# Solutions to problem set 5

### 5.1 Pure and mixed states

a) A pure state is the most accurate description possible of a quantum system. It is represented by a state vector $|\psi\rangle$ in Hilbert space. A mixed state is used when we do not know the exact quantum state, but only the probabilities $p_1$ for a set of possible states $|\psi_i\rangle$. It is represented by a density matrix $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$. Mixed states also occur for composite systems in pure states. The reduced density matrix of one component is then a mixed state when there is entanglement between this component and the rest of the system.

b) We measure the spin in the $x$-direction. $|\rightarrow\rangle$ is an eigenstate of $\sigma_x$ with eigenvalue $+1$, which means that we will measure spin up in $x$ for all particles in ensamble A. For ensamble B we will measure spin up and spin down randomly with equal probabilities.

c) We prove that the density matrices are the same:

$$\rho_B = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow|$$

$$\begin{aligned}
\rho_C &= \frac{1}{2}|\rightarrow\rangle\langle\rightarrow| + \frac{1}{2}|\leftarrow\rangle\langle\leftarrow| \\
&= \frac{1}{4}\left(|\uparrow\rangle + |\downarrow\rangle\right)\left(\langle\uparrow| + \langle\downarrow|\right) + \frac{1}{4}\left(|\uparrow\rangle - |\downarrow\rangle\right)\left(\langle\uparrow| - \langle\downarrow|\right) \\
&= \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow|
\end{aligned}$$

Since the density matrices are the same will we get the same statistics for al possible measurements, and we can distinguish the ensembles.

d) The state is $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$. It is clear that if we measure the first particle along the $z$-axis we have equal probabilities of measuring up or down, and the second particle will collapse to the opposite state, generating ensemble B. Ensemble C is generated by measuring the first particle in the $x$-direction. to see this we rewrite $|\psi\rangle$ in terms of the states $|\rightarrow\rangle$ and $|\leftarrow\rangle$. We have

$$\begin{aligned}
|\uparrow\rangle &= \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle + |\leftarrow\rangle\right) \\
|\downarrow\rangle &= \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle - |\leftarrow\rangle\right)
\end{aligned}$$

which we use to get

$$\begin{aligned}
|\psi\rangle &= \frac{1}{2\sqrt{2}}\left(|\rightarrow\rangle + |\leftarrow\rangle\right)\left(|\rightarrow\rangle - |\leftarrow\rangle\right) - \frac{1}{2\sqrt{2}}\left(|\rightarrow\rangle - |\leftarrow\rangle\right)\left(|\rightarrow\rangle + |\leftarrow\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(|\leftarrow\rightarrow\rangle - |\rightarrow\leftarrow\rangle\right)
\end{aligned}$$

e) Consider the case where person 1 measures spin along the $z$-axis and therefore prepares ensemble B. If person 2 also measures along the $z$-axis, the outcomes of the two measurements will always be perfectly anticorrelated. If instead person 1 measures $x$-spin and prepares ensemble C while person 2 still measures $z$-spin, teh results will be uncorrelated. Nothing changes if person 1 measures after person 2.

## 5.2 Entanglement

a)
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|++\rangle + |--\rangle\right)$$

This stateis a pure state, and thus has the density matrix:

$$\begin{aligned}\hat{\rho} &= \frac{1}{2}\left(|++\rangle\langle++| + |--\rangle\langle--| + |++\rangle\langle--| + |--\rangle\langle++|\right)\\ &= \frac{1}{2}\sum_{n,m\in\{+,-\}} |nn\rangle\langle mm|\end{aligned}$$

The entropy is then given by:

$$S_{\mathcal{A}} = S_{\mathcal{B}} = \mathrm{Tr}_{\mathcal{A}}\left(\hat{\rho}_{\mathcal{A}}\log\hat{\rho}_{\mathcal{A}}\right)\ \left(= \mathrm{Tr}_{\mathcal{B}}\left(\hat{\rho}_{\mathcal{B}}\log\hat{\rho}_{\mathcal{B}}\right)\right)$$

where $\hat{\rho}_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}\left(\hat{\rho}\right)$. The trace of a matrix in the product space is:

