Solutions to problem set 7

7.1 The canonical commutation relations.

$$\hat{A}_{\mathbf{k}a} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}\left(\hat{a}_{\mathbf{k}a} + \hat{a}^\dagger_{-\mathbf{k}\bar{a}}\right), \quad \hat{E}_{\mathbf{k}a} = i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}\left(\hat{a}_{\mathbf{k}a} - \hat{a}^\dagger_{-\mathbf{k}\bar{a}}\right)$$

$$\left[\hat{A}_{\mathbf{k}a}, \hat{E}^\dagger_{\mathbf{k}'a'}\right] = -i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'}$$

We need to express the ladder operators in terms of $\hat{A}$ and $\hat{E}^\dagger$, by inspection, we see that:

$$\hat{a}_{\mathbf{k}a} = \frac{1}{2}\left(\sqrt{\frac{2\omega_k\epsilon_0}{\hbar}}\hat{A}_{\mathbf{k}a} - i\sqrt{\frac{2\epsilon_0}{\hbar\omega_k}}\hat{E}_{\mathbf{k}a}\right)$$

from which we calculate:

$$\hat{a}^\dagger_{\mathbf{k}a} = \frac{1}{2}\left(\sqrt{\frac{2\omega_k\epsilon_0}{\hbar}}\hat{A}^\dagger_{\mathbf{k}a} + i\sqrt{\frac{2\epsilon_0}{\hbar\omega_k}}\hat{E}^\dagger_{\mathbf{k}a}\right)$$

Then:

$$\begin{aligned}
\left[\hat{a}_{\mathbf{k}a}, \hat{a}^\dagger_{\mathbf{k}'a'}\right] &= \left[\frac{1}{2}\left(\sqrt{\frac{2\omega_k\epsilon_0}{\hbar}}\hat{A}_{\mathbf{k}a} - i\sqrt{\frac{2\epsilon_0}{\hbar\omega_k}}\hat{E}_{\mathbf{k}a}\right), \frac{1}{2}\left(\sqrt{\frac{2\omega_{k'}\epsilon_0}{\hbar}}\hat{A}^\dagger_{\mathbf{k}'a'} + i\sqrt{\frac{2\epsilon_0}{\hbar\omega_{k'}}}\hat{E}^\dagger_{\mathbf{k}'a'}\right)\right] \\
&= \frac{1}{4}\left(\frac{2\epsilon_0\sqrt{\omega_k\omega_{k'}}}{\hbar}\underbrace{\left[\hat{A}_{\mathbf{k}a}, \hat{A}^\dagger_{\mathbf{k}'a'}\right]}_{=0} + i\frac{2\epsilon_0}{\hbar}\sqrt{\frac{\omega_k}{\omega_{k'}}}\left[\hat{A}_{\mathbf{k}a}, \hat{E}^\dagger_{\mathbf{k}'a'}\right]\right) \\
&\quad + \frac{1}{4}\left(-i\frac{2\epsilon_0}{\hbar}\sqrt{\frac{\omega_{k'}}{\omega_k}}\left[\hat{E}_{\mathbf{k}a}, \hat{A}^\dagger_{\mathbf{k}'a'}\right] + \frac{2\epsilon_0}{\hbar\sqrt{\omega_k\omega_{k'}}}\underbrace{\left[\hat{E}_{\mathbf{k}a}, \hat{E}^\dagger_{\mathbf{k}'a'}\right]}_{=0}\right) \\
&= i\frac{\epsilon_0}{2\hbar}\sqrt{\frac{\omega_k}{\omega_{k'}}}\underbrace{\left[\hat{A}_{\mathbf{k}a}, \hat{E}^\dagger_{\mathbf{k}'a'}\right]}_{=-i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'}} - i\frac{\epsilon_0}{2\hbar}\sqrt{\frac{\omega_{k'}}{\omega_k}}\left[\hat{E}_{\mathbf{k}a}, \hat{A}^\dagger_{\mathbf{k}'a'}\right]
\end{aligned}$$

Looking at the last commutator:

$$\begin{aligned}
\left[\hat{E}_{\mathbf{k}a}, \hat{A}^\dagger_{\mathbf{k}'a'}\right] &= \hat{E}_{\mathbf{k}a}\hat{A}^\dagger_{\mathbf{k}'a'} - \hat{A}^\dagger_{\mathbf{k}'a'}\hat{E}_{\mathbf{k}a} = \left(\hat{A}_{\mathbf{k}'a'}\hat{E}^\dagger_{\mathbf{k}a}\right)^\dagger - \left(\hat{E}^\dagger_{\mathbf{k}a}\hat{A}_{\mathbf{k}'a'}\right)^\dagger \\
&= \left(\hat{A}_{\mathbf{k}'a'}\hat{E}^\dagger_{\mathbf{k}a} - \hat{E}^\dagger_{\mathbf{k}a}\hat{A}_{\mathbf{k}'a'}\right)^\dagger = \left[\hat{A}_{\mathbf{k}'a'}, \hat{E}^\dagger_{\mathbf{k}a}\right]^\dagger \\
&= \left(-i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{k}'\mathbf{k}}\delta_{a'a}\right)^\dagger = i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{k}'\mathbf{k}}\delta_{a'a}
\end{aligned}$$

