Solutions to problem set 12

12.1 Quantum gates for teleportation

a) We have to calculate the action of each gate on the state. The initial state is

$$|\psi_0\rangle = (c_0|0\rangle + c_1|1\rangle) \otimes |0\rangle \otimes |0\rangle = c_0|000\rangle + c_1|100\rangle.$$

We write H^i for the Hadamard gate on qubit i, and C^{ij}_{NOT} for the CNOT gate with i as control bit and j as target bit. After each gate we then get

$$|\psi_1\rangle = H^b|\psi_0\rangle = \frac{1}{\sqrt{2}}\Big[c_0|000\rangle + c_0|010\rangle + c_1|100\rangle + c_1|110\rangle\Big]$$

$$|\psi_2\rangle = C^{bc}_{NOT}|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big[c_0|000\rangle + c_0|011\rangle + c_1|100\rangle + c_1|111\rangle\Big]$$

$$|\psi_3\rangle = C^{ab}_{NOT}|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big[c_0|000\rangle + c_0|011\rangle + c_1|110\rangle + c_1|101\rangle\Big]$$

$$\begin{aligned}|\psi_4\rangle = H^a|\psi_3\rangle = \frac{1}{2}\Big[&c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle \\ &+c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle\Big]\end{aligned}$$

$$\begin{aligned}|\psi_5\rangle = C^{bc}_{NOT}|\psi_4\rangle = \frac{1}{2}\Big[&c_0|000\rangle + c_0|100\rangle + c_0|010\rangle + c_0|110\rangle \\ &+c_1|011\rangle - c_1|111\rangle + c_1|001\rangle - c_1|101\rangle\Big]\end{aligned}$$

$$\begin{aligned}|\psi_6\rangle = H^c|\psi_5\rangle = \frac{1}{2\sqrt{2}}\Big[&(c_0+c_1)|000\rangle + (c_0-c_1)|001\rangle + (c_0-c_1)|100\rangle + (c_0+c_1)|101\rangle \\ &+(c_0+c_1)|010\rangle + (c_0-c_1)|011\rangle + (c_0-c_1)|110\rangle + (c_0+c_1)|111\rangle\Big]\end{aligned}$$

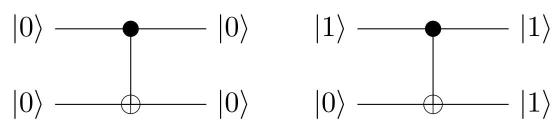
$$\begin{aligned}|\psi_7\rangle = C^{ac}_{NOT}|\psi_6\rangle = \frac{1}{2\sqrt{2}}\Big[&(c_0+c_1)|000\rangle + (c_0-c_1)|001\rangle + (c_0-c_1)|101\rangle + (c_0+c_1)|100\rangle \\ &+(c_0+c_1)|010\rangle + (c_0-c_1)|011\rangle + (c_0-c_1)|111\rangle + (c_0+c_1)|110\rangle\Big]\end{aligned}$$

$$\begin{aligned}|\psi_8\rangle = H^c|\psi_7\rangle &= \frac{1}{2}\Big[c_0|000\rangle + c_1|001\rangle + c_0|100\rangle + c_1|101\rangle \\ &\quad +c_0|010\rangle + c_1|011\rangle + c_0|110\rangle + c_1|111\rangle\Big] \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes (c_0|0\rangle + c_1|1\rangle)\end{aligned}$$

b) Measuring qubits a and b at the dashed line collapses the wavefunction at that point. But since a and b only acts as control bits forthe last four gates, their states do not change. Then the state will be the same as if we measure a and b on the final state $|\psi_8\rangle$ instead. The only difference is that now the CNOT gates will not be nonlocal two-qubit gates, but rather local one-qubit gates on qubit c conditioned on the measurement outcomes for a and b. This has to be transmitted from a and b to c as in the usual teleportation protocol. Then we still get $|c'\rangle = |a\rangle$ at the end. and only need local operations after the dashed line.

