FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2022. Solution

Problem 1: Supersymmetric quantum mechanics

a) We have

$$A^\dagger A = \left(-\frac{ip}{\sqrt{2m}} + W(x)\right)\left(\frac{ip}{\sqrt{2m}} + W(x)\right) = \frac{p^2}{2m} - \frac{i}{\sqrt{2m}}[p, W(x)] + W^2.$$

Using (remember that the derivative should act on all functions to the right)

$$\frac{\partial}{\partial x}W = \frac{\partial W}{\partial x} + W\frac{\partial}{\partial x}$$

we get

$$[p, W(x)] = -i\frac{\partial}{\partial x}W + iW\frac{\partial}{\partial x} = -i\frac{\partial W}{\partial x} -$$

Thus we have

$$W^2 - \frac{1}{\sqrt{2m}}\frac{dW}{dx} = V_-.$$

b) This is just multiplying matrices.

c) For a system of two particles, the total Hilbert space would be the tensor product of the individual Hilbert spaces. In this case, the Hamiltonian is the direct sum of the individual Hamiltonians, and corresponds to a single particle that is confined in one of the two potentials with no amplitude for tunneling between them.

d) The ground state energy is

$$E_0 = \langle\Psi_0|H|\Psi_0\rangle = \langle\Psi_0|\left\{Q, Q^\dagger\right\}|\Psi_0\rangle = \langle\Psi_0|QQ^\dagger|\Psi_0\rangle + \langle\Psi_0|Q^\dagger Q|\Psi_0\rangle = |Q^\dagger|\Psi_0\rangle|^2 + |Q|\Psi_0\rangle|^2 \geq 0$$

e)

$$AH_- = AA^\dagger A = H_+ A,$$
$$A^\dagger H_+ = A^\dagger A A^\dagger = H_- A^\dagger.$$

f)

$$H_+ A|\psi_n^-\rangle = AH_-|\psi_n^-\rangle = E_n^- A|\psi_n^-\rangle.$$

g) For unbroken SUSY we have

$$H|\Psi_0\rangle = H_-|\psi_0^-\rangle = 0.$$

This implies that

$$\langle\psi_0^-|H_-|\psi_0^-\rangle = \langle\psi_0^-|A^\dagger A|\psi_0^-\rangle = |A|\psi_0^-\rangle|^2 = 0$$

which means that $A|\psi_0^-\rangle = 0$.

h) We order the eigenstates $|\psi_0^\pm\rangle$ according to increasing energy, and we know that the ground state $|\psi_0^-\rangle$ of H_- does not have a corresponding eigenstate of H_+ while all the other states do. So $E_{n-1}^+ = E_n^-$ and

$$|\psi_n^+\rangle = NA|\psi_{n+1}^-\rangle$$

with some normalization N. To determine this we calculate

$$1 = \langle\psi_n^+|\psi_n^+\rangle = N^2\langle\psi_{n+1}^-|A^\dagger A|\psi_{n+1}^-\rangle = N^2 E_{n+1}^-$$

which gives $N^2 = 1/E_{n+1}^- = 1/E_n^+$ and we get

$$|\psi_n^+\rangle = \frac{A}{\sqrt{E_n^+}}|\psi_{n+1}^-\rangle. \tag{1}$$

i) We know that $A|\psi_0^-\rangle = 0$, which in the position basis takes the form

$$\left[\frac{1}{\sqrt{2m}}\frac{\partial}{\partial x} + W\right]\psi_0^-(x) = 0.$$

Solving this differential equation gives

$$\psi_0^-(x) = Ne^{-\sqrt{2m}\int_0^x W(x')dx'}.$$

j)

$$V_\pm = W^2 \pm \frac{1}{\sqrt{2m}}\frac{\partial W}{\partial x} = b^2\frac{\cos^2 x}{\sin^2 x} \pm \frac{b}{\sqrt{2m}}\frac{1}{\sin^2 x} = -b^2 + b\left(b \pm \frac{1}{\sqrt{2m}}\right)\frac{1}{\sin^2 x}.$$

If we choose $b = \frac{1}{\sqrt{2m}}$ we get

$$V_- = -\frac{1}{2m}$$
$$V_+ = -\frac{1}{2m} + \frac{1}{m\sin^2 x}$$

on the interval $0 \le x \le \pi$ with both potentials being ∞ outside this interval.

k) Normally the potential is 0 at the bottom of the well and the eigenstates are written as $\sqrt{2/\pi}\sin nx$ with $n = 1, 2, \ldots$ with the eigenvalues $n^2/2m$. Since we start numbering from $n = 0$ we write

$$\psi_n^-(x) = \sqrt{\frac{2}{\pi}}\sin(n+1)x$$

with

$$E_n^- = \frac{(n+1)^2 - 1}{2m}.$$

where we have subtracted the $1/2m$ energy at the bottom of the potential.

l) To simplify the expresions, we define $n' = n+1$. We have

$$A\psi^-_{n'}(x) = \frac{1}{\sqrt{2m}}\left[\frac{\partial}{\partial x} - \frac{1}{\tan x}\right]\psi^-_{n'}(x) = \frac{1}{\sqrt{\pi m}}\left[n'\cos n'x - \frac{\sin n'x}{\tan x}\right].$$