So far, we have:

$$\begin{aligned}\left[\hat{a}_{\mathbf{k}a}, \hat{a}^{\dagger}_{\mathbf{k}'a'}\right] &= i\frac{\epsilon_0}{2\hbar}\sqrt{\frac{\omega_k}{\omega_{k'}}}\left(-i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{kk}'}\delta_{aa'}\right) - i\frac{\epsilon_0}{2\hbar}\sqrt{\frac{\omega_{k'}}{\omega_k}}\left(i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{k}'\mathbf{k}}\delta_{a'a}\right) \\ &= \frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\delta_{\mathbf{kk}'}\delta_{aa'} + \frac{1}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}}\left(\delta_{\mathbf{k}'\mathbf{k}}\delta_{a'a}\right) \\ &= \frac{1}{2}\left(\sqrt{\frac{\omega_k}{\omega_{k'}}} + \sqrt{\frac{\omega_{k'}}{\omega_k}}\right)\delta_{\mathbf{kk}'}\delta_{aa'}\end{aligned}$$

$\delta_{\mathbf{kk}'}$ will either return zero if $\mathbf{k} \neq \mathbf{k}'$, or 1 if $\mathbf{k} = \mathbf{k}'$, in the latter case, $\sqrt{\frac{\omega_k}{\omega_{k'}}} + \sqrt{\frac{\omega_{k'}}{\omega_k}} = 2$, and if not, the expression has no contribution, thus, we can neglect it and get the desired result:

$$\left[\hat{a}_{\mathbf{k}a}, \hat{a}^{\dagger}_{\mathbf{k}'a'}\right] = \delta_{\mathbf{kk}'}\delta_{aa'}$$

7.2 Charged particle in a strong magnetic field (Midterm Exam 2005).

a) From Newtons second law:

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{v} \times \mathbf{B}\right) \tag{1}$$

for a particle moving in a magnetic field ($\mathbf{E} = 0$). The velocity is restricted to the xy-plane, and the magnetic field is in the z-direction. Thus, by integration:

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \frac{eB}{m}\left(\mathbf{v} \times \mathbf{k}\right) \\ \Rightarrow \mathbf{v} &= \frac{eB}{m}\left(\mathbf{r} \times \mathbf{k}\right) + \mathbf{C} \\ &= -\frac{eB}{m}\mathbf{k} \times \mathbf{r} + \mathbf{C}\end{aligned}$$

We recognize this as the expression for angular velocity with $\omega = -\frac{eB}{m}$ where $\mathbf{C}$ is a constant that can be determined from the initial conditions. We can parametrize $\mathbf{C}$ to a vector on the same form $\mathbf{C} = -\omega \times \mathbf{r}_0$ where $\mathbf{r}_0$ is a constant:

$$\begin{aligned}\mathbf{v} &= \vec{\omega} \times \mathbf{r} - \vec{\omega} \times \mathbf{r}_0 \\ \mathbf{v} &= \vec{\omega} \times \left(\mathbf{r} - \mathbf{r}_0\right)\end{aligned} \tag{2}$$

We see that this represents constant angular motion around the centre $\mathbf{r}_0$ with angular frequency $\omega = -\frac{eB}{m}$.

To check if $L_{mek} = m(xv_y - yv_x)$ is a constant of motion, we start by calculating:

$$\begin{aligned}m\frac{d\mathbf{v}}{dt} &= e\mathbf{v} \times \mathbf{B} \\ &= q\begin{vmatrix}\hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & 0 \\ 0 & 0 & B\end{vmatrix} \\ &= v_y B\mathbf{i} - v_x B\mathbf{j}\end{aligned}$$

Then:

$$\begin{aligned}
\frac{dL_{mek}}{dt} &= m\left(\frac{dx}{dt}v_y + x\frac{dv_y}{dt} - \frac{dy}{dt}v_x - y\frac{dv_x}{dt}\right) \\
&= m\frac{dx}{dt}v_y + xm\frac{dv_y}{dt} - m\frac{dy}{dt}v_x - ym\frac{dv_x}{dt} \\
&= mv_xv_y - eBxv_x - mv_yv_x - eByv_y \\
&= -eB\left(xv_x + yv_y\right) \\
&= -eB\mathbf{r}\cdot\mathbf{v} \\
&= -eB\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right) \\
&= -\frac{eB}{2}\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt}\cdot\mathbf{r}\right) \\
&= -\frac{eB}{2}\frac{d}{dt}r^2
\end{aligned}$$

