UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in:	**FYS4110/9110 Modern Quantum Mechanics**
Day of exam:	**8. December 2022**
Exam hours:	**15.00-19.00, 4 hours**
This examination paper consists of 3 pages	
Permitted materials:	**Approved electronic calculator.**
	Rottmann: Matematisk formelsamling
	One sheet (2 pages) A4 paper of notes

Language: The solutions may be written in Norwegian or English depending on your own preference.
Make sure that your copy of this examination paper is complete before answering.
All answers should be justified

PROBLEM 1
Approximate quantum cloning

The no-cloning theorem tells us that it is impossible to copy an unknown quantum state. In this problem we will study a protocol which takes the quantum state of a two-level system and produces two two-level systems with the state of both approximating as well as possible the original state.

We consider three two-level systems. The first (system A) is the original to be copied, the second (system B) is the system that will receive a copy of the state, and the third (system C) is an auxiliary system (often called an ancilla). We label the states in the usual way, $|000\rangle = |0\rangle_A|0\rangle_B|0\rangle_C = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C$ and similar for the other basis states. We define a unitary operation through the equations

$$U|000\rangle = \sqrt{\frac{2}{3}}|000\rangle + \sqrt{\frac{1}{6}}\left(|011\rangle + |101\rangle\right) \tag{1}$$

$$U|100\rangle = \sqrt{\frac{2}{3}}|111\rangle + \sqrt{\frac{1}{6}}\left(|010\rangle + |100\rangle\right) \tag{2}$$

Let the initial state of system A be $|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$, and apply the above operation to the system if the inital state of B and C is $|00\rangle_{BC}$.

a) Calculate the reduced density matrices ρ_A and ρ_B of systems A and B.

b) Determine the Bloch vector of both the final states ρ_A and ρ_B and find how they are related to the Bloch vector of the initial state $|\psi\rangle$.

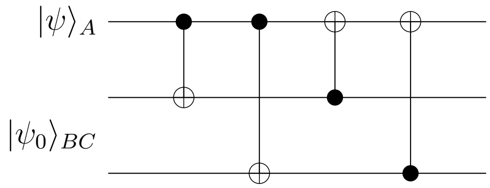
c) We define the fidelity of the copying operation as the overlap of the copied state with the original

$$F = \langle\psi|\rho_B|\psi\rangle.$$

Calculate the fidelity for this operation.

d) The relations (1) and (2) do not define the operation U completely, and it is necessary to show that it can be completed as a unitary operation on all the basis vactors. We can do this by demonstrating that it is produced by a quantum circuit.

Show that the following quantum circuit will implement the unitary operation U on the required input states.



The systems B and C must initially be prepared in the state

$$|\psi_0\rangle_{BC} = \sqrt{\frac{2}{3}}|00\rangle_{BC} + \sqrt{\frac{1}{6}}\left(|01\rangle_{BC} + |11\rangle_{BC}\right).$$

There is a simple circuit to do this step also, starting from the state $|00\rangle_{BC}$, but for the exam we just assume that it has been prepared.

PROBLEM 2
Lindblad equation for pure dephasing

We are interested in studying a two level system subject to pure phase noise. That is, the interaction with the environment does not induce any transitions between the eigenstates of the system. This can be described by a Lindblad equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] - \frac{\gamma}{2}(L^\dagger L\rho + \rho L^\dagger L - 2L\rho L^\dagger)$$

with one Lindblad operator $L = \sigma_z$. The Hamiltonian is

$$H = \frac{1}{2}\hbar\omega_0\sigma_z.$$

a) Solve the Lindblad equation and find the components of the Bloch vector as functions of time for a general initial state. Describe the motion of the Bloch vector.

b) Find an expression for the entropy as a function of time. Sketch the form of the entropy as a function of time and relate the form of the curve to the trajectory of the Bloch vector.

PROBLEM 3
Absolutely maximally entangled states

We start by studying a quantum system that consists of two subsystems, which we cal system A and system B.

a) A product state has a density matrix of the form $\rho = \rho_A \otimes \rho_B$. Show that the entropy of this state is the sum of the individual entropies, $S = S_A + S_B$.

b) We now assume that the Hilbert spaces of the two subsystems have dimensions n_A and n_B. If we have the total system in some pure state $|\psi\rangle$, what is the maximal entanglement entropy that can exist between the two subsystems? You should demonstrate your result, not just state the answer.

Consider the following state of four three-level systems

$$|\psi\rangle = \frac{1}{3} \sum_{i,j=0,1,2} |i\rangle|j\rangle|i+j\rangle|i+2j\rangle$$

where all addtions of the indices i and j are to be taken mod(3). We now select any two of the three-level systems as system A and the remaining two as system B.

c) Calculate the reduced density matrix for all possible divisions of the system i two halves (all possible combinations of two three-level systems in subsystem A) and show that the entanglement entropy is maximal in all cases.

d) From the result of the previous question, what can we say about the entanglement entropy between any of the three-level systems and the remaining three? What can we say about the entanglement between the two three-level systems that constitute subsystem A?