$$\begin{aligned}\hat{\rho}_{\mathcal{A}} &= \mathrm{Tr}_{\mathcal{B}}\left(\frac{1}{2}\sum_{n,m\in\{+,-\}}|nn\rangle\langle mm|\right) = \frac{1}{2}\sum_{n,m\in\{+,-\}}\mathrm{Tr}_{\mathcal{B}}\left(|nn\rangle\langle mm|\right)\\ &= \frac{1}{2}\sum_{n,m\in\{+,-\}}\mathrm{Tr}_{\mathcal{B}}\left(\left(|n\rangle_{\mathcal{A}}\otimes|n\rangle_{\mathcal{B}}\right)\left(\langle m|_{\mathcal{A}}\otimes\langle m|_{\mathcal{B}}\right)\right)\\ &= \frac{1}{2}\sum_{n,m\in\{+,-\}}\mathrm{Tr}_{\mathcal{B}}\left(\left(|n\rangle\langle m|\right)_{\mathcal{A}}\otimes\left(|n\rangle\langle m|\right)_{\mathcal{B}}\right)\\ &= \frac{1}{2}\sum_{n,m\in\{+,-\}}\left(\left(|n\rangle\langle m|\right)_{\mathcal{A}}\otimes\mathrm{Tr}\left(|n\rangle\langle m|\right)_{\mathcal{B}}\right)\\ &= \frac{1}{2}\sum_{n,m\in\{+,-\}}\left(\left(|n\rangle\langle m|\right)_{\mathcal{A}}\otimes\delta_{mn}\right)\end{aligned}$$

Due to the trace only sums the diagonal elements ( $\mathrm{Tr}\left(|n\rangle\langle m|\right) = \langle m|n\rangle = \delta_{mn}$. Since $\delta_{mn}$ is a number, the tensor product reduces down to simple multiplication:

$$\hat{\rho}_{\mathcal{A}} = \frac{1}{2}\sum_{n,m\in\{+,-\}}\left(\delta_{mn}|n\rangle\langle m|\right)$$

Thus,

$$\hat{\rho}_{\mathcal{A}} = \frac{1}{2}\left(|+\rangle\langle+| + |-\rangle\langle-|\right)$$

This is a matrix with both eigenvalues $\frac{1}{2}$, thus we find the entropy:

$$S_{\mathcal{A}} = S_{\mathcal{B}} = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \log 2 \tag{1}$$

Thus, they are maximally entangeled.

b) The operation $\hat{U}_B = \mathbb{1} \otimes \hat{U}_B$, and $\hat{U}_A = \hat{U}_A \otimes \mathbb{1}$, thus, applying both yields:

$$\hat{U}_A\hat{U}_B = \left(\hat{U}_A \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes \hat{U}_B\right) = \hat{U}_A \otimes \hat{U}_B$$

Applying this as a transformation, we get:

$$|\psi\rangle \to |\psi'\rangle = \hat{U}_A \otimes \hat{U}_B|\psi\rangle$$

$$\hat{\rho} \to \hat{\rho}' = \left(\hat{U}_A \otimes \hat{U}_B\right)\hat{\rho}\left(\hat{U}_A \otimes \hat{U}_B\right)^\dagger$$

Then:

$$\begin{aligned}
\hat{\rho}'_A &= \mathrm{Tr}_B\left[\left(\hat{U}_A \otimes \hat{U}_B\right)\hat{\rho}\left(\hat{U}_A \otimes \hat{U}_B\right)^\dagger\right] \\
&= \mathrm{Tr}_B\left[\left(\hat{U}_A \otimes \hat{U}_B\right)\left(\frac{1}{2}\sum_{n,m\in\{+,-\}}\left(|n\rangle_A \otimes |n\rangle_B\right)\left(\langle m|_A \otimes \langle m|_B\right)\right)\left(\hat{U}_A \otimes \hat{U}_B\right)^\dagger\right] \\
&= \mathrm{Tr}_B\left[\frac{1}{2}\sum_{n,m\in\{+,-\}}\left(\left[\hat{U}_A|n\rangle_A\right] \otimes \left[\hat{U}_B|n\rangle_B\right]\right)\left(\left[\langle m|_A\hat{U}_A^\dagger\right] \otimes \left[\langle m|_B\hat{U}_B^\dagger\right]\right)\right] \\
&= \frac{1}{2}\mathrm{Tr}_B\left[\sum_{n,m\in\{+,-\}}\left(\left[\hat{U}_A|n\rangle_A\langle m|_A\hat{U}_A^\dagger\right] \otimes \left[\hat{U}_B|n\rangle_B\langle m|_B\hat{U}_B^\dagger\right]\right)\right] \\
&= \frac{1}{2}\left[\sum_{n,m\in\{+,-\}}\left(\left[\hat{U}_A|n\rangle_A\langle m|_A\hat{U}_A^\dagger\right] \otimes \mathrm{Tr}\left[\hat{U}_B|n\rangle_B\langle m|_B\hat{U}_B^\dagger\right]\right)\right]
\end{aligned}$$

From problem set 1, we showed that $\mathrm{Tr}\left(\hat{U}A\hat{U}^\dagger\right) = \mathrm{Tr}\left(A\right)$ by

$$\mathrm{Tr}\left(\hat{U}A\hat{U}^\dagger\right) = \mathrm{Tr}\left(\hat{U}\left[A\hat{U}^\dagger\right]\right) = \mathrm{Tr}\left(\left[A\hat{U}^\dagger\right]\hat{U}\right) = \mathrm{Tr}\left(A\right)$$