As we see, $\frac{d}{dt}L_{mek} \neq 0$, and thus not a constant of motion. Instead we have that $L = L_{mek} + (eB/2)r^2$ is conserved as:

$$\begin{aligned}
\frac{dL}{dt} &= \frac{dL_{mek}}{dt} + \frac{eB}{2}\frac{d}{dt}r^2 \\
&= \frac{eB}{2}\frac{d}{dt}\left(-r^2 + r^2\right) \\
&= 0
\end{aligned}$$

b) To check if $\mathbf{R}$ is a constant of motion, we take the derivative:

$$\begin{aligned}
\frac{d\mathbf{R}}{dt} &= \frac{d\mathbf{r}}{dt} + \frac{1}{\omega}\frac{d}{dt}\left(\mathbf{k}\times\mathbf{v}\right) \\
&= \frac{d\mathbf{r}}{dt} + \frac{1}{\omega}\mathbf{k}\times\frac{d\mathbf{v}}{dt} \\
&\overset{(1)}{=} \mathbf{v} + \frac{e}{m\omega}\mathbf{k}\times\left(\mathbf{v}\times\mathbf{B}\right) \\
&= \mathbf{v} + \frac{e}{m\omega}\left(\mathbf{v}\left(\mathbf{k}\cdot\mathbf{B}\right) - \mathbf{B}\left(\mathbf{k}\cdot\mathbf{v}\right)\right) \\
&= \mathbf{v} + \frac{eB}{m\omega}\mathbf{v} \\
&= \mathbf{v} + \frac{eB}{m}\frac{m}{-eB}\mathbf{v} \\
&= \mathbf{0}
\end{aligned}$$

Which it is. Inserting (2) into the expression for $\mathbf{R}$ we have:

$$\begin{aligned}
\mathbf{R} &= \mathbf{r} + \frac{1}{\omega}\mathbf{k}\times\mathbf{v} \\
&= \mathbf{r} + \frac{1}{\omega}\mathbf{k}\times\vec{\omega}\times(\mathbf{r}-\mathbf{r}_0) \\
&= \mathbf{r} + \frac{1}{\omega}\left(\vec{\omega}\cdot\left(\underbrace{\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)}_{=0}\right) - (\mathbf{r}-\mathbf{r}_0)\underbrace{(\mathbf{k}\cdot\vec{\omega})}_{=\omega}\right) \\
&= \mathbf{r} - (\mathbf{r}-\mathbf{r}_0) \\
\mathbf{R} &= \mathbf{r}_0 \qquad (3)
\end{aligned}$$

So $\mathbf{R}$ points to the centre of the circular orbit. $\vec{\rho}$ is given by:

$$\begin{aligned}
\vec{\rho} &= \mathbf{R} - \mathbf{r} \\
&= \mathbf{r}_0 - \mathbf{r}
\end{aligned}$$

So ρ points from the particle to the centre of orbit.

c) If we use (from the problem set):

$$\begin{aligned}
\mathbf{v} &= \frac{(\mathbf{p}-e\mathbf{A})}{m} \\
&= \frac{1}{m}\left(\mathbf{p} + \frac{e}{2}\mathbf{r}\times\mathbf{B}\right) \\
&= \frac{1}{m}\left(\mathbf{p} + \frac{eB}{2}\mathbf{r}\times\mathbf{k}\right)
\end{aligned}$$

where $\mathbf{p}$ denotes the canonical momentum, we can express $\mathbf{R}$ with $\mathbf{p}$ and $\mathbf{r}$ only :

$$\begin{aligned}
\mathbf{R} &= \mathbf{r} + \frac{1}{\omega}\mathbf{k}\times\mathbf{v} \\
&= \mathbf{r} + \frac{1}{m\omega}\mathbf{k}\times\left(\mathbf{p} + \frac{eB}{2}\mathbf{r}\times\mathbf{k}\right) \\
&= \mathbf{r} + \frac{1}{m\omega}\left(\mathbf{k}\times\mathbf{p} + \frac{eB}{2}\underbrace{\mathbf{k}\times[\mathbf{r}\times\mathbf{k}]}_{=\mathbf{r}}\right) \\
&= \mathbf{r} + \frac{1}{m\omega}\underbrace{\mathbf{k}\times\mathbf{p}}_{=-p_y\hat{i}+p_x\hat{j}} + \underbrace{\frac{eB}{2m\omega}}_{=-\frac{1}{2}}\mathbf{r} \\
&= \mathbf{r}\left(1-\frac{1}{2}\right) + \frac{1}{m\omega}\left(-p_y\hat{i}+p_x\hat{j}\right) \\
&= \left(\frac{1}{2}x - \frac{1}{m\omega}p_y\right)\mathbf{i} + \left(\frac{1}{2}y + \frac{1}{m\omega}p_x\right)\mathbf{j} \\
&\equiv X\mathbf{i} + Y\mathbf{j}
\end{aligned}$$

We can now express these as QM operators by replacing $r \rightarrow \hat{r}$ and $p \rightarrow \hat{p}$ with the commutation relations

$$[\hat{r}_j, \hat{p}_k] = i\hbar\delta_{jk}$$

This gives:

$$\begin{aligned}\hat{X} &= \frac{1}{2}\hat{x} - \frac{1}{m\omega}\hat{p}_y \\ \hat{Y} &= \frac{1}{2}\hat{y} - \frac{1}{m\omega}\hat{p}_x\end{aligned}$$