12.2 Quantum cloning of orthogonal states

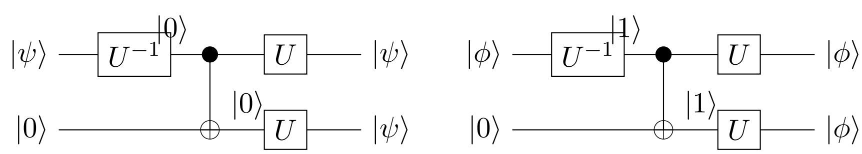
a) Assume first that $|\psi\rangle = |0\rangle$ and $|\phi\rangle = |1\rangle$. Then we can check that a single CNOT gate gives the desired result (upper line is the original, lower line is the copy)



Since $|\psi\rangle$ and $|\phi\rangle$ are orthogonal, there exist a unitary transformation U such that

$$|\psi\rangle = U|0\rangle$$
$$|\phi\rangle = U|1\rangle$$

The inverse of this transforms $|\psi\rangle$ and $|\phi\rangle$ to $|0\rangle$ and $|1\rangle$, and we can then use the CNOT as above and transform the result back, giving the final circuit



b) Here we can use the simple circuit with a single CNOT gate. The input is (qubits are written from top down)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

giving the final state

$$|\psi_1\rangle = C_{NOT}|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

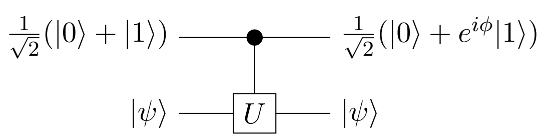
12.3 Quantum circuit for controlled R_k

a) We define $\phi = 2\pi/2^k$ and get

$$
\begin{aligned}
|\psi_1\rangle \otimes |\psi_2\rangle = (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)& \\
\overset{R_{k+1}}{\rightarrow} (a_0|0\rangle + a_1 e^{i\phi/2}|1\rangle) \otimes (b_0|0\rangle + b_1 e^{i\phi/2}|1\rangle)& \\
\overset{CNOT}{\rightarrow} a_0|0\rangle \otimes (b_0|0\rangle + b_1 e^{i\phi/2}|1\rangle) + a_1 e^{i\phi/2}|1\rangle \otimes (b_0|1\rangle + b_1 e^{i\phi/2}|0\rangle)& \\
\overset{R^\dagger_{k+1}}{\rightarrow} a_0|0\rangle \otimes (b_0|0\rangle + b_1|1\rangle) + a_1 e^{i\phi/2}|1\rangle \otimes (b_0 e^{-i\phi/2}|1\rangle + b_1 e^{i\phi/2}|0\rangle)& \\
\overset{CNOT}{\rightarrow} a_0|0\rangle \otimes (b_0|0\rangle + b_1|1\rangle) + a_1|1\rangle \otimes (b_0|0\rangle + b_1 e^{i\phi}|1\rangle)& \\
= a_0|0\rangle \otimes |\psi_2\rangle + a_1|1\rangle \otimes R_k|\psi_2\rangle&
\end{aligned}
$$

This is the controlled R_k operation.

b) Let $U|\psi\rangle = e^{i\phi}|\psi\rangle$. The situation is described by this circuit



The evolution of the state is

$$
\begin{aligned}
\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \otimes |\psi\rangle \overset{\text{control}-U}{\rightarrow} \frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes U|\psi\rangle)& \\
= \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle) \otimes \psi.&
\end{aligned}
$$

c) Since multiplying by a phase factor does not change a quantum state, U does not really change the state of the target if the initial state is an eigenstate. However, the relative phase between two states does make a physical difference. Therefore, when the control is in a superposition, there is a phase difference between the two states after the control-U operation. Since the state of the target is the same in both cases, it factors out, leaving a product state with the relative phase between the two states of the control qubit.