We arrive at

$$\hat{\rho}'_A = \frac{1}{2}\left[\sum_{n,m\in\{+,-\}}\hat{U}_A|n\rangle\langle m|\hat{U}_A^\dagger\delta_{mn}\right] = \left[\sum_{n\in\{+,-\}}\hat{U}_A|n\rangle\langle n|\hat{U}_A^\dagger\right] = \hat{U}_A\hat{\rho}_A\hat{U}_A^\dagger$$

The entropy is then given as:

$$S'_A = -\mathrm{Tr}\left(\hat{\rho}'_A\log\left(\hat{\rho}'_A\right)\right)$$

Since $\hat{\rho}'_A = \hat{U}_A\hat{\rho}_A\hat{U}_A^\dagger$, they have the same eigenvalues, and therefore the entropy is the same.

c) After the measurement, part A of the system is projected on one of the eigenstates of the operator being measured (it does not matter which operator this is). It is then in a well defined pure state and not entangled with part B any more. The entropy of entanglement after the measurement is 0.

## 5.3 Matrix representation of tensor products

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$|c\rangle = |a\rangle \otimes |b\rangle \Rightarrow |c\rangle = \sum_{ij} a_i b_j |ij\rangle, \quad |ij\rangle = |i\rangle_A \otimes |j\rangle_B$$

a) We have

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix} = \begin{pmatrix} a_1 \mathbf{b} \\ a_2 \mathbf{b} \end{pmatrix} \tag{2}$$

For the basis vectors, we can assume

$$|1\rangle_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

And similarily in the $B$ space. We can use the result (2), to have:

$$|ij\rangle = |i\rangle_A \otimes |j\rangle_B = \begin{pmatrix} i_1 \mathbf{j} \\ i_2 \mathbf{j} \end{pmatrix}$$

Then:

$$|11\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |12\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |21\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |22\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

b) We have

$$\mathbf{C} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}\mathbf{B} & A_{12}\mathbf{B} \\ A_{21}\mathbf{B} & A_{22}\mathbf{B} \end{pmatrix}$$

c)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We show three examples

$$\sigma_1 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

$$\sigma_1 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$\sigma_2 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}$$

d) We have

$$\mathbf{Cc} = \begin{pmatrix} A_{11}\mathbf{B} & A_{12}\mathbf{B} \\ A_{21}\mathbf{B} & A_{22}\mathbf{B} \end{pmatrix} \begin{pmatrix} a_1\mathbf{b} \\ a_2\mathbf{b} \end{pmatrix} = \begin{pmatrix} A_{11}a_1\mathbf{Bb} + A_{12}a_2\mathbf{Bb} \\ A_{21}a_1\mathbf{Bb} + A_{22}a_2\mathbf{Bb} \end{pmatrix}$$

We use that $\hat{A} \otimes \hat{B}|a\rangle \otimes |b\rangle = \hat{A}|a\rangle \otimes \hat{B}|b\rangle$ and that the matrix representing $\hat{A}|a\rangle$ is

$$\mathbf{Aa} = \begin{pmatrix} A_{11}a_1 + A_{12}a_2 \\ A_{21}a_1 + A_{22}a_2 \end{pmatrix}$$

Then the matrix representing $\hat{A} \otimes \hat{B}|a\rangle \otimes |b\rangle$ is

$$\begin{pmatrix} (A_{11}a_1 + A_{12}a_2)\mathbf{Bb} \\ (A_{21}a_1 + A_{22}a_2)\mathbf{Bb} \end{pmatrix}$$

which is the same as $\mathbf{Cc}$

## 5.4 Schmidt decomposition 1

We have a system consisting of two spin-$\frac{1}{2}$ particles. For each of the following states, study the reduced density matrix of of one of the particles and determine if the state is entangled or not. For the states which are not entangled, find a factorization of the state as a tensor product of one state for each particle. For the entagled states, find the Schmidt decomposition of the state.

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\ |\psi_2\rangle &= \frac{1}{2}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\ |\psi_3\rangle &= a_+|\uparrow\uparrow\rangle + a_-|\uparrow\downarrow\rangle + a_-|\downarrow\uparrow\rangle + a_+|\downarrow\downarrow\rangle \\ |\psi_4\rangle &= a_-|\uparrow\uparrow\rangle + a_+|\uparrow\downarrow\rangle + a_+|\downarrow\uparrow\rangle + a_-|\downarrow\downarrow\rangle \end{aligned}$$

where

$$a_\pm = \frac{\sqrt{3} \pm 1}{4}$$

$|\psi_1\rangle = \frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$**:**

The density matrix

$$\rho_1 = |\psi_1\rangle\langle\psi_1| = \frac{1}{4}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)\left(\langle\uparrow\uparrow| - \langle\uparrow\downarrow| + \langle\downarrow\uparrow| - \langle\downarrow\downarrow|\right)$$

$$\rho_1^A = \mathrm{Tr}_B\,\rho_1 = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The eigenvalues are 0 and 1, which shows that $|\psi_1\rangle$ is not entangled. To find the factorization of the state we need the eigenvectors of the reduced density matrix $\rho_1^A$. The one with eigenvalue 1 is $|1\rangle_A = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$, while the one with eigenvalue 0 is $|0\rangle_A = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$ (since this has eigenvalue 0 it will not appear in the factorization). We can now express the state $|\psi_1\rangle$ in terms of these eigenvectors and find that

$$|\psi_1\rangle = |1\rangle_A \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