These commute as:

$$\begin{aligned}\left[\hat{X}, \hat{Y}\right] &= \left[\frac{1}{2}\hat{x} - \frac{1}{m\omega}\hat{p}_y, \frac{1}{2}\hat{y} + \frac{1}{m\omega}\hat{p}_x\right] \\ &= \frac{1}{4}\underbrace{[\hat{x}, \hat{y}]}_{=0} + \frac{1}{2m\omega}\underbrace{[\hat{x}, \hat{p}_x]}_{=i\hbar} - \frac{1}{2m\omega}\underbrace{[\hat{p}_y, \hat{y}]}_{=-i\hbar} - \frac{1}{m^2\omega^2}\underbrace{[\hat{p}_y, \hat{p}_x]}_{=0} \\ &= \frac{i\hbar}{m\omega}\end{aligned}$$

For $\rho = \mathbf{R} - \mathbf{r}$ we have the component operators:

$$\begin{aligned}\hat{\rho}_x &= -\frac{1}{2}\hat{x} - \frac{1}{m\omega}\hat{p}_y \\ \hat{\rho}_y &= -\frac{1}{2}\hat{y} + \frac{1}{m\omega}\hat{p}_x\end{aligned}$$

That gives:

$$\begin{aligned}[\rho_x, \rho_y] &= \left[-\frac{1}{2}\hat{x} - \frac{1}{m\omega}\hat{p}_y, -\frac{1}{2}\hat{y} + \frac{1}{m\omega}\hat{p}_x\right] \\ &= \frac{1}{4}\underbrace{[\hat{x}, \hat{y}]}_{=0} - \frac{1}{2m\omega}\underbrace{[\hat{x}, \hat{p}_x]}_{=i\hbar} + \frac{1}{2m\omega}\underbrace{[\hat{p}_y, \hat{y}]}_{=-i\hbar} - \frac{1}{m^2\omega^2}\underbrace{[\hat{p}_y, \hat{p}_x]}_{=0} \\ &= -\frac{i\hbar}{m\omega}\end{aligned}$$

Here $\hat{X}$ and $\hat{Y}$ and $\hat{\rho}_x$ and $\hat{\rho}_y$ respectively commute as a phase space where we have replaced $\hbar \to \hbar/m\omega$. This means that there are unceartainty relations between the operators, and that they can not be known simultaniously. We now introduce $l_B^2 = \hbar/m\omega$ such that

$$[\hat{X}, \hat{Y}] = [\hat{\rho}_y, \hat{\rho}_x] = il_B^2 \tag{4}$$

d)

$$\hat{a} = \frac{1}{\sqrt{2}l_B}\left(\hat{X} + i\hat{Y}\right) \quad \hat{b} = \frac{1}{\sqrt{2}l_B}\left(\hat{\rho}_x - i\hat{\rho}_y\right)$$

We know that $\hat{X}$,$\hat{Y}$ and $\hat{\rho}_x, \hat{\rho}_y$ are made up of hermitian operators, and thus:

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}l_B}\left(\hat{X} - i\hat{Y}\right), \quad \hat{b}^\dagger = \frac{1}{\sqrt{2}l_B}\left(\hat{\rho}_x + i\hat{\rho}_y\right)$$

where $l_B = \sqrt{\frac{\hbar}{\mid eB \mid}}$. Then:

$$\begin{aligned}
\left[\hat{a}, \hat{a}^\dagger\right] &= \frac{1}{2l_B^2}\left[\hat{X} + i\hat{Y}, \hat{X} - i\hat{Y}\right] \\
&= \frac{1}{2l_B^2}\left(\underbrace{\left[\hat{X}, \hat{X}\right]}_{=0} + \underbrace{\left[\hat{X}, -i\hat{Y}\right] + \left[i\hat{Y}, \hat{X}\right]}_{=-2i\left[\hat{X}, \hat{Y}\right]} + \underbrace{\left[i\hat{Y}, -i\hat{Y}\right]}_{=0}\right) \\
&= \frac{-2i^2 l_B^2}{2l_B^2} \\
&= 1 \\
\left[\hat{b}, \hat{b}^\dagger\right] &= \frac{1}{2l_B^2}\left[\hat{\rho}_x - i\hat{\rho}_y, \hat{\rho}_x + i\hat{\rho}_y\right] \\
&= \frac{1}{2l_B^2}\left(\underbrace{\left[\hat{\rho}_x, \hat{\rho}_x\right]}_{=0} + \underbrace{\left[\hat{\rho}_x, i\hat{\rho}_y\right] + \left[-i\hat{\rho}_y, \hat{\rho}_x\right]}_{=2i\left[\hat{\rho}_x, \hat{\rho}_y\right]} + \underbrace{\left[-i\hat{\rho}_y, i\hat{\rho}_y\right]}_{=0}\right) \\
&= \frac{-2i^2 l_B^2}{2l_B^2} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\left[\hat{a}, \hat{b}^\dagger\right] &= \left[\hat{X} + i\hat{Y}, \hat{\rho}_x + i\hat{\rho}_y\right] \\
&= \left[\hat{X}, \hat{\rho}_x\right] + i\left[\hat{X}, \hat{\rho}_y\right] + i\left[\hat{Y}, \hat{\rho}_x\right] - \left[\hat{Y}, \hat{\rho}_y\right]
\end{aligned}$$

