Problem set 5

5.1 Pure and mixed states

a) Explain what is the difference between pure and mixed quantum states. How are they represented mathematically?

b) An ensemble of spin-$\frac{1}{2}$ particles are produced by some (to you) unknown procedure. You are informed that the particles will be either (ensemble A) in the state $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ or (ensemble B) in a random statistical mixture with 50% of the particles in the state $|\uparrow\rangle$ and 50% of the particles in the state $|\downarrow\rangle$. You are allowed to measure the spin of each particle along an axis of your choice (you do not have to choose the same axis for each particle). Describe an experiment which would reveal whether the particles are prepared in ensemble A or ensemble B. Explain what will be the probabilities of different measurement oucomes for both ensembles when using your measurement procedure.

c) Consider now a third enemble (ensemble C), where the particles are prepared in a random statistical mixture with 50% of the particles in the state $|\rightarrow\rangle$ and 50% of the particles in the state $|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$. Prove that we can not distiguish ensembles B and C by any measurements on the particles.

Instead of direct preparation as described above, we can prepare the ensembles B or C remotely by entanglement in the following way. Person 1 (the preparer) prepares an ensemble of pairs of entangled particles in the state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. He keeps one particle from each pair to himself and sends the other particle from each pair to person 2 (you). By doing appropriate measurements on his particles, person 1 can now decide at a later point if he would like your particles to belong to ensemble B or C.

d) Which measurement should person 1 perform to generate ensemble B and which to generate ensemble C? Justify your answer.

e) Even if the ensembles B and C are indistiguishable by local measurements by person 2, as you showed in question c), they can be distiguished by the correlations between the measurement outcomes of persons 1 and 2. Explain which measurements person 2 should do, and how the difference between ensembles B and C are visible in the correlations. Assume that the pairs are labeled, so that we can compare the measurement oucomes for the two particles belonging to the same pair. What changes if person 1 decides to wait with his measurements until after person 2 makes the measurements, so that the two ensembles are not prepared until after they are measured.

5.2 Entanglement

Two persons A and B communicate with the help of quantum entanglement. They share a set of pairs of particles with spins in the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle) \tag{1}$$

where $|++\rangle = |+\rangle \otimes |+\rangle$ is a state where both particles of the pair have *spin up* in the z-direction, and similarly $|--\rangle = |-\rangle \otimes |-\rangle$ is the state where both particles have *spin down* in the z-direction.

a) What is the quantity used as measure for the degree of entanglement in such a two-partite system, and what is the degree of entanglement in the given spin state?

b) Assume A and B perform independent spin operations on their particles in a given pair, each operation described by a unitary operator, $\hat{U}_A$ or $\hat{U}_B$. What happens to the entanglement of the two-particle system under such an operation.

c) Assume A performs an ideal measurement of the spin component in the x- direction, which projects the spin to an eigenstate of the x-component of the spin operator. What happens to the entanglement in this case?

5.3 Matrix representation of tensor products

Assume $|a\rangle = \sum_{i=1}^{2} a_i \, |i\rangle_A$ is a vector in an 2-dimensional Hilbert space $\mathcal{H}_A$ and $|b\rangle = \sum_{j=1}^{2} b_j \, |j\rangle_B$ is a vector in another 2-dimensional Hilbert space $\mathcal{H}_B$, with $\{|i\rangle_A\}$ as an orhonormal basis set in $\mathcal{H}_A$ and $\{|j\rangle_B\}$ as a similar vector set in $\mathcal{H}_B$. The composite vector $|c\rangle = |a\rangle \otimes |b\rangle$ is a product vector in the tensor product space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Expanded in the product basis it has the form $|c\rangle = \sum_{ij} a_i b_j \, |ij\rangle$ with $|ij\rangle = |i\rangle_A \otimes |j\rangle_B$.