$|\psi_2\rangle = \frac{1}{2}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$**:**

The density matrix

$$\rho_2 = |\psi_2\rangle\langle\psi_2| = \frac{1}{4}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)\left(\langle\uparrow\uparrow| + \langle\uparrow\downarrow| + \langle\downarrow\uparrow| - \langle\downarrow\downarrow|\right)$$

$$\rho_2^A = \mathrm{Tr}_B\,\rho_2 = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is not a pure state, so $|\psi_2\rangle$ is entangled. The eigenvalues are both $\frac{1}{2}$ and all vectors are eigenvectors. Because of that we can choose which basis to use for part A, and the Schmidt decomposition is not unique. Let us take the basis to be $|\uparrow\rangle$ and $|\downarrow\rangle$ for simplicity, and we find

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) + \frac{1}{\sqrt{2}}|\downarrow\rangle \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

$|\psi_3\rangle = a_+|\uparrow\uparrow\rangle + a_-|\uparrow\downarrow\rangle + a_-|\downarrow\uparrow\rangle + a_+|\downarrow\downarrow\rangle$**:**

The density matrix

$$\rho_3 = |\psi_3\rangle\langle\psi_3| = \left(a_+|\uparrow\uparrow\rangle + a_-|\uparrow\downarrow\rangle + a_-|\downarrow\uparrow\rangle + a_+|\downarrow\downarrow\rangle\right)\left(a_+\langle\uparrow\uparrow| + a_-\langle\uparrow\downarrow| + a_-\langle\downarrow\uparrow| + a_+\langle\downarrow\downarrow|\right)$$

$$\rho_3^A = \mathrm{Tr}_B\,\rho_3 = (a_+^2 + a_-^2)|\uparrow\rangle\langle\uparrow| + 2a_+a_-|\uparrow\rangle\langle\downarrow| + 2a_+a_-|\downarrow\rangle\langle\uparrow| + (a_+^2 + a_-^2)|\downarrow\rangle\langle\downarrow| = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

Diagonalizing we find the eigenvalues $p^+ = \frac{3}{4}$ with eigenvector $|\uparrow_x\rangle$ and $p^- = \frac{1}{4}$ with eigenvector $|\downarrow_x\rangle$. This is not a pure state, so $|\psi_3\rangle$ is entangled. Expressing the state in terms of the eigenvectors we find

$$|\psi_3\rangle = \frac{\sqrt{3}}{2}|\uparrow_x\uparrow_x\rangle + \frac{1}{2}|\downarrow_x\downarrow_x\rangle$$

$|\psi_4\rangle = a_-|\uparrow\uparrow\rangle + a_+|\uparrow\downarrow\rangle + a_+|\downarrow\uparrow\rangle + a_-|\downarrow\downarrow\rangle$**:**

The density matrix

$$\rho_4 = |\psi_3\rangle\langle\psi_3| = (a_-|\uparrow\uparrow\rangle + a_+|\uparrow\downarrow\rangle + a_+|\downarrow\uparrow\rangle + a_-|\downarrow\downarrow\rangle)\,(a_-\langle\uparrow\uparrow| + a_+\langle\uparrow\downarrow| + a_+\langle\downarrow\uparrow| + a_-\langle\downarrow\downarrow|)$$

$$\rho_4^A = \mathrm{Tr}_B\,\rho_4 = (a_+^2+a_-^2)|\uparrow\rangle\langle\uparrow| + 2a_+a_-|\uparrow\rangle\langle\downarrow| + 2a_+a_-|\downarrow\rangle\langle\uparrow| + (a_+^2+a_-^2)|\downarrow\rangle\langle\downarrow| = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

which is the same as we found for $\rho_3^A$. Thus we get the same eigenvalues and eigenvectors and we find

$$|\psi_4\rangle = \frac{\sqrt{3}}{2}|\uparrow_x\uparrow_x\rangle - \frac{1}{2}|\downarrow_x\downarrow_x\rangle.$$

## 5.5 Schmidt decomposition 2

a) The Schmidt decomposition rewrites a general state in the product space, as a sum of states expressed in an orthonormal basis for each Hilbert space:

$$\Psi(x) = c_1\chi_1\phi_1(x) + c_2\chi_2\phi_2(x) \tag{3}$$

Thus, the spinors and wavefunctions must satisfy the orthonormality conditions

$$\chi_i^\dagger\chi_j = \int dx\phi_i^*\phi_j = \delta_{ij}$$

b) The normalization factor is given by $\langle\Psi|\Psi\rangle = 1$.