As $[\hat{X}, \hat{\rho}_x] = \hat{Y}, \hat{\rho}_y]$ are trivially zero (they contain only operators that commute) we get:

$$\begin{aligned}
-i\left[\hat{a}, \hat{b}^\dagger\right] &= \left[\hat{X}, \hat{\rho}_y\right] + \left[\hat{Y}, \hat{\rho}_x\right] \\
&= \left[\hat{X}, \hat{\rho}_y\right] + \left[\hat{Y}, \hat{\rho}_x\right]
\end{aligned}$$

The relevant commutators are:

$$\begin{aligned}
\left[\hat{X}, \hat{\rho}_y\right] &= -\frac{1}{4}\left[\hat{x}, \hat{y}\right] + \frac{1}{2m\omega}\left[\hat{x}, \hat{p}_x\right] + \frac{1}{2m\omega}\left[\hat{p}_y, \hat{y}\right] - \frac{1}{m^2\omega^2}\left[\hat{p}_x, \hat{p}_y\right] \\
&= \frac{1}{2m\omega}\left(\left[\hat{x}, \hat{p}_x\right] - \left[\hat{y}, \hat{p}_y\right]\right) \\
&= 0 \\
\left[\hat{Y}, \hat{\rho}_x\right] &= \frac{1}{2m\omega}\left(\left[\hat{y}, \hat{p}_y\right] - \left[\hat{x}, \hat{p}_x\right]\right) \\
&= 0
\end{aligned}$$

such that

$$-i\left[\hat{a}, \hat{b}^\dagger\right] = 0$$

And similarily $\left[\hat{a}, \hat{b}\right] = \left[\hat{a}^\dagger, \hat{b}^\dagger\right] = 0$.

This means that the operators $\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$ follow the same algebra (and the same physics) as two independent harmonic oscillators.

e) The hamiltonian is:

$$H = \frac{1}{2m}\left(\mathbf{p} - a\mathbf{A}\right)^2 = \frac{1}{2}m\mathbf{v}^2$$

We found the velocity in (2), and by using the result (3), we see we get:

$$\begin{aligned} H &= \frac{1}{2}m\left(\tilde{\omega} \times (\mathbf{r} - \mathbf{R})\right)^2 \\ &= \frac{1}{2}m\left(-\tilde{\omega} \times \tilde{\rho}\right)^2 \\ &\overset{\vec{\omega}\perp\vec{\rho}}{=} -\frac{1}{2}m\omega^2\rho^2 \\ &= -\frac{1}{2}m\omega^2\left(\rho_x^2 + \rho_y^2\right) \end{aligned} \tag{5}$$

Expressing these in terms of the ladder operators $\hat{b}$ and $\hat{b}^\dagger$ yields:

$$\hat{\rho}_x = \frac{l_B}{\sqrt{2}}\left(\hat{b} + \hat{b}^\dagger\right), \quad \hat{\rho}_y = -\frac{il_B}{\sqrt{2}}\left(\hat{b} - \hat{b}^\dagger\right) \tag{6}$$

$$\begin{aligned} \hat{H} &= \frac{1}{2}m\omega^2\frac{l_B^2}{2}\left(\left(\hat{b} + \hat{b}^\dagger\right)^2 - \left(\hat{b} - \hat{b}^\dagger\right)^2\right) \\ &= \frac{1}{4}m\omega^2 l_B^2\left(2\hat{b}\hat{b}^\dagger + 2\hat{b}^\dagger\hat{b}\right) \\ &= \frac{1}{4}m\omega^2 l_B^2\left(2\left[1 + \hat{b}^\dagger\hat{b}\right] + 2\hat{b}^\dagger\hat{b}\right) \\ &= m\omega^2 l_B^2\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) \\ &= m\omega^2\frac{\hbar}{m\mid\omega\mid}\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) \\ &= \hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) \end{aligned} \tag{7}$$

This is the hamiltonian for the harmonic oscillator, and has the energy spectrum $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ independent of m. This means that for each energy there are m degenrate states. The angular momentum operator is:

$$L = m\left(xv_y - yv_x\right) + \frac{eB}{2}r^2 = m\mathbf{r} \times \mathbf{v} + \frac{eB}{2}\mathbf{r}^2$$

