $\downarrow$: $|\psi\rangle \to \frac{1}{\sqrt{2}}|\downarrow\rangle(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$. $S_{BC} = \ln 2$.

c) The eigenstates of σ_x are

$$\begin{aligned}|\rightarrow\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \qquad \sigma_x|\rightarrow\rangle = |\rightarrow\rangle \\ |\leftarrow\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle) \qquad \sigma_x|\leftarrow\rangle = -|\leftarrow\rangle\end{aligned}$$

If we solve these equations we get

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\rightarrow\rangle + |\leftarrow\rangle), \qquad |\downarrow\rangle = \frac{1}{\sqrt{2}}(|\rightarrow\rangle - |\leftarrow\rangle).$$

The state can then be expressed as

$$|\psi\rangle = \frac{1}{\sqrt{6}}(|\rightarrow\downarrow\downarrow\rangle + |\leftarrow\downarrow\downarrow\rangle + |\rightarrow\uparrow\downarrow\rangle - |\leftarrow\uparrow\downarrow\rangle + |\rightarrow\downarrow\uparrow\rangle - |\leftarrow\downarrow\uparrow\rangle)$$

Measure $\rightarrow$: $|\psi\rangle \rightarrow \frac{1}{\sqrt{3}}|\rightarrow\rangle(|\downarrow\downarrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$.

Measure $\leftarrow$:$|\psi\rangle \rightarrow \frac{1}{\sqrt{3}}|\leftarrow\rangle(|\downarrow\downarrow\rangle - |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

For BC we have (up to a global phase)

$$|\psi_{BC}\rangle = \frac{1}{\sqrt{3}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \pm |\downarrow\downarrow\rangle).$$

The density matrix is

$$\rho_{BC} = |\psi_{BC}\rangle\langle\psi_{BC}| = \frac{1}{3}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \pm |\downarrow\downarrow\rangle)(\langle\uparrow\downarrow| + \langle\downarrow\uparrow| \pm \langle\downarrow\downarrow|)$$

$$\rho_B = \mathrm{Tr}_C\, \rho_{BC} = \frac{1}{3}(2|\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow| \pm |\uparrow\rangle\langle\downarrow| \pm |\downarrow\rangle\langle\uparrow|)$$

In matrix form

$$\rho_B = \frac{1}{3}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 2 \end{pmatrix}$$

The eigenvalues are found from the characteristic equation

$$\begin{vmatrix} \frac{1}{3} - \lambda & \pm\frac{1}{3} \\ \pm\frac{1}{3} & \frac{2}{3} - \lambda \end{vmatrix} = (\lambda - \frac{1}{3})(\lambda - \frac{2}{3}) - \frac{1}{9} = 0$$

which gives

$$\lambda_\pm = \frac{1 \pm \frac{1}{3}\sqrt{5}}{2}.$$

The entanglement entropy is

$$S = -\frac{1 + \frac{1}{3}\sqrt{5}}{2} \ln \frac{1 + \frac{1}{3}\sqrt{5}}{2} - \frac{1 - \frac{1}{3}\sqrt{5}}{2} \ln \frac{1 - \frac{1}{3}\sqrt{5}}{2}.$$