$$\begin{aligned}
\langle\Psi|\Psi\rangle &= \int_{-\infty}^{\infty} |\,\Psi(x)\,|^2\,dx \\
&= \int_{-\infty}^{\infty} dx\,|\,\psi_1(x)\,|^2 + \int_{-\infty}^{\infty} dx\,|\,\psi_2(x)\,|^2 \\
&= |\,N\,|^2\left(\int_{-\infty}^{\infty} e^{-2\lambda(x-x_0)^2}dx + \int_{-\infty}^{\infty} e^{-2\lambda(x+x_0)^2}dx\right)
\end{aligned}$$

Substituting $y = x \pm x_0$ in the first and second integral respectively yields :

$$\begin{aligned}
\langle\Psi|\Psi\rangle &= 2\,|\,N\,|^2\int_{-\infty}^{\infty} e^{-2\lambda y^2}dx \\
&= 2\,|\,N\,|^2\sqrt{\frac{\pi}{2\lambda}} \\
\Rightarrow N &= \sqrt[4]{\frac{\lambda}{2\pi}}, \quad \text{when choosing } N \in \mathbb{R}
\end{aligned}$$

Then it follows:

$$\begin{aligned}
\Delta &= \langle\psi_1|\psi_2\rangle \\
&= N^2\int_{-\infty}^{\infty} e^{-\lambda(x-x_0)^2}e^{-\lambda(x+x_0)^2}dx \\
&= N^2\int_{-\infty}^{\infty} e^{-2\lambda\left(x^2+x_0^2\right)}dx \\
&= N^2e^{-2\lambda x_0^2}\int_{-\infty}^{\infty} dxe^{-2\lambda x^2} \\
&= N^2e^{-2\lambda x_0^2}\sqrt{\frac{\pi}{2\lambda}} \\
\Delta &= \frac{1}{2}e^{-2\lambda x_0^2}
\end{aligned}$$

c) To find the Schmidt decompostion, we have to find the eigenstates of the reduced density matrix for at least one of the subsystems (spin or position). It is simplest to work with spin, since it has the smallest Hilbert space. Therefore we will trace over the position

$$\rho_{spin} = \int dx\langle x|\Phi\rangle\langle\Phi|x\rangle = \int dx\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\begin{pmatrix}\psi_1^* & \psi_2^*\end{pmatrix} = \int dx\begin{pmatrix}\psi_1\psi_1^* & \psi_1\psi_2^*\\ \psi_2\psi_1^* & \psi_2\psi_2^*\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \Delta\\ \Delta & \frac{1}{2}\end{pmatrix}$$

The eigenvalues of this are

$$p_1 = \frac{1}{2}\left(1+e^{-2\lambda x_0^2}\right) \qquad p_2 = \frac{1}{2}\left(1-e^{-2\lambda x_0^2}\right)$$

with the corresponding eigenvectors

$$\chi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \chi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

The coefficients in the Schmidt decomposition are the square roots of the eigenvectors, $c_i = \sqrt{p_i}$ and we get from Eq (3) that

$$\begin{aligned}
\psi_1 &= \frac{1}{\sqrt{2}}c_1\phi_1 + \frac{1}{\sqrt{2}}c_2\phi_2 \\
\psi_2 &= \frac{1}{\sqrt{2}}c_1\phi_1 - \frac{1}{\sqrt{2}}c_2\phi_2
\end{aligned}$$

which we can solve to find

$$\begin{aligned}
\phi_1 &= \frac{1}{\sqrt{2}c_1}(\psi_1+\psi_2) = \frac{N}{\sqrt{1+e^{-2\lambda x_0^2}}}\left(e^{-\lambda(x-x_0)^2}+e^{-\lambda(x+x_0)^2}\right) \\
\phi_2 &= \frac{1}{\sqrt{2}c_2}(\psi_1-\psi_2) = \frac{N}{\sqrt{1-e^{-2\lambda x_0^2}}}\left(e^{-\lambda(x-x_0)^2}-e^{-\lambda(x+x_0)^2}\right)
\end{aligned}$$

### 5.6 Coupled two-level systems

$$\hat{H} = \frac{\epsilon}{2}\left(3\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z\right) + \lambda\left(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+\right)$$

$$\sigma_\pm = \frac{1}{2}\left(\sigma_x \pm i\sigma_y\right)$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a)

$$\begin{aligned} \sigma_+ &= \frac{1}{2}\begin{pmatrix} 0 & 1+1 \\ 1-1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \sigma_- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \hat{H} &= \frac{\epsilon}{2}\left[\begin{pmatrix} 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & -3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}\right] && (4) \\ &+\lambda\left[\begin{pmatrix} 0 & 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} 2\epsilon & 0 & 0 & 0 \\ 0 & \epsilon & \lambda & 0 \\ 0 & \lambda & -\epsilon & 0 \\ 0 & 0 & 0 & -2\epsilon \end{pmatrix} && (5) \end{aligned}$$