We consider the matrix representation of the vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{2}$$

The vector $|c\rangle$ can be representet as a single column matrix of dimension 4. We define the matrix elements $c_k, k = 1, ..., 4$, of such a matrix by the following relation

$$c_{j+2(i-1)} = a_i b_j \tag{3}$$

a) Express the column matrix $\mathbf{c}$ (4×1 matrix) in terms of the matrix elements of $\mathbf{a}$ and $\mathbf{b}$, and show that it can be written in a compact form as

$$\mathbf{c} = \begin{pmatrix} a_1 \mathbf{b} \\ a_2 \mathbf{b} \end{pmatrix} \tag{4}$$

What are the vector representations for the four basis vectors $|ij\rangle$?

We consider next operators $\hat{A}$, $\hat{B}$ and $\hat{C} = \hat{A} \otimes \hat{B}$, which act in $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}$ respectively. The corresponding 2×2 matrix $\mathbf{A}$ represents $\hat{A}$ in the basis $\{|i\rangle_A\}$ and the 2×2 matrix $\mathbf{B}$ represents $\hat{B}$ in the basis $\{|j\rangle_B\}$. The tensor product of the operators can be represented as a 4×4 matrix $\mathbf{C}$, with elements

$$C_{j+2(i-1), j'+2(i'-1)} = A_{ii'} B_{jj'} \tag{5}$$

similar to the column matrix c_i, defined in (3).

b) Show that the matrix $\mathbf{C}$ can be written in a form similar to (4). in terms of the components of $\mathbf{A}$ and the matrix $\mathbf{B}$.

c) Find the 4×4 matrix representations of the tensor products $\sigma_k \otimes \sigma_l$, where $\sigma_k, k = 1, 2, 3$, are the Pauli matrices. Write them in a form similar to (4). It is sufficient to do this for three different choices of the Pauli matrices.

d) Show that the matrix representations are consistent with normal matrix multiplication in the sense that the matrix product $\mathbf{Cc}$ gives the vector representing the state $\hat{A} \otimes \hat{B}|a\rangle \otimes |b\rangle$.

5.4 Schmidt decomposition 1

We have a system consisting of two spin-$\frac{1}{2}$ particles. For each of the following states, study the reduced density matrix of of one of the particles and determine if the state is entangled or not. For the states which are not entangled, find a factorization of the state as a tensor product of one state for each particle. For the entagled states, find the Schmidt decomposition of the state.

$$
\begin{aligned}
|\psi_1\rangle &= \frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\
|\psi_2\rangle &= \frac{1}{2}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\
|\psi_3\rangle &= a_+|\uparrow\uparrow\rangle + a_-|\uparrow\downarrow\rangle + a_-|\downarrow\uparrow\rangle + a_+|\downarrow\downarrow\rangle \\
|\psi_4\rangle &= a_-|\uparrow\uparrow\rangle + a_+|\uparrow\downarrow\rangle + a_+|\downarrow\uparrow\rangle + a_-|\downarrow\downarrow\rangle
\end{aligned}
$$

where

$$a_\pm = \frac{\sqrt{3} \pm 1}{4}$$

5.5 Schmidt decomposition 2

Entanglement can occur not only between distinct particles, but also between different observables fot the same particle, like position and spin. Here we will find the Schmidt decomposition of one continuos and one discrete Hilbert space. A spin-half particle moving in one dimension is described by a two-component wave function

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{6}$$

where the upper matrix position is assumed to correspond to ”spin up” in the z-direction and the lower matrix position to ”spin down” in the same direction. The scalar product of the two wave functions will generally be different from zero, and we write it as

$$\langle\psi_1|\psi_2\rangle = \int dx\, \psi_1^*(x)\psi_2(x) \equiv \Delta \tag{7}$$

a) The Schmidt decomposition of the two-component wave function has the form

$$\Psi(x) = c_1\chi_1\,\phi_1(x) + c_2\chi_2\,\phi_2(x) \tag{8}$$

where c_1 and c_2 are expansion coefficients, χ_1 and χ_2 are normalized, two-component spinors, and $\phi_1(x)$ and $\phi_2(x)$ are normalized, scalar (one-component) wave functions. What are the conditions that the spinors and wave functions should satisfy?

