FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2022

Return of solutions:
The problem set is available from Friday morning, 30 September.
You may submit handwritten solutions, but they have to be scanned and included in one single file, which is submitted in Inspera before Friday, 7 October, at 14:00.

Language:

Solutions may be written in Norwegian or English depending on your preference.

Questions concerning the problems:

Please ask Joakim Bergli or Maria Markova.

Problem 1: Supersymmetric quantum mechanics

In this problem we will study a construction that is known as supersymmetric quantum mechanics (SUSYQM). For the harmonic oscillator, we can construct the operators $\hat{a}$ and $\hat{a}^\dagger$ that allows us to find the energy spectrum algabraically without explicitly solving the Schödinger equation. SUSYQM will allow us a similar construction for certain other potentials.

We study the motion of a particle in a potential $V_-(x)$ with the Hamiltonian in the position basis (we use units where $\hbar = 1$)

$$H_- = \frac{p^2}{2m} + V_-(x) = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V_-(x).$$

Assume that there exists a function $W(x)$ such that the operators

$$A = \frac{ip}{\sqrt{2m}} + W(x)$$
$$A^\dagger = -\frac{ip}{\sqrt{2m}} + W(x)$$

factorize the Hamiltonian so that $H_- = A^\dagger A$. The function $W(x)$ is known as the superpotential.

a) Show that $W(x)$ must satisfy the equation

$$W^2 - \frac{1}{\sqrt{2m}}\frac{dW}{dx} = V_-.$$

We define the partner Hamiltonian to H_- as

$$H_+ = AA^\dagger = \frac{p^2}{2m} + V_+,$$

where V_+ is called the partner potential to V_-. We now define an extended system with the Hamiltonian

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \tag{1}$$

and the two operators

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \qquad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}.$$

b) Show that we have the following supersymmetry algebra (which is identical to Eq. (1.233) in the lecture notes)

$$\begin{aligned} \left\{Q, Q^\dagger\right\} &= H \\ [Q, H] &= \left[Q^\dagger, H\right] = 0 \\ \{Q, Q\} &= \left\{Q^\dagger, Q^\dagger\right\} = 0. \end{aligned}$$

For this reason we say that the extended system is supersymmetric.

c) Explain what is the difference between the extended system described by the Hamiltonian (1) and a system of two particles, one in the potential V_- and one on the potential V_+.

d) Show that the ground state energy of H is nonnegative. That is, if $H|\Psi_0\rangle = E_0|\Psi_0\rangle$ then $E_0 \geq 0$.

If $E_0 = 0$ we say that SUSY is unbroken. If $E_0 > 0$ we have broken SUSY. The spectra of the partner Hamiltonians are related.

e) Show that we have the intertwinig relations

$$AH_- = H_+A, \qquad A^\dagger H_+ = H_-A^\dagger.$$

f) Show that this implies that if we have an eigenstate $|\psi_n^-\rangle$ of H_- with eigenvalue E_n^- then $A|\psi_n^-\rangle$ is an eigenstate of H_+ with the same eigenvalue.

The partner Hamiltonians H_- and H_+ therefore have the same eigenvalues and are called essentially isospectral. The only exception to this is the ground state of one of the Hamiltonians for a system with unbroken SUSY. Let us assume that this is the case for H_-, in which case we can choose the ground state of H on the form

$$|\Psi_0\rangle = \begin{pmatrix} |\psi_0^-\rangle \\ 0 \end{pmatrix}$$

g) Show that if we are to have unbroken SYSY with this ground state we must have $A|\psi_0^-\rangle = 0$ so that there is no corresponding eigenstate of H_+.

h) Show that we have the relation

$$|\psi_n^+\rangle = \frac{A}{\sqrt{E_n^+}}|\psi_{n+1}^-\rangle$$

between the eigenstates of H_+ and H_- with the eigenvalues related by

$$E_{n-1}^+ = E_n^- .$$

i) Show that the ground state wave function is then determined by the equation

$$\psi_0^-(x) = \langle x|\psi_0^-\rangle = Ne^{-\sqrt{2m}\int_0^x W(x')dx'} .$$

We will now study the superpotential

$$W(x) = \begin{cases} \infty & x < 0 \\ -\frac{b}{\tan x} & 0 \leq x \leq \pi \\ -\infty & x > \pi \end{cases} \tag{2}$$

j) Show that for a specific choice of b, the partner potentials take the form

$$V_- = -\frac{1}{2m}$$
$$V_+ = -\frac{1}{2m} + \frac{1}{m\sin^2 x}$$

on the interval $0 \leq x \leq \pi$ with both potentials being ∞ outside this interval. Specify the value of b that gives these potentials.

k) The potential V_- corresponds to the well known case of a particle in an infinite square well. Either solve this problem or write down the solution (eigenstates and eigenvalues) from your favorite textbook.

l) Use the relations between V_- and V_+ derived above to determine the eigenvalues and eigenstates of H_+.

The above calculation gives an example where we could determine exact analytic expressions for the eigenstates and eigenvalues of the potential V_+ from the fact that we knew the corresponding results for the simpler potential V_-. In some cases we can do even better and determine the full spectrum and eigenstates directly. Assume that we have a family of partner potentials $V_\pm(a_o, x)$ that depends on some real parameter a_0. This family is called shape invariant if there exist real functions $a_1 = f(a_0)$ and $g(a_0)$ such that

$$V_+(a_0, x) + g(a_0) = V_-(a_1, x) + g(a_1)$$

with the corresponding relation

$$H_+(a_0, x) + g(a_0) = H_-(a_1, x) + g(a_1)$$

for the Hamiltonians.

m) Show that for a shape invariant superpotential, the Hamiltonians $H_+(a_0, x)$ and $H_-(a_1, x)$ have the same eigenstates, and that the energy eigenvalues are related by

$$E_n^+(a_0) = E_n^-(a_1) + g(a_1) - g(a_0).$$

n) Assume that SUSY is unbroken for all a_n generated recursively from a_0 by $a_{n+1} = f(a_n)$ and that $E_0^-(a_n) = 0$. Show that the energy spectrum of $H_-(a_0)$ is given by

$$E_n^-(a_0) = g(a_n) - g(a_0). \tag{3}$$

o) Show that the corrsponding energy eigenstates are

$$|\psi_n^-(a_0)\rangle = \frac{A^\dagger(a_0)}{\sqrt{E_{n-1}^+(a_0)}} \cdots \frac{A^\dagger(a_{n-2})}{\sqrt{E_1^+(a_{n-2})}} \frac{A^\dagger(a_{n-1})}{\sqrt{E_0^+(a_{n-1})}} |\psi_0^-(a_n)\rangle \tag{4}$$

We now return to the superpotential (2) that we studied above. To simplify the expressions, we will choose the mass so that $\sqrt{2m} = 1$.

p) Show that we have

$$V_+(b, x) = V_-(b+1, x) + (b+1)^2 - b^2.$$

Use this to show that we can generate a family of shape invariant potentials by choosing the initial parameter value a_0 to be any b, and determine the associated functions $f(b)$ and $g(b)$.

q) Choosing the starting value $a_0 = 1$ corresponds to the infinite square well. Using the relation (3), find the energy eigenvalues of this system. Use (4) to find the wave functions for the two lowest energy eigenstates. Admittedly, this is a very complicated way to calculate something that we know in advance how to derive in a simple way.