Solutions to problem set 3

3.1 Ladder operators in the Heisenberg picture

Alternative 1

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \quad \hat{\mathcal{U}}(t,0) = e^{-it\hat{H}/\hbar}$$

One way to find the time evolution of an operator in the heisenberg picture, is to transform them as follows:

$$\begin{aligned}
\hat{a}(t) &= \hat{\mathcal{U}}(t,0)^\dagger\hat{a}\hat{\mathcal{U}}(t,0) = e^{it\hat{H}/\hbar}\hat{a}e^{-it\hat{H}/\hbar} \\
\hat{a}^\dagger(t) &= \hat{\mathcal{U}}(t,0)^\dagger\hat{a}^\dagger\hat{\mathcal{U}}(t,0) = e^{it\hat{H}/\hbar}\hat{a}e^{-it\hat{H}/\hbar}
\end{aligned}$$

We know that

$$e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} = \hat{B} + \lambda\left[\hat{A},\hat{B}\right] + \frac{\lambda^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \cdots$$

We need the commutators

$$\begin{aligned}
\left[\hat{H},\hat{a}\right] &= \hbar\omega\left[\hat{a}^\dagger\hat{a} + \frac{1}{2},\hat{a}\right] = \hbar\omega\left[\hat{a}^\dagger\hat{a},\hat{a}\right] = \hbar\omega\left(\left[\hat{a}^\dagger,\hat{a}\right]\hat{a} + \hat{a}^\dagger\left[\hat{a},\hat{a}\right]\right) = -\hbar\omega\hat{a} \\
\left[\hat{H},\hat{a}^\dagger\right] &= \hbar\omega\left[\hat{a}^\dagger\hat{a} + \frac{1}{2},\hat{a}^\dagger\right] = \hbar\omega\left[\hat{a}^\dagger\hat{a},\hat{a}^\dagger\right] = \hbar\omega\left(\left[\hat{a}^\dagger,\hat{a}^\dagger\right]\hat{a} + \hat{a}^\dagger\left[\hat{a},\hat{a}^\dagger\right]\right) = \hbar\omega\hat{a}^\dagger
\end{aligned}$$

in order to get:

$$\begin{aligned}
e^{it\hat{H}/\hbar}\hat{a}e^{-it\hat{H}/\hbar} &= \hat{a} + \frac{it}{\hbar}\left[\hat{H},\hat{a}\right] + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2\left[\hat{H},\left[\hat{H},\hat{a}\right]\right] + \cdots \\
&= \hat{a} - it\omega\hat{a} + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2(-\hbar\omega)^2\hat{a} + \cdots \\
&= \hat{a}\left(1 + (-it\omega) + \frac{(-it\omega)^2}{2!} + \cdots\right) \\
&= \hat{a}e^{-it\omega}
\end{aligned}$$

and

$$\begin{aligned}
e^{it\hat{H}/\hbar}\hat{a}^\dagger e^{-it\hat{H}/\hbar} &= \hat{a}^\dagger + \frac{it}{\hbar}\left[\hat{H},\hat{a}^\dagger\right] + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2\left[\hat{H},\left[\hat{H},\hat{a}^\dagger\right]\right] + \cdots \\
&= \hat{a}^\dagger + it\omega\hat{a}^\dagger + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2(\hbar\omega)^2\hat{a}^\dagger + \cdots \\
&= \hat{a}^\dagger\left(1 + (it\omega) + \frac{(it\omega)^2}{2!} + \cdots\right) \\
&= \hat{a}^\dagger e^{it\omega}
\end{aligned}$$

So we have:

$$\hat{a}(t) = \hat{a}e^{-it\omega}, \quad \hat{a}^\dagger(t) = \hat{a}^\dagger e^{i\omega t}$$

Alternative 2 Another option is to use the Heisenberg equation of motion:

$$\dot{\hat{a}} = \frac{i}{\hbar}[H, \hat{a}] = i\omega\left[\hat{a}^\dagger\hat{a}, \hat{a}\right] = -i\omega\hat{a} \tag{1}$$

The solution to this equation is the same if a is an operator as if it was a number, and we get

$$\hat{a}(t) = \hat{a}e^{-it\omega},$$

and similarly for $\hat{a}^\dagger$.