The eigenvalue equation becomes:

$$\begin{aligned} |\,\hat{H} - \mathbb{1}e\,| &= 0 \\ \begin{vmatrix} 2\epsilon - e & 0 & 0 & 0 \\ 0 & \epsilon - e & \lambda & 0 \\ 0 & \lambda & -\epsilon - e & 0 \\ 0 & 0 & 0 & -2\epsilon - e \end{vmatrix} &= 0 \\ (2\epsilon - e)\begin{vmatrix} \epsilon - e & \lambda & 0 \\ \lambda & -\epsilon - e & 0 \\ 0 & 0 & -2\epsilon - e \end{vmatrix} &= 0 \\ (2\epsilon - e)(-2\epsilon - e)\left[(\epsilon - e)(-\epsilon - e) - \lambda^2\right] &= 0 \end{aligned}$$

From here, we immidiately see the value of the first two eigenvalues, the rest is determined by:

$$\begin{aligned} -\left(\epsilon^2 - e^2\right) - \lambda^2 &= 0 \\ e &= \pm\sqrt{\epsilon^2 + \lambda^2} \end{aligned}$$

The eigenvalues are thus:

$$e_1 = 2\epsilon, \quad e_2 = -2\epsilon, \quad e_3 = \sqrt{\epsilon^2 + \lambda^2}, \quad e_4 = -\sqrt{\epsilon^2 + \lambda^2}$$

We see that $e_1$ and $e_2$ are independent of $\lambda$, and from the hamiltonian (5), it is easy to to see that the eigenvectors are:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Then setting $\epsilon = \mu \cos\theta$ and $\lambda = \mu \sin\theta$, we get:

$$e_1 = 2\mu\cos\theta, \quad e_2 = -2\mu\cos\theta, \quad e_3 = \mu, \quad e_4 = -\mu$$

The hamiltonian takes the form:

$$\hat{H} = \begin{pmatrix} 2\mu\cos\theta & 0 & 0 & 0 \\ 0 & \mu\cos\theta & \mu\sin\theta & 0 \\ 0 & \mu\sin\theta & -\mu\cos\theta & 0 \\ 0 & 0 & 0 & -2\mu\cos\theta \end{pmatrix}$$

For the remaining subspace, the eigenvector equation is:

$$\begin{pmatrix} \mu\cos\theta & \mu\sin\theta \\ \mu\sin\theta & -\mu\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm\mu \begin{pmatrix} a \\ b \end{pmatrix}$$

Then:

$$\begin{aligned} a\cos\theta + b\sin\theta &= \pm a \\ a\cos\theta - b\sin\theta &= \pm b \end{aligned}$$

Staring with the first equation:

$$a\cos\theta + b\sin\theta = \pm a \Rightarrow b = a\frac{\pm 1 - \cos\theta}{\sin\theta}$$

Then, if $a = \sin\theta$, we get the following eigenvectors:

$$\mathbf{e}_3' = \begin{pmatrix} 0 \\ \sin\theta \\ 1 - \cos\theta \\ 0 \end{pmatrix}, \quad \mathbf{e}_4' = \begin{pmatrix} 0 \\ \sin\theta \\ -1 - \cos\theta \\ 0 \end{pmatrix}$$

I marked them as to say that they are not the final eigenvectors, they need to be normalized first:

$$\begin{aligned} \sqrt{\mathbf{e}_3' \cdot \mathbf{e}_3'} &= \sqrt{\sin^2\theta + (1-\cos\theta)^2} = \sqrt{\sin^2\theta + 1 - 2\cos\theta + \cos\theta^2} = \sqrt{2 - 2\cos\theta} \\ \sqrt{\mathbf{e}_4' \cdot \mathbf{e}_4'} &= \sqrt{\sin^2\theta + (1+\cos\theta)^2} = \sqrt{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta} = \sqrt{2 + 2\cos\theta} \end{aligned}$$

Then:

$$\mathbf{e}_3 = \frac{1}{\sqrt{2-2\cos\theta}}\begin{pmatrix}0\\ \sin\theta\\ 1-\cos\theta\\ 0\end{pmatrix} = \begin{pmatrix}0\\ \cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\\ 0\end{pmatrix}, \quad \mathbf{e}_4 = \frac{1}{\sqrt{2+2\cos\theta}}\begin{pmatrix}0\\ \sin\theta\\ -1-\cos\theta\\ 0\end{pmatrix} = \begin{pmatrix}0\\ \sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\\ 0\end{pmatrix}$$

Just to summarize, the eigenvectors are:

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix}0\\ \cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\\ 0\end{pmatrix}, \quad \mathbf{e}_4 = \begin{pmatrix}0\\ \sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\\ 0\end{pmatrix}$$