From earlier, we had $\mathbf{r} = \mathbf{R} - \vec{\rho}$, $\mathbf{v} = -\vec{\omega} \times \vec{\rho}$. Then:

$$\begin{aligned} L &= m\left(\mathbf{R} - \vec{\rho}\right) \times \left(-\vec{\omega} \times \vec{\rho}\right) + \frac{eB}{2}\left(\mathbf{R} - \vec{\rho}\right)^2 \\ &= m\left(-\vec{\omega}\left[\left(\mathbf{R} - \vec{\rho}\right) \cdot \vec{\rho}\right] - \vec{\rho}\left[\underbrace{\left(\mathbf{R} - \vec{\rho}\right) \cdot \left(-\vec{\omega}\right)}_{=0}\right]\right) + \frac{eB}{2}\left(\mathbf{R} - \vec{\rho}\right)^2 \\ &= -m\vec{\omega}\left(\vec{\rho} \cdot \mathbf{R} - \vec{\rho}^2\right) + \frac{eB}{2}\left(\mathbf{R}^2 - 2\mathbf{R}\vec{\rho} + \vec{\rho}^2\right) \\ &= m\vec{\omega}\left(\vec{\rho}^2 - \vec{\rho} \cdot \mathbf{R}\right) - m\omega\left(\frac{\mathbf{R}^2}{2} - \vec{\rho} \cdot \mathbf{R} + \frac{\vec{\rho}^2}{2}\right) \end{aligned}$$

Since L is zero except for the z component, we can drop the vector notation and have

$$L = \frac{1}{2}m\omega\left(\vec{\rho}^2 - \mathbf{R}^2\right)$$

Quantizing this, and remembering from (5) and (7), we have:

$$\begin{aligned} \vec{\rho}^2 &= \frac{l_B^2}{2}\left(\left(\hat{b} + \hat{b}^\dagger\right)^2 - \left(\hat{b} - \hat{b}^\dagger\right)^2\right) \\ &= \frac{l_B^2}{2}\left(2\left[1 + \hat{b}^\dagger\hat{b}\right] + 2\hat{b}^\dagger\hat{b}\right) \\ &= \frac{l_B^2}{2}\left(4\hat{b}^\dagger\hat{b} + 2\right) \\ &= 2l_B^2\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) \end{aligned}$$

Then, for $\mathbf{R}$, we can write X and Y in terms of ladder operators as:

$$\hat{X} = \frac{l_B}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right), \quad \hat{Y} = -\frac{il_B}{\sqrt{2}}\left(\hat{a} - \hat{a}^\dagger\right) \tag{8}$$

Which will result in by the same calculation as above in:

$$\mathbf{R}^2 = 2l_B^2\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

Then:

$$\hat{L} = m\omega l_B^2\left(\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a}\right) = \hbar\left(\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a}\right)$$

with eigenvalues:

$$l_{mn} = \hbar\left(n - m\right)$$

f) We want to calculate the expectation values $\langle z|\hat{x}|z\rangle$ and $\langle z|\hat{y}|z\rangle$ for the coherent degenerate ground-state $|z\rangle$ that fullfills $\hat{a}|z\rangle = z|z\rangle$ and $\hat{b}|z\rangle = 0$. From $\hat{\mathbf{r}} = \hat{\mathbf{R}} - \hat{\rho}$, we get:

$$\hat{x} = \hat{X} - \hat{\rho}_x, \quad \hat{y} = \hat{Y} - \hat{\rho}_y$$

Using the expressions in (6) and (8):

$$\begin{aligned}\langle z|\hat{x}|z\rangle &= \langle z|\hat{X}|z\rangle - \langle z|\hat{\rho}_x|z\rangle \\ &= \langle z|\frac{l_B}{\sqrt{2}}\left(\hat{a}+\hat{a}^\dagger\right)|z\rangle - \langle z|\frac{l_B}{\sqrt{2}}\left(\hat{b}+\hat{b}^\dagger\right)|z\rangle \\ &= \frac{l_B}{\sqrt{2}}\left(\langle z|\hat{a}|z\rangle + \langle z|\hat{a}^\dagger|z\rangle - \langle z|\hat{b}|z\rangle + \langle z|\hat{b}^\dagger|z\rangle\right)\end{aligned}$$

The trick, is to let the let the hermitian conjugated operators act on the bras, and the regular operators on the kets such that $\hat{a}|z\rangle = z|z\rangle$, $\langle z|\hat{a}^\dagger = (\hat{a}|z\rangle)^\dagger = \langle z|z^*$ and $\hat{b}|z\rangle = 0$. We get:

$$\begin{aligned}\langle z|\hat{x}|z\rangle &= \frac{l_B}{\sqrt{2}}\left(\langle z|z|z\rangle + \langle z|z^*|z\rangle\right) \\ &= \frac{l_B}{\sqrt{2}}\left(z+z^*\right) \\ &= \sqrt{2}l_B\,\mathrm{Re}\,(z)\end{aligned}$$