3.2 Displacement operators in phase space

We have the Campbell-Baker-Hausdorff expansion: $e^{\lambda\hat{A}}e^{\lambda\hat{B}} = e^{\lambda\hat{A}+\lambda\hat{B}+\frac{\lambda^2}{2}[\hat{A},\hat{B}]+\cdots}$. Let's examine the commutators, with $\hat{A} = z_a\hat{a}^\dagger - z_a^*\hat{a}$, $\hat{B} = z_b\hat{a}^\dagger - z_b^*\hat{a}$.

$$\begin{aligned}
\left[\hat{A}, \hat{B}\right] &= \left[z_a\hat{a}^\dagger - z_a^*\hat{a}, z_b\hat{a}^\dagger - z_b^*\hat{a}\right] = \underbrace{\left[z_a\hat{a}^\dagger, z_b\hat{a}^\dagger\right]}_{=0} + \left[z_a\hat{a}^\dagger, -z_b^*\hat{a}\right] + \left[-z_a^*\hat{a}, z_b\hat{a}^\dagger\right] + \underbrace{\left[-z_a^*\hat{a}, -z_b^*\hat{a}\right]}_{=0} \\
&= z_a z_b^* - z_a^* z_b = \underbrace{\mathrm{Re}\left(z_a z_b^*\right) - \mathrm{Re}\left(z_a^* z_b\right)}_{=0} + \underbrace{i\,\mathrm{Im}\left(z_a z_b^*\right) - i\,\mathrm{Im}\left(z_a^* z_b\right)}_{=i\,\mathrm{Im}\left(z_a z_b^*\right) - i\,\mathrm{Im}\left(\left[z_a z_b^*\right]^*\right)} = 2i\,\mathrm{Im}\left(z_a z_b^*\right) \\
\left[\hat{B}, \hat{A}\right] &= -\left[\hat{A}, \hat{B}\right] = -2i\,\mathrm{Im}\left(z_a z_b^*\right)
\end{aligned}$$

These are complex numbers (and not operators), which means higher order terms vanish, as everything commutes with numbers. We're ready to examine the problem:

$$\begin{aligned}
\hat{\mathcal{D}}\left(z_a\right)\hat{\mathcal{D}}\left(z_b\right) &= e^{\lambda\hat{A}+\lambda\hat{B}+\frac{\lambda^2}{2}[\hat{A},\hat{B}]} \\
&\overset{\lambda=1}{=} e^{z_a\hat{a}^\dagger - z_a^*\hat{a} + z_b\hat{a}^\dagger - z_b^*\hat{a} + \frac{1}{2}\left(z_a z_b^* - z_a^* z_b\right)} \\
&= e^{\left(z_a+z_b\right)\hat{a}^\dagger - \left(z_a^*+z_b^*\right)\hat{a} + i\,\mathrm{Im}\left(z_a z_b^*\right)}
\end{aligned}$$

If we change the order, the only thing that changes is the order in the commutator:

$$\hat{\mathcal{D}}\left(z_b\right)\hat{\mathcal{D}}\left(z_a\right) = e^{\left(z_a+z_b\right)\hat{a}^\dagger - \left(z_a^*+z_b^*\right)\hat{a} - i\,\mathrm{Im}\left(z_a z_b^*\right)}$$

From this wee see that

$$\hat{\mathcal{D}}\left(z_a\right)\hat{\mathcal{D}}\left(z_b\right) = e^{2i\,\mathrm{Im}\left(z_a z_b^*\right)}\hat{\mathcal{D}}\left(z_b\right)\hat{\mathcal{D}}\left(z_a\right)$$