The energies are:

$$E_1 = 2\mu\cos\theta, \quad E_2 = -2\mu\cos\theta, \quad E_3 = \mu, \quad E_4 = -\mu$$

b) The two interesting eigenstates are $\mathbf{e}_3$ and $\mathbf{e}_4$

$$\begin{aligned}\hat{\rho}_3 &= \mathbf{e}_3\mathbf{e}_3^T = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2} & 0\\ 0 & \cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\\ \hat{\rho}_4 &= \mathbf{e}_4\mathbf{e}_4^T = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & \sin^2\frac{\theta}{2} & -\cos\frac{\theta}{2}\sin\frac{\theta}{2} & 0\\ 0 & -\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \cos^2\frac{\theta}{2} & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\end{aligned}$$

Before considering the partial traces, let's look at how this works out in the matrix representation. A general 4x4 matrix can be written as a sum over tensor products between 2x2 matrices (also called "Kronecker product"):

$$\begin{aligned}C &= \sum_{ij} c_{ij} A_i \otimes B_j\\ &= \sum_{ij} c_{ij} \begin{pmatrix} A^i_{11}\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}_j & A^i_{12}\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}_j\\ A^i_{21}\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}_j & A^i_{22}\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}_j\end{pmatrix}\\ &\equiv \sum_{ij} c_{ij}\begin{pmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}_{ij}\end{aligned}$$

Then the partial traces become:

$$\begin{aligned}\mathrm{Tr}_A C &= \sum_{ij} c_{ij}\left(C_{11}+C_{22}\right)_{ij}\\ \mathrm{Tr}_B C &= \sum_{ij} c_{ij}\begin{pmatrix}\mathrm{Tr}C_{11} & \mathrm{Tr}C_{12}\\ \mathrm{Tr}C_{21} & \mathrm{Tr}C_{22}\end{pmatrix}_{ij}\end{aligned}$$

And since $\text{Tr}\,(A+B) = \text{Tr}A + \text{Tr}B$, we see that in our case:

$$\begin{aligned}
\hat{\rho}_3^A = \text{Tr}_B\hat{\rho}_3 &= \begin{pmatrix} \cos^2\frac{\theta}{2} & 0 \\ 0 & \sin^2\frac{\theta}{2} \end{pmatrix} \\
\hat{\rho}_3^B = \text{Tr}_A\hat{\rho}_3 &= \begin{pmatrix} \sin^2\frac{\theta}{2} & 0 \\ 0 & \cos^2\frac{\theta}{2} \end{pmatrix} \\
\hat{\rho}_4^A = \text{Tr}_B\hat{\rho}_4 &= \begin{pmatrix} \sin^2\frac{\theta}{2} & 0 \\ 0 & \cos^2\frac{\theta}{2} \end{pmatrix} \\
\hat{\rho}_4^B = \text{Tr}_A\hat{\rho}_4 &= \begin{pmatrix} \cos^2\frac{\theta}{2} & 0 \\ 0 & \sin^2\frac{\theta}{2} \end{pmatrix}
\end{aligned}$$

c) We see that all the reduced density matrices have the same eigenvalues, and the von Neuman entropy is thus the same and given by:

$$S = -\cos^2\frac{\theta}{2}\ln\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\ln\sin^2\frac{\theta}{2}$$

The entropy is maximal when $\cos^2\frac{\theta}{2} = \sin^2\frac{\theta}{2} = \frac{1}{2}$, which means

$$\theta = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

## 5.7 Entanglement in the Jaynes-Cummings model

a) We have

$$|\psi(t)\rangle = c_n^-(t)|-, n+1\rangle + c_n^+(t)|+, n\rangle$$

which gives the density matrix

$$\begin{aligned}
\rho &= |\psi(t)\rangle\langle\psi(t)| \\
&= |c_n^-(t)|^2|-, n+1\rangle\langle -, n+1| + c_n^-(t)c_n^+(t)^*|-, n+1\rangle\langle +, n| \\
&\quad + c_n^-(t)^*c_n^+(t)|+, n\rangle\langle -, n+1| + |c_n^+(t)|^2|+, n\rangle\langle +, n|
\end{aligned}$$

Tracing over the photon mode we find

$$\rho_{TLS} = \sum_m \langle m|\rho|m\rangle = |c_n^-(t)|^2|-\rangle\langle -| + |c_n^+(t)|^2|+\rangle\langle +|.$$

This is diagonal, and we have the probabilities for the two states

$$\begin{aligned}
p^+ &= |c_n^+(t)|^2 = \sin^2\frac{\Omega_n t}{2}\sin^2\theta_n \\
p^- &= |c_n^-(t)|^2 = 1 - \sin^2\frac{\Omega_n t}{2}\sin^2\theta_n
\end{aligned}$$

