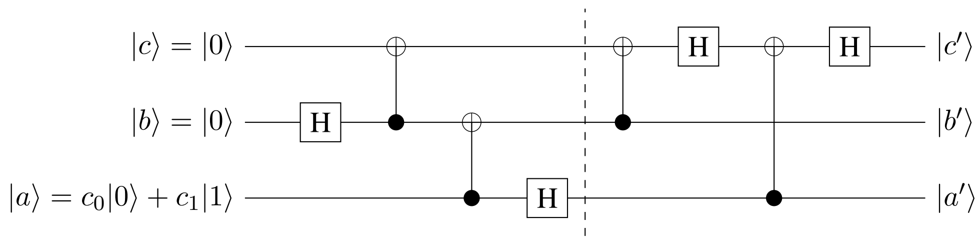
Problem set 12

12.1 Quantum gates for teleportation

We consider the following quantum circuit (ignore the vertical dashed line for the moment)



a) Confirm that the final state is a product state despite the entanglement created in the middle of the circuit and that $|c'\rangle = |a\rangle$ so that the state of the lowest qubit is teleported to the upper qubit.

b) We now measure the a and b qubits at the vertical dashed line and perform the two remaining CNOT gates based on the outcomes of these measurements. That is, they are local at qubit c. Check that we still find $|c'\rangle = |a\rangle$ at the end.

12.2 Quantum cloning of orthogonal states

The no-cloning theorem tells us that it is not possible to make a copy of an arbitrary initial state. However, if we know that the states we have to copy are not general, but selected from a set of orthogonal states, we can find a way to copy them.

a) Given two orthgonal states $|\psi\rangle$ and $|\phi\rangle$ for a single qubit, design a quantum circuit with two input qubits with the following properties. If the first qubit is in the state to be copied, which is always either $|\psi\rangle$ or $|\phi\rangle$ and the second qubit is in a standard state $|0\rangle$, the output is $|\psi\rangle|\psi\rangle$ or $|\phi\rangle|\phi\rangle$ depending on whether $|\psi\rangle$ or $|\phi\rangle$ was input on the first qubit (the circuit is not general, it will depend on which states $|\psi\rangle$ and $|\phi\rangle$ are used). Assume that you can use as elementary gates in the circuit all single qubit gates and CNOT.

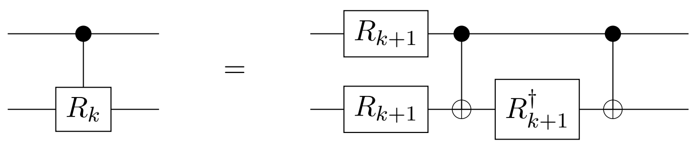
b) Assume for simplicity that the two orthogonal states are $|0\rangle$ and $|1\rangle$. What is the output of the circuit if the input is the superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$?

12.3 Quantum circuit for controlled R_k

a) In the quantum Fourier transformation, we needed to perform a controlled R_k operation. The one-qubit operator

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

is then performed on the target qubit if the control qubit is in the state $|1\rangle$. When the control qubit is in the state $|0\rangle$ no operation is performed on the target qubit. We know that all two-qubit operators can be decomposed in single qubit operators and controlled NOT (CNOT) operations. Show that the following quantum circuit is one such decompostion for the controlled R_k operation



b) We consider now general controlled U operations, with U a one-qubit operator. This means that the operation U is performed on the target qubit if the control qubit is in the state $|1\rangle$. When the control qubit is in the state $|0\rangle$ no operation is performed on the target qubit. In both cases, the control qubit is not changed. If this was a classical system, this would be all the possibilities, but in a quantum system, one can have a control qubit that is in a superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ of the two basis states. In general, the two qubits will be entangled by this operation, so no definite quantum state can be ascribed to any of them. However, a special situation arises if the initial state of the target qubit is an eigenstate of U. Draw a quantum circuit desribing this situation. Show that in this case, the two qubits are not entangled by the operation. Show also that in this case, it is the target qubit that is not changed, while the state of the control qubit is changed. Find the final state of the control qubit in terms of the eigenvalues of U.

c) This result is surprising if we only are used to the classical world, and deserves an explanation. Explain in words why the target is not changed while the state of the control does change.