This makes: $\alpha\left(z_a, z_b\right) = 2\,\mathrm{Im}\left(z_a z_b^*\right)$, and the condition for the displacements to commute, is $\mathrm{Im}\left(z_a z_b^*\right) = 0$. Written out, this becomes:

$$\begin{aligned}
\frac{1}{2m\hbar\omega}\mathrm{Im}\left[\left(m\omega x_c^a + ip_c^a\right)\left(m\omega x_c^b - ip_c^b\right)\right] &= 0 \\
\mathrm{Im}\left[\left(m\omega\right)^2 x_c^a x_c^b - im\omega x_c^a p_c^b + im\omega p_c^a x_c^b + p_c^a p_c^b\right] &= 0 \\
-m\omega x_c^a p_c^b + m\omega p_c^a x_c^b &= 0
\end{aligned}$$

The resulting condition is:

$$\frac{x_c^a}{p_c^a} = \frac{x_c^b}{p_c^b}$$

Geometrically it means that $z_b = cz_a$ with $c \in \mathbb{R}$ a real constant, which means that z_a and z_b lie on the same line passing through the origin in the complex plane.

3.3 Eigenvectors for $\hat{a}^\dagger$?

We assume $\hat{a}^\dagger|\bar{z}\rangle = \bar{z}|\bar{z}\rangle$ and expand in the basis $|n\rangle$.
Starting with the left hand side:

$$\begin{aligned}
\hat{a}^\dagger|\bar{z}\rangle &= \hat{a}^\dagger \sum_{n=0}^{\infty} |n\rangle\langle n|\bar{z}\rangle \\
&= \sum_{n=0}^{\infty} \langle n|\bar{z}\rangle \sqrt{n+1}|n+1\rangle \\
&= \sum_{n=1}^{\infty} \langle n-1|\bar{z}\rangle \sqrt{n}|n\rangle
\end{aligned}$$

Then the right:

$$\bar{z}|\bar{z}\rangle = \bar{z}\sum_{n=0}^{\infty}\langle n|\bar{z}\rangle|n\rangle$$

Equating:

$$\sum_{n=1}^{\infty} \langle n-1|\bar{z}\rangle \sqrt{n}|n\rangle = \bar{z}\sum_{n=0}^{\infty}\langle n|\bar{z}\rangle|n\rangle$$

This gives for all n:

$$\langle n|\bar{z}\rangle = \langle n-1|\bar{z}\rangle\frac{\sqrt{n}}{\bar{z}}$$

and

$$\langle 0|\bar{z}\rangle = 0$$

as there is no corresponding term. This means that $\langle 1|\bar{z}\rangle = \langle 0|\bar{z}\rangle\frac{\sqrt{1}}{\bar{z}} = 0$, and thus all coefficients are 0 by induction. As the sum of coefficients that are exactly 0 is still 0, we cannot satisfy $\sum_{n=0}^{\infty}|\langle n|\bar{z}\rangle|^2 = 1$, and therfore not normalize it.

3.4 A driven harmonic oscillator (Exam 2010)

$$\hat{H} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) + \hbar\lambda \left(\hat{a}^\dagger e^{-i\omega t} + \hat{a} e^{i\omega t}\right), \qquad \left[\hat{a}, \hat{a}^\dagger\right] = 1$$

$$\hat{x} = \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right), \qquad \hat{p} = -\frac{i}{2}\left(\hat{a} - \hat{a}^\dagger\right)$$

a) We're given the Heisenberg equation of motion, which can be used on operators in order to get a differential equation that determines the time evolution of the operator

$$\frac{d}{dt}\hat{A} = \frac{i}{\hbar}\left[\hat{H}, \hat{A}\right] + \frac{\partial \hat{A}}{\partial t}$$

Note that $\frac{\partial \hat{A}}{\partial t}$ is the explicit time derivative of the operator in the Schrödinger picture. The ladder operators have no explicit time dependende in the Schrödinger picture, and thus $\dfrac{\partial \hat{a}}{\partial t} = 0$.