Onto the next:

$$\begin{aligned}\langle z|\hat{y}|z\rangle &= \langle z|\hat{Y}|z\rangle - \langle z|\hat{\rho}_y|z\rangle \\ &= -\frac{il_B}{\sqrt{2}}\left(\langle z|\hat{a}|z\rangle - \langle z|\hat{a}^\dagger|z\rangle - \underbrace{\langle z|\hat{b}|z\rangle}_{=0} + \underbrace{\langle z|\hat{b}^\dagger|z\rangle}_{=0}\right) \\ &= -\frac{il_B}{\sqrt{2}}\left(z-z^*\right) \\ &= \sqrt{2}l_B\,\mathrm{Im}(z)\end{aligned}$$

Writing $|z\rangle$ in the $|m\rangle$ basis gives:

$$|z\rangle = \sum_m |m\rangle\langle m|z\rangle = e^{-\frac{1}{2}|z|^2}\sum_m \frac{z^m}{\sqrt{m!}}|m\rangle$$

This is gotten by use of equation 1.216 in the lecture notes. When considering how many states that fit in $|z\rangle$ in the lowest landau level for a give z, we use :

$$\langle\hat{x}\rangle^2 + \langle\hat{y}\rangle^2 = \langle\hat{r}\rangle^2 = 2l_B^2\left(\mathrm{Re}\,(z)^2 + \mathrm{Im}\,(z)^2\right) = 2l_B^2 \mid z\mid^2$$

This corresponds to a circle in the complex plane with radius $\sqrt{2}l_B \mid z \mid$ and an area

$$\begin{aligned}A &= \pi\left(\sqrt{2}l_B \mid z\mid\right)^2 \\ &= 2\pi l_B^2 \mid z\mid^2\end{aligned}$$

The state $|z\rangle$, corresponding to the edge of said circle, has an overlap with the $|m\rangle$ state :

$$\begin{aligned}|\langle m|z\rangle|^2 &= \mid e^{-\frac{1}{2}|z|^2}\sum_{m'}\frac{z^{m'}}{\sqrt{m'!}}\langle m|m'\rangle\mid^2 \\ &= e^{-|z|^2}\frac{z^{2m}}{m!} \\ &= e^{-|z|^2}\frac{\left(\mid z\mid^2\right)^m}{m!}\end{aligned} \tag{9}$$

To find the m state with maximum overlap with $|z\rangle$ we find:

$$\begin{aligned}\frac{d}{d\mid z\mid^2}\mid\langle m|z\rangle\mid^2 &= \frac{d}{d\mid z\mid^2}\left(e^{-|z|^2}\frac{\left(\mid z\mid^2\right)^m}{m!}\right)\\ &= -e^{-|z|^2}\frac{\left(\mid z\mid^2\right)^m}{m!}+e^{-|z|^2}\frac{m\left(\mid z\mid^2\right)^{m-1}}{m!}\\ &= \frac{e^{-|z|^2}\left(\mid z\mid^2\right)^{m-1}}{m!}\left(-\mid z\mid^2+m\right)\end{aligned}$$

Then we get that $m=\mid z\mid^2$. Since the overlap falls exponentially, we can up to a good approximation take the state with —z— to be in the pure state m, which means that if we restrict the available space to a circle with radius $r^2=2l_B^2|z|^2$ in the z plane we have that $2l_B^2|z|^2\leq r^2$ and we can restrict m to $2l_B^2 m\leq r^2$. As the area is proportional to r^2 we see that the number of states increases linearly with the available area in the complex plane. The density is

$$\rho=\frac{N}{A}=\frac{m}{2\pi l_B^2 m}=\frac{1}{2\pi l_B^2}$$

g) When introducing the electric field, we get an energy contribution:

$$H_E=-e\vec{E}\cdot\vec{r}=-eE\mathbf{x}$$

Quantizing this and using the relation $\hat{x}=\hat{X}-\hat{\rho}_x$, we get:

$$\hat{H}_E=-eE\left(\hat{X}-\hat{\rho}_x\right)=-\frac{l_B}{\sqrt{2}}eE\left(\hat{a}+\hat{a}^\dagger-\hat{b}-\hat{b}^\dagger\right)$$

Then the total hamiltonian is:

$$\hat{H}=\hat{H}_0+\hat{H}_E=\hbar\omega\left(\hat{b}^\dagger\hat{b}+\frac{1}{2}\right)-\frac{l_B}{\sqrt{2}}eE\left(\hat{a}+\hat{a}^\dagger-\hat{b}-\hat{b}^\dagger\right)$$

In order to only consider the lowest landau level, i.e $|m,0\rangle\equiv|m\rangle$, we need the "effective" hamiltonian for this level:

$$\hat{H}|m,0\rangle = \hbar\omega\left(\underbrace{\hat{b}^\dagger\hat{b}|m,0\rangle}_{=0}+\frac{1}{2}|m,0\rangle\right)-\frac{l_B}{\sqrt{2}}eE\left(\hat{a}|m,0\rangle+\hat{a}^\dagger|m,0\rangle-\underbrace{\hat{b}|m,0\rangle}_{=0}-\underbrace{\hat{b}^\dagger|m,0\rangle}_{=\sqrt{0+1}|m,1\rangle}\right)$$