b) Assume the two wave functions of (6) are real Gaussian functions of the form

$$\psi_1(x) = Ne^{-\lambda(x-x_0)^2}\,, \quad \psi_2(x) = Ne^{-\lambda(x+x_0)^2} \tag{9}$$

Determine the normalization factor N and the overlap Δ, expressed in terms of λ and x_0.

c) Determine the coefficients, spinors and wave functions in (8). (Since the wave function $\Psi(x)$ is real, you may assume the variables in Eq.(8) all to be real.)

5.6 Coupled two-level systems

Two coupled two-level systems A and B are described by the following Hamiltonian

$$\hat{H} = \frac{\epsilon}{2}(3\,\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) + \lambda(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \tag{10}$$

where the first factor in the tensor product refers to system A and the second factor to system B. In the equation we use the definition $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

a) Write the Hamiltonian as a 4x4 matrix and show that two of the eigenvalues and eigenvectors are independent of λ. Introduce new variables, defined by $\epsilon = \mu\cos\theta$ and $\lambda = \mu\sin\theta$. Solve the eigenvalue problem for the remaining two-dimensional subspace and determine both the energies and eigenvectors as functions of μ and θ.

b) Express the two eigenstates as 4x4 density matrices and determine the reduced density matrices for the two subsystems A and B.

c) Determine the entropy of the reduced density matrices as functions of θ. For what parameter value is the entanglement of the two subsystems maximal?

5.7 Entanglement in the Jaynes-Cummings model

We have in the lectures discussed Rabi oscillations of a Two Level System (TLS) driven by an external oscillating field. In this case the field is treated as a classical quantity with a given time dependence which is not affected by the dynamics of the TLS. We have also studied the Jaynes-Cummings model which is an extension of the Rabi problem to a quantized field (in a cavity, so that emitted photons are not lost, but return and can be reabsorbed). The two models gave to some extent similar results (see the lecture notes Sec. 1.4 for definitions of all symbols).

For the Rabi problem we get that if the TLS starts in the ground state, the time evolution (in the rotating frame) is

$$|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$$

with

$$c_0(t) = \cos\frac{\Omega t}{2} + i\sin\frac{\Omega t}{2}\cos\theta, \qquad c_1(t) = i\sin\frac{\Omega t}{2}\sin\theta.$$

In the Jaynes-Cummings model, starting from the state $|-, n+1\rangle$ where the TLS is the ground state and that there are $n+1$ photons in the cavity, we have

$$|\psi(t)\rangle = c_n^-(t)|-, n+1\rangle + c_n^+(t)|+, n\rangle$$

with (up to a global phase)

$$c_n^-(t) = \cos\frac{\Omega_n t}{2} + i\sin\frac{\Omega_n t}{2}\cos\theta_n, \qquad c_n^+(t) = i\sin\frac{\Omega_n t}{2}\sin\theta_n.$$

a) If we study the situation in more detail, we will see that there are differences between the two models. For the Jaynes-Cummings model, assume that the initial state of the TLS is the ground state and that there are $n+1$ photons in the cavity. Find the reduced density matrix of the TLS as a function of time. Find the entanglement entropy as a function of time. What is the maximal entanglement for given parameters and when is the state maximally entangled?

b) Find the Bloch vector for the state as a function of time both for the Rabi problem and the Jaynes-Cummings model. Draw the motion of the Bloch vector in the Bloch sphere and compare the two. Describe the differences between the two models.

c) We usually think that quantum physics should approach classical in the limit where the energy of the system is much larger than the level spacing, which in this case means in the limit $n \to \infty$ where the number of photons is large. Consider you results in this limit, and discuss to what extent we have a reasonable classical limit in this case. Do you have any ideas for what could be changed to make the behaviour more classical-like in a certain limit?