Using

$$\left[\hat{H}, \hat{a}\right] = -\hbar\omega_0 \hat{a} - \hbar\lambda e^{-i\omega t} \Rightarrow \frac{d^2 \hat{a}}{dt^2} = -i\frac{d}{dt}\left(\omega_0 \hat{a} + \lambda e^{-i\omega t}\right)$$

we have that

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{a}\right] = -i\left(\omega_0 \hat{a} + \lambda e^{-i\omega t}\right)$$

Taking the derivative we get

$$\begin{aligned} \frac{d^2 \hat{a}}{dt^2} &= -i\omega_0 \frac{d\hat{a}}{dt} - \lambda\omega e^{-i\omega t} \\ &= -\omega_0^2 \hat{a} - \left(\omega_0 + \omega\right)\lambda e^{-i\omega t} \end{aligned}$$

Conjugating we get

$$\frac{d^2 \hat{a}^\dagger}{dt^2} = -\omega_0^2 \hat{a}^\dagger - \left(\omega_0 + \omega\right)\lambda e^{i\omega t}.$$

Then,

$$\begin{aligned} \frac{d^2 \hat{x}}{dt^2} &= \frac{1}{2}\left(\frac{d^2 \hat{a}}{dt^2} + \frac{d^2 \hat{a}^\dagger}{dt^2}\right) = \frac{1}{2}\left(-\omega_0^2 \hat{a} - \left(\omega_0 + \omega\right)\lambda e^{-i\omega t} - \omega_0^2 \hat{a}^\dagger - \left(\omega_0 + \omega\right)\lambda e^{i\omega t}\right) \\ &= \frac{1}{2}\left(-\omega_0^2\left(\hat{a} + \hat{a}^\dagger\right) - \lambda\left(\omega_0 + \omega\right)\left(e^{i\omega t} + e^{-i\omega t}\right)\right) \\ &= -\omega_0^2 \hat{x} - \lambda\left(\omega_0 + \omega\right)\frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \\ &= -\omega_0^2 \hat{x} - \lambda\left(\omega_0 + \omega\right)\cos\omega t \end{aligned}$$

This is indeed on the form:

$$\frac{d^2 \hat{x}}{dt^2} + \omega_0^2 \hat{x} = C\cos\omega t, \quad C = -\lambda\left(\omega_0 + \omega\right)$$

b) Using that $\hat{H}_T|\psi_T(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi_T(t)\rangle$, we get

$$\begin{aligned}
\hat{H}_T|\psi_T(t)\rangle &= i\hbar\frac{\partial}{\partial t}\hat{T}(t)|\psi(t)\rangle \\
&= i\hbar\frac{\partial\hat{T}}{\partial t}|\psi(t)\rangle + i\hbar\hat{T}(t)\frac{\partial}{\partial t}|\psi(t)\rangle \\
&= i\hbar\frac{\partial\hat{T}}{\partial t}|\psi(t)\rangle + \hat{T}(t)\hat{H}(t)|\psi(t)\rangle \\
\hat{H}_T|\psi_T(t)\rangle &= i\hbar\frac{\partial\hat{T}}{\partial t}\hat{T}(t)^\dagger|\psi_T(t)\rangle + \hat{T}(t)\hat{H}(t)\hat{T}(t)^\dagger|\psi_T(t)\rangle \\
\Rightarrow \hat{H}_T &= \hat{T}(t)\hat{H}(t)\hat{T}(t)^\dagger + i\hbar\frac{\partial\hat{T}}{\partial t}\hat{T}(t)^\dagger
\end{aligned}$$

Then we use that: $\hat{T}(t) = e^{i\omega t\hat{a}^\dagger\hat{a}} \Rightarrow \frac{\partial\hat{T}}{\partial t} = i\omega\hat{a}^\dagger\hat{a}\hat{T}(t)$ and $\hat{T}(t)^\dagger = e^{-i\omega\hat{a}^\dagger\hat{a}}$, the last one is due to $\hat{a}^\dagger\hat{a} = \hat{N}$ which is hermitian. Then

$$i\hbar\frac{\partial\hat{T}}{\partial t}\hat{T}(t)^\dagger = i\hbar i\omega\hat{a}^\dagger\hat{a}\hat{T}(t)\hat{T}(t)^\dagger = -\hbar\omega\hat{a}^\dagger\hat{a}$$