We see that the last term isn't in the lowest Landau level, thus, we may neglect it. We're left with:

$$\begin{aligned}\hat{H}'|m,0\rangle &= \frac{1}{2}\hbar\omega|m,0\rangle-\frac{l_B}{\sqrt{2}}eE\left(\hat{a}+\hat{a}^\dagger\right)|m,0\rangle\\ \hat{H}' &= \frac{1}{2}\hbar\omega-\frac{l_B}{\sqrt{2}}eE\left(\hat{a}+\hat{a}^\dagger\right)\end{aligned}$$

In the Heisenberg picture we can find the time evolution of $\hat{a}$ and $\hat{a}^\dagger$ and get:

$$\begin{aligned}\hat{a}(t)|z\rangle &= \hat{\mathcal{U}}(t,0)\hat{a}\hat{\mathcal{U}}(0,t)|z\rangle\\ &= \hat{\mathcal{U}}(t,0)\hat{a}|z(t)\rangle\\ &= \hat{\mathcal{U}}(t,0)z(t)|z(t)\rangle\\ &= z(t)|z\rangle\end{aligned}$$

So:

$$\begin{aligned}
\hat{a}(t) &= \hat{\mathcal{U}}(t,0)\hat{a}\hat{\mathcal{U}}(0,t) \\
&= e^{-it\hat{H}'}\hat{a}e^{i\hbar\hat{H}'} \\
&= \hat{a} + \frac{it}{\hbar}\left[\hat{H}',\hat{a}\right] + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2\left[\hat{H}',\left[\hat{H}',\hat{a}\right]\right] + \cdots
\end{aligned}$$

Calculating the commutator:

$$\begin{aligned}
\left[\hat{H}',\hat{a}\right] &= \left[\frac{1}{2}\hbar\omega - \frac{l_B}{\sqrt{2}}eE\left(\hat{a}+\hat{a}^\dagger\right),\hat{a}\right] \\
&= -\frac{l_B}{\sqrt{2}}eE\left[\hat{a}+\hat{a}^\dagger,\hat{a}\right] \\
&= -\frac{l_B}{\sqrt{2}}eE\left(\underbrace{[\hat{a},\hat{a}]}_{=0} + \underbrace{\left[\hat{a}^\dagger,\hat{a}\right]}_{=-1}\right) \\
&= \frac{l_B}{\sqrt{2}}eE
\end{aligned}$$

Since all operators commute with a scalar, and the “higher order” commutators vanish:

$$\hat{a}(t) = \hat{a} + \frac{itl_B}{\hbar\sqrt{2}}eE$$

Then:

$$\begin{aligned}
\hat{a}(t)|z\rangle &= \left(z + \frac{itl_B}{\sqrt{2}}eE\right)|z\rangle \\
z(t) &= z + \frac{itl_B}{\sqrt{2}}eE
\end{aligned}$$

which shows that $|z\rangle$ gets a time dependence. In order to show that this corresponds to a drift in the y-direction, let’s consider:

$$\hat{X}(t) = \frac{l_B}{\sqrt{2}}\left(\hat{a}(t)+\hat{a}^\dagger(t)\right), \quad \text{and} \quad \hat{Y}(t) = -\frac{il_B}{\sqrt{2}}\left(\hat{a}(t)-\hat{a}^\dagger(t)\right)$$

Where $\hat{a}^\dagger(t)$ is:

$$\begin{aligned}
(\hat{a}(t))^\dagger &= (\hat{a} + \frac{itl_B}{\hbar\sqrt{2}}eE)^\dagger \\
\hat{a}^\dagger(t) &= \hat{a}^\dagger - \frac{itl_B}{\hbar\sqrt{2}}eE
\end{aligned}$$

Then:

$$\hat{X}(t) = \frac{l_B}{\sqrt{2}}\left(\hat{a}(t)+\hat{a}^\dagger(t)\right) = \frac{l_B}{\sqrt{2}}\left(\hat{a} + \frac{itl_B}{\hbar\sqrt{2}}eE + \hat{a}^\dagger - \frac{itl_B}{\hbar\sqrt{2}}eE\right) = \hat{X}$$

No drift in the $\hat{X}$ direction, onto $\hat{Y}$:

$$\begin{aligned}\hat{Y}(t) &= -\frac{il_B}{\sqrt{2}}\left(\hat{a} + \frac{itl_B}{\hbar\sqrt{2}}eE - \hat{a}^\dagger + \frac{itl_B}{\hbar\sqrt{2}}eE\right) \\ &= \hat{Y}(0) + \frac{l_B^2}{\hbar}eEt = \hat{Y}(0) + \frac{e}{\mid eB \mid}Et = \hat{Y}(0) - \frac{E}{\mid B \mid}t\end{aligned}$$

Then the velocity is:

$$v_{\text{drift}} = -\frac{E}{\mid B \mid}$$

in the y direction, as $\hat{Y}(t)$ is the movement of the guiding center (center of orbit).