The entanglement entropy is

$$S = -p^+ \ln p^+ - p^- \ln p^-$$

This is maximal when $p^+$ and $p^-$ are as equal as possible. If $\sin^2\theta_n > 1/2$, which means that $\theta_n > \pi/4$, we can get $p^+ = p^- = \frac{1}{2}$ with

$$S_{max} = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \ln 2.$$

This will happen when

$$\sin^2\frac{\Omega_n t}{2}\sin^2\theta_n = \frac{1}{2}$$

which means

$$t = \frac{2}{\Omega_n}\arcsin\left[\frac{1}{\sqrt{2}\sin\theta_n}\right] = \frac{2}{\Omega_n}\arcsin\left[\frac{\Omega_n}{\sqrt{2}g_n}\right]$$

If $\sin^2\theta_n < 1/2$ we have $p^+ < \frac{1}{2}$ and maximal when $\frac{\Omega_n t}{2} = \frac{\pi}{2} + m\pi$, with $m$ an integer. $p^+_{max} = \sin^2\theta_n$ and

$$S_{max} = -\sin^2\theta_n \ln\sin^2\theta_n - \cos^2\theta_n \ln\cos^2\theta_n.$$

b) For the Rabi model (in the rotating frame) we have

$$|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$$

with

$$c_0(t) = \cos\frac{\Omega t}{2} + i\sin\frac{\Omega t}{2}\cos\theta, \qquad c_1(t) = i\sin\frac{\Omega t}{2}\sin\theta.$$

This is a pure state and the Bloch vector has components

$$\begin{aligned} m_x^R &= 2\,\mathrm{Re}(c_0^* c_1) = \sin 2\theta \sin^2\frac{\Omega t}{2} \\ m_y^R &= 2\,\mathrm{Im}(c_0^* c_1) = \sin\theta\sin\Omega t \\ m_z^R &= |c_0|^2 - |c_1|^2 = 1 - 2\sin^2\theta\sin^2\frac{\Omega t}{2}. \end{aligned}$$

For the Jaynes-Cummings model we use

$$\rho_{TLS} = \frac{1}{2}(\mathbb{1} + \mathbf{m}^{JC}\cdot\sigma)$$

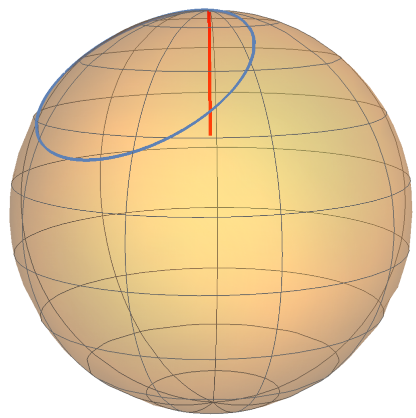
We know that

$$\rho_{TLS} = p^-|-\rangle\langle-| + p^+|+\rangle\langle+| = \frac{1}{2}(\mathbb{1} + (p^- - p^+)\sigma_z)$$

from which we read out

$$\begin{aligned} m_x^{JC} &= m_y^{JC} = 0 \\ m_z^{JC} &= p^- - p^+ = 1 - 2\sin^2\theta_n\sin^2\frac{\Omega_n t}{2}. \end{aligned}$$

In the Rabi model, the state is always pure, and the Bloch vector presesses in a circle on the surface of the Bloch sphere. In the Jaynes-Cummings model, the qubit is entangled with the photon mode, and the reduced density matrix describes a mixed state. The Bloch vector oscillates along the axis of the Bloch sphere with $m_z^{JC} = m_z^R$ This is shown in the following figure with the Rabi model in Blue and the Jaynes-Cummings model in red.

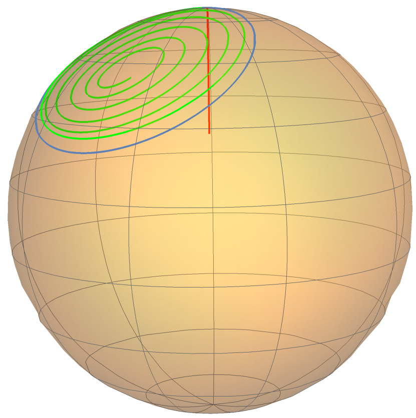


c) In the limit $n \to \infty$ we have that $g_n = 2\lambda\sqrt{n+1}$ grows. This means that

$$\Omega_n = \sqrt{\Delta^2 + g_n^2} \to g_n$$

and $\sin\theta_n = \frac{g_n}{\Omega_n} \to 1$. So the amplitude and frequency of the oscillations grow, but the Bloch vector is always on the axis of the Bloch sphereand entanglement is not reduced.

An idea for a better classical limit is to assume that the photon mode starts in a coherent state instead of an energy eigenstate. We know that coherent states are the link to classical mechanics for the harmonic oscillator, and we can hope that it will extend to the Jaynes-Cummings model as well. In the figure the result is shown in green



As we can see, it works to some extent, but it becomes a spiral instead of a circle. Here I used an average photon number of 9 in the initial state. Maybe it should be bigger for the limit, but numerics gets slow. More work is needed......