We define the two terms in the Hamiltonian as

$$\hat{H} = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\lambda\left(\hat{a}^\dagger e^{-i\omega t} + \hat{a}e^{i\omega t}\right) \equiv \hat{H}_0 + \hat{H}_1$$

We see that $\left[\hat{H}_0, \hat{T}(t)\right] = 0$ which means that

$$\hat{T}(t)\hat{H}_0\hat{T}(t)^\dagger = \hat{H}_0$$

We have that

$$e^{i\omega t\hat{a}^\dagger\hat{a}}\hat{H}_1 e^{-i\omega\hat{a}^\dagger\hat{a}} = \hbar\lambda\left(e^{i\omega t\hat{a}^\dagger\hat{a}}\hat{a}^\dagger e^{-i\omega\hat{a}^\dagger\hat{a}}e^{-i\omega t} + e^{i\omega t\hat{a}^\dagger\hat{a}}\hat{a}e^{-i\omega\hat{a}^\dagger\hat{a}}e^{i\omega t}\right)$$

Applying $e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} = \hat{B} + \lambda\left[\hat{A}, \hat{B}\right] + \frac{\lambda^2}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \cdots$, with the commutators:

$$\begin{aligned}
\left[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger\right] &= \left[\hat{a}^\dagger, \hat{a}^\dagger\right]\hat{a} + \hat{a}^\dagger\left[\hat{a}, \hat{a}^\dagger\right] \\
&= \hat{a}^\dagger \\
\Rightarrow \left[\hat{a}^\dagger\hat{a}, \left[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger\right]\right] &= \left[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger\right] \\
&= \hat{a}^\dagger \\
\left[\hat{a}^\dagger\hat{a}, \hat{a}\right] &= \left[\hat{a}^\dagger, \hat{a}\right]\hat{a} + \hat{a}^\dagger\left[\hat{a}, \hat{a}\right] \\
&= -\hat{a} \\
\Rightarrow \left[\hat{a}^\dagger\hat{a}, \left[\hat{a}^\dagger\hat{a}, \hat{a}\right]\right] &= \hat{a}
\end{aligned}$$

we get

$$\begin{aligned}
e^{i\omega t\hat{a}^\dagger\hat{a}}\hat{H}_1 e^{-i\omega\hat{a}^\dagger\hat{a}} &= \hbar\lambda\left(\hat{a}^\dagger\left(\sum_{n=0}^{\infty}\frac{(i\omega t)^n}{n!}\right)e^{-i\omega t} + \hat{a}\left(\sum_{n=0}^{\infty}\frac{(-i\omega t)^n}{n!}\right)e^{i\omega t}\right) \\
&= \hbar\lambda\left(\hat{a}^\dagger e^{i\omega t}e^{-i\omega t} + \hat{a}e^{-i\omega t}e^{i\omega t}\right) \\
&= \hbar\lambda\left(\hat{a}^\dagger + \hat{a}\right).
\end{aligned}$$

Then we find

$$\begin{aligned}\hat{T}(t)\hat{H}(t)\hat{T}(t)^\dagger &= \hat{H}_0 + \hbar\lambda\left(\hat{a}^\dagger + \hat{a}\right) \\ &= \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\lambda\left(\hat{a}^\dagger + \hat{a}\right)\end{aligned}$$

The full Hamiltonian is:

$$\begin{aligned}\hat{H}_T &= \hat{T}(t)\hat{H}(t)\hat{T}(t)^\dagger + i\hbar\frac{\partial\hat{T}}{\partial t}\hat{T}(t)^\dagger \\ &= \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\lambda\left(\hat{a}^\dagger + \hat{a}\right) - \hbar\omega\hat{a}^\dagger\hat{a} \\ &= \hbar\left(\omega_0 - \omega\right)\hat{a}^\dagger\hat{a} + \hbar\lambda\left(\hat{a}^\dagger + \hat{a}\right) + \frac{1}{2}\hbar\omega_0\end{aligned}$$