With

$$E^+_n = E^-_{n'} = \frac{(n'+1)^2 - 1}{2m}$$

we get

$$\psi^+_n(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{(n'+1)^2-1}}\left[n'\cos n'x - \frac{\sin n'x}{\tan x}\right].$$

m) Since

$$H_+(a_0, x) - H_-(a_1, x) = g(a_1) - g(a_0)$$

is a number, the two Hamiltonians will have the same eigenfuntions and the difference between the eigenvalues is the same number so that

$$E^+_n(a_0) = E^-_n(a_1) + g(a_1) - g(a_0).$$

n) If SUSY is unbroken for all n we have $E^-_0(a_n) = 0$. From shape invariance(SI) we have

$$E^+_0(a_0) = E^-_0(a_1) + g(a_1) - g(a_0) = g(a_1) - g(a_0).$$

From SUSY we know that

$$E^-_1(a_0) = E^+_0(a_0) = g(a_1) - g(a_0).$$

We can repeat the same process

$$\begin{aligned} E^-_2(a_0) &\overset{SUSY}{=} E^+_1(a_0) \overset{SI}{=} E^-_1(a_1) + g(a_1) - g(a_0) \\ &\overset{SUSY}{=} E^+_0(a_1) + g(a_1) - g(a_0) \overset{SI}{=} E^-_0(a_2) + g(a_2) - g(a_1) + g(a_1) - g(a_0) = g(a_2) - g(a_0). \end{aligned}$$

The same continues for higher levels so that

$$E^-_n(a_0) = g(a_n) - g(a_0).$$

o) From Eq. (1) we get

$$A^\dagger|\psi^+_n\rangle = \frac{A^\dagger A}{\sqrt{E^+_n}}|\psi^-_{n+1}\rangle = \frac{H_-}{\sqrt{E^+_n}}|\psi^-_{n+1}\rangle = \frac{E^-_{n+1}}{\sqrt{E^+_n}}|\psi^-_{n+1}\rangle = \sqrt{E^+_n}|\psi^-_{n+1}\rangle$$

which means

$$|\psi^-_n(a_0)\rangle = \frac{A^\dagger(a_0)}{\sqrt{E^+_{n-1}(a_0)}}|\psi^+_{n-1}(a_0)\rangle.$$

We then have

$$|\psi_1^-(a_0)\rangle = \frac{A^\dagger(a_0)}{\sqrt{E_0^+(a_0)}}|\psi_0^+(a_0)\rangle \overset{SI}{=} \frac{A^\dagger(a_0)}{\sqrt{E_0^+(a_0)}}|\psi_0^-(a_1)\rangle$$

$$\begin{aligned}|\psi_2^-(a_0)\rangle &= \frac{A^\dagger(a_0)}{\sqrt{E_1^+(a_0)}}|\psi_1^+(a_0)\rangle \overset{SI}{=} \frac{A^\dagger(a_0)}{\sqrt{E_1^+(a_0)}}|\psi_1^-(a_1)\rangle \\ &= \frac{A^\dagger(a_0)}{\sqrt{E_1^+(a_0)}}\frac{A^\dagger(a_1)}{\sqrt{E_0^+(a_1)}}|\psi_0^+(a_1)\rangle \overset{SI}{=} \frac{A^\dagger(a_0)}{\sqrt{E_1^+(a_0)}}\frac{A^\dagger(a_1)}{\sqrt{E_0^+(a_1)}}|\psi_0^-(a_2)\rangle.\end{aligned}$$

Repeating this procedure we get

$$|\psi_n^-(a_0)\rangle = \frac{A^\dagger(a_0)}{\sqrt{E_{n-1}^+(a_0)}}\cdots\frac{A^\dagger(a_{n-2})}{\sqrt{E_1^+(a_{n-2})}}\frac{A^\dagger(a_{n-1})}{\sqrt{E_0^+(a_{n-1})}}|\psi_0^-(a_n)\rangle$$

p) With $\sqrt{2m} = 1$ we have

$$V_\pm(b,x) = -b^2 + b(b \pm 1)\frac{1}{\sin^2 x}.$$

This means that

$$V_+(b,x) = -b^2+b(b+1)\frac{1}{\sin^2 x} = -b^2+(b+1)^2-(b+1)^2+(b+1)(b+1-1)\frac{1}{\sin^2 x} = V_(b+1,x)+(b+1)^2-b^2.$$

If we choose the funtions

$$f(b) = b+1 \qquad g(b) = b^2$$

we satisfy the conditions for shape invariance.

q) Choosing the value $b = 1$ corresponds to the infinite square well for $V_-(1,x)$. We choose $a_0 = 1$ and get $a_n = a_{n-1} + 1 = n + 1$. This means that $g(a_n) = (n+1)^2$. Using (**??**) this gives

$$E_n^-(1) = (n+1)^2 - 1$$

which are the energy eigenvalues if $\sqrt{2m} = 1$. The wavefunctions can b edetermined since we know that

$$\psi_0^-(a_n, x) = Ne^{-\int_0^x W(a_n,x')dx'}$$

We need the integral

$$-\int W(a_n, x')dx' = a_n \int \frac{dx}{\tan x} = a_n \ln|\sin x| + C$$

where C is the integration constant. This gives that up to normalization we have

$$\psi_0^-(1,x) = N \sin x.$$

We can also find

$$\psi_1^-(1,x) = NA^\dagger(a_0)e^{-\int_0^x W(a_1,x')dx'} = NA^\dagger(1)e^{-\int_0^x W(2,x')dx'}\left(\frac{\partial}{\partial x} - \frac{1}{\tan x}\right)e^{2\ln|\sin x|} = N\sin 2x.$$