c) If $|\psi(t)\rangle$ evolves as a coherent state, it satisifies: $\hat{a}|\psi(t)\rangle = \psi(t)|\psi(t)\rangle$. Let's note that the hamiltonian in the Schrödinger picture has an explicit time dependence, and in order to get the time evolution operator, we would want it to be time-independent. We know that $\hat{H}_T$ is time independent and generates the time dependence through $\hat{\mathcal{U}}_T(t) = e^{-it\hat{H}_T/\hbar}$. Thus, in order to get the time depence in the original picture, we need to write $\hat{\mathcal{U}}(t)$ in terms of $\hat{\mathcal{U}}_T(t)$:

$$\begin{aligned}|\psi_T(t)\rangle &= \hat{T}(t)|\psi(t)\rangle = \hat{T}(t)\hat{\mathcal{U}}(t)|\psi(0)\rangle \\ |\psi_T(t)\rangle &= \hat{\mathcal{U}}_T(t)|\psi_T(0)\rangle \\ \Rightarrow \hat{\mathcal{U}}_T(t) &= \hat{T}(t)\hat{\mathcal{U}}(t) \Rightarrow \hat{\mathcal{U}}(t) = \hat{T}(t)^\dagger\hat{\mathcal{U}}_T(t)\end{aligned}$$

If $|\psi(t)\rangle$ evolves as a coherent state, then it is an eigenstate of $\hat{a}$:

$$\hat{a}|\psi(t)\rangle = \hat{a}\hat{\mathcal{U}}(t)|0\rangle = \hat{\mathcal{U}}(t)\hat{\mathcal{U}}(t)^\dagger\hat{a}\hat{\mathcal{U}}(t)|0\rangle = \hat{\mathcal{U}}(t)\hat{\mathcal{U}}_T(t)^\dagger\hat{T}(t)\hat{a}\hat{T}(t)^\dagger\hat{\mathcal{U}}_T(t)|0\rangle$$

From exercise b), we got that $\hat{T}(t)\hat{a}\hat{T}(t)^\dagger = \hat{a}e^{-i\omega t}$, which leaves us with:

$$\hat{a}|\psi(t)\rangle = e^{-i\omega t}\hat{\mathcal{U}}(t)\hat{\mathcal{U}}_T(t)^\dagger\hat{a}\hat{\mathcal{U}}_T(t)|0\rangle = e^{-i\omega t}\hat{\mathcal{U}}(t)e^{it\hat{H}_T/\hbar}\hat{a}e^{-it\hat{H}_T/\hbar}|0\rangle \tag{2}$$

Expanding $e^{it\hat{H}_T/\hbar}\hat{a}e^{-it\hat{H}_T/\hbar}$ yields:

$$e^{it\hat{H}_T/\hbar}\hat{a}e^{-it\hat{H}_T/\hbar} = \hat{a} + \frac{it}{\hbar}\left[\hat{H}_T, \hat{a}\right] + \left(\frac{it}{\hbar}\right)^2\left[\hat{H}_T, \left[\hat{H}_T, \hat{a}\right]\right] + \cdots$$

The commutators are:

$$\begin{aligned}\left[\hat{H}_T, \hat{a}\right] &= \left[\hbar\left(\omega_0 - \omega\right)\hat{a}^\dagger\hat{a} + \hbar\lambda\left(\hat{a}^\dagger + \hat{a}\right) + \frac{1}{2}\hbar\omega_0, \hat{a}\right] \\ &= \left[\hbar\left(\omega_0 - \omega\right)\hat{a}^\dagger\hat{a}, \hat{a}\right] + \left[\hbar\lambda\left(\hat{a}^\dagger + \hat{a}\right), \hat{a}\right] + \underbrace{\left[\frac{1}{2}\hbar\omega_0, \hat{a}\right]}_{=0} \\ &= \hbar\left(\omega_0 - \omega\right)\left[\hat{a}^\dagger\hat{a}, \hat{a}\right] + \hbar\lambda\left(\left[\hat{a}^\dagger, \hat{a}\right] + \left[\hat{a}, \hat{a}\right]\right) \\ &= -\hbar\left(\omega_0 - \omega\right)\hat{a} - \hbar\lambda \\ &= \hbar\left(\omega - \omega_0\right)\hat{a} - \hbar\lambda\end{aligned}$$

$$\begin{aligned}
\left[\hat{H}_T,\left[\hat{H}_T,\hat{a}\right]\right] &= \left[\hat{H}_T,\hbar\left(\omega-\omega_0\right)\hat{a}-\hbar\lambda\right] \\
&= \hbar\left(\omega_0-\omega\right)\left[\hat{H}_T,\hat{a}\right] \\
\left[\hat{H}_T,\left[\hat{H}_T,\left[\hat{H}_T,\hat{a}\right]\right]\right] &= \hbar\left(\omega-\omega_0\right)\left[\hat{H}_T,\left[\hat{H}_T,\hat{a}\right]\right] \\
&= \left[\hbar\left(\omega-\omega_0\right)\right]^2\left[\hat{H}_T,\hat{a}\right]
\end{aligned}$$

We see a pattern emerging:

$$\begin{aligned}
e^{it\hat{H}_T/\hbar}\hat{a}e^{-it\hat{H}_T/\hbar} &= \hat{a}+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{it}{\hbar}\right)^n\left[\hbar\left(\omega-\omega_0\right)\right]^{n-1}\left[\hat{H}_T,\hat{a}\right] \\
&= \hat{a}+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{it}{\hbar}\right)^n\left[\hbar\left(\omega-\omega_0\right)\right]^{n-1}\left(\hbar\left(\omega-\omega_0\right)\hat{a}-\hbar\lambda\right) \\
&= \hat{a}+\hat{a}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{it}{\hbar}\right)^n\left[\hbar\left(\omega-\omega_0\right)\right]^{n}-\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{it}{\hbar}\right)^n\left[\hbar\left(\omega-\omega_0\right)\right]^{n-1}\hbar\lambda \\
&= \hat{a}\sum_{n=0}^{\infty}\frac{1}{n!}\left[it\left(\omega-\omega_0\right)\right]^n-\sum_{n=1}^{\infty}\frac{1}{n!}\left(it\right)^n\left[\left(\omega-\omega_0\right)\right]^{n-1}\lambda \\
&= \hat{a}\sum_{n=0}^{\infty}\frac{1}{n!}\left[it\left(\omega-\omega_0\right)\right]^n-\frac{\lambda}{\left(\omega-\omega_0\right)}\sum_{n=1}^{\infty}\frac{1}{n!}\left[it\left(\omega-\omega_0\right)\right]^n \\
&= \hat{a}e^{it(\omega-\omega_0)}-\frac{\lambda}{\left(\omega-\omega_0\right)}\left(e^{it(\omega-\omega_0)}-1\right)
\end{aligned}$$

Inserting back into (2) gives:

$$\hat{a}|\psi(t)\rangle = e^{-i\omega t}\hat{\mathcal{U}}(t)\left(\hat{a}e^{it(\omega-\omega_0)}-\frac{\lambda}{\left(\omega-\omega_0\right)}\left(e^{it(\omega-\omega_0)}-1\right)\right)|0\rangle$$

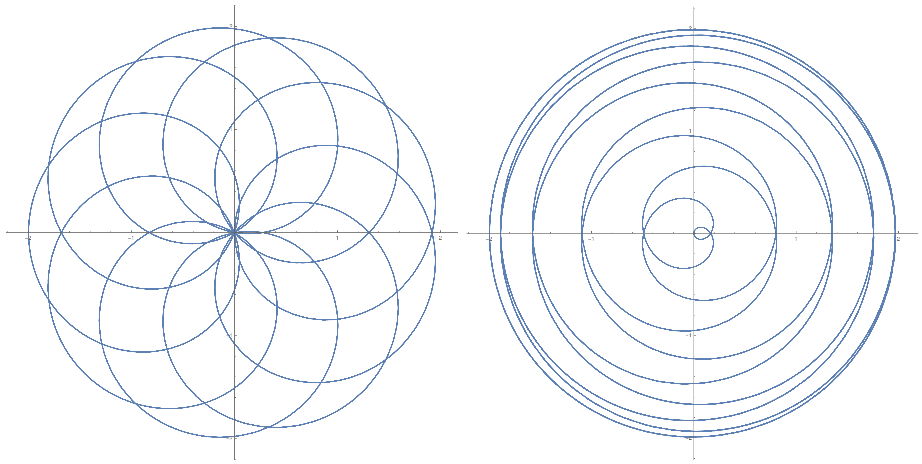
We note that $\hat{a}|0\rangle = 0$:

$$\begin{aligned}
\hat{a}|\psi(t)\rangle &= e^{-i\omega t}\hat{\mathcal{U}}(t)\frac{\lambda}{\left(\omega-\omega_0\right)}\left(1-e^{it(\omega-\omega_0)}\right)|0\rangle \\
&= e^{-i\omega t}\frac{\lambda}{\left(\omega-\omega_0\right)}\left(1-e^{it(\omega-\omega_0)}\right)\hat{\mathcal{U}}(t)|0\rangle \\
&= e^{-i\omega t}\frac{\lambda}{\left(\omega-\omega_0\right)}\left(1-e^{it(\omega-\omega_0)}\right)|\psi(t)\rangle \\
&= \frac{\lambda}{\left(\omega-\omega_0\right)}\left(e^{-i\omega t}-e^{-it\omega_0}\right)|\psi(t)\rangle \\
&\equiv z(t)|z(t)\rangle
\end{aligned}$$

We see that the number $z(t)$ is the difference of two complex numbers which have the same constant absolute value while rotating with different frequencies. It will be 0 when the two are in phase and maximal when they are out of phase. To get some better understanding we can rewrite

$$e^{-i\omega t}-e^{-it\omega_0} = 2i\sin\frac{\omega_0-\omega}{2}te^{i\frac{\omega_0+\omega}{2}t}$$

This means that $z(t)$ will follow a spiral path with the phase increasing with a freqency equal to $\frac{\omega_0+\omega}{2}$ while the amplitude changes with the frequency $\frac{\omega_0-\omega}{2}$. Here are two examples (Left: $\omega_0 = 1$, $\omega = 0.1$, Right: $\omega_0 = 1$, $\omega = 0.9$)



The real part is given as:

$$x(t) = \frac{1}{2}\left(z(t) + z(t)^*\right) = \frac{1}{2}\frac{\lambda}{(\omega-\omega_0)}\left(e^{-i\omega t} - e^{-it\omega_0} + e^{i\omega t} - e^{it\omega_0}\right) = \frac{\lambda}{(\omega-\omega_0)}\left(\cos\omega t - \cos\omega_0 t\right)$$

Finding the equation of motion:

$$\frac{d^2x}{dt^2} = \frac{\lambda}{(\omega-\omega_0)}\left(-\omega^2\cos\omega t + \omega_0^2\cos\omega_0 t\right)$$

We want this to be equal to $C\cos\omega t - \omega_0^2\frac{\lambda}{(\omega-\omega_0)}\left(\cos\omega t - \cos\omega_0 t\right)$. The trick is then to add in 0:

$$\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{\lambda}{(\omega-\omega_0)}\left(-\omega^2\cos\omega t + \omega_0^2\left(\cos\omega_0 t + \cos\omega t - \cos\omega t\right)\right) \\
&= \frac{\lambda}{(\omega-\omega_0)}\left(\left(-\omega^2 + \omega_0^2\right)\cos\omega t + \omega_0^2\left(\cos\omega_0 t - \cos\omega t\right)\right) \\
&= \frac{\lambda}{(\omega-\omega_0)}\left(-\omega^2 + \omega_0^2\right)\cos\omega t - \omega_0^2 x = -\lambda\left(\omega+\omega_0\right)\cos\omega t - \omega_0^2 x
\end{aligned}$$

Thus, we see it satisfies the equation with $C = -\lambda\left(\omega + \omega_0\right)$.