

# Problem set 1

We begin the weekly sets with some problems concerning basic and useful mathematical relations.

## 1.1 Commutators and anti-commutators

We use the standard notation for commutators and anticommutators

$$[A, B] = AB - BA \quad \{A, B\} = AB + BA \quad (1)$$

where  $A$  and  $B$  are two operators or matrices. Show the following identities,

$$\begin{aligned} [A, BC] &= [A, B]C + B[A, C] \\ [A, BC] &= \{A, B\}C - B\{A, C\} \end{aligned} \quad (2)$$

## 1.2 Trace and determinant

We remind you about the following relations

$$\text{Tr}(AB) = \text{Tr}(BA), \quad \det(AB) = \det A \det B \quad (3)$$

- a) Assume  $\hat{A}$  to be the operator for a quantum observable and  $A$  to be the matrix representation of this operator in an orthonormalized basis  $\{|n\rangle\}$ , which means

$$A_{mn} = \langle m|\hat{A}|n\rangle \quad (4)$$

We define the trace and determinant of the (abstract) operator as

$$\text{Tr } \hat{A} = \text{Tr } A, \quad \det \hat{A} = \det A \quad (5)$$

Show that if we change to a new basis  $\{|n'\rangle\}$ , which is related to the first by a unitary transformation, that will not change the values of the trace and determinant.

- b) Assume  $\hat{A}$  is a hermitian operator with eigenvalues  $a_n, n = 1, 2, \dots$ . Explain why the trace and determinant can be expressed in terms of the eigenvalues as

$$\text{Tr } \hat{A} = \sum_n a_n \quad \det \hat{A} = \prod_n a_n \quad (6)$$

- c) The *spectral decomposition* of an hermitian operator  $\hat{A}$  is a sum of the form

$$\hat{A} = \sum_n a_n |n\rangle\langle n| \quad (7)$$

where  $a_n$  are the eigenvalues and  $|n\rangle$  are the corresponding eigenvectors of the operator. A function  $f(a)$  defines an *operator function*  $\hat{f} \equiv f(\hat{A})$  of  $\hat{A}$  by the related decomposition

$$\hat{f} \equiv \sum_n f(a_n) |n\rangle\langle n| \quad (8)$$

Use this definition and the results of problem b) to show that we have the following relation

$$\det e^{\hat{A}} = e^{\text{Tr} \hat{A}} \quad (9)$$

We assume the trace of  $\hat{A}$  to be well defined and finite (which may not necessarily be the case in an infinite dimensional Hilbert space).

d) Show that for general state vectors  $|\psi\rangle$  and  $|\phi\rangle$  we have the relation

$$\langle\psi|\phi\rangle = \text{Tr}(|\phi\rangle\langle\psi|) \quad (10)$$

### 1.3 Dirac's delta function

The basic relation defining the delta functions is the following

$$f(x) = \int_{-\infty}^{\infty} dx' \delta(x - x') f(x') \quad (11)$$

with  $f(x)$  as any chosen function. Clearly  $\delta(x)$  is not a function in the usual sense, and in particular it has the property that  $\delta(x) = 0$  for  $x \neq 0$  and  $\delta(0) = \infty$ . Nevertheless it is possible (with some care) to treat it as a function, and as we know from the wavefunction description of quantum physics it is in many cases a very useful concept.

We remind you about the formulas for Fourier transformation in one dimension

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{f}(k) e^{ikx} \quad (12)$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ikx} \quad (13)$$

a) Show that the delta function has the following Fourier expansion

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \quad (14)$$

b) Assume  $g(x)$  is a differentiable function with zero at one point  $x_0$ ,

$$g(x_0) = 0 \quad (15)$$

Assume also that the derivative does not vanish at this point,  $g'(x_0) \neq 0$ . Show by use of the definition (11), and by studying the integral  $\int dx \delta(g(x)) f(x)$ , that we have the following relation

$$\delta(g(x)) = \frac{1}{|g'(x_0)|} \delta(x - x_0) \quad (16)$$

(Hint, make change of variable  $x \rightarrow g$  in the integral.) Assume that the function  $g(x)$  has several zeros, at the points  $x = x_i$ . Explain why this gives the generalized formula

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \delta(x - x_i) \quad (17)$$

### 1.4 Position and momentum eigenstates

The position and momentum eigenstates are given by the relations

$$\hat{x}|x\rangle = x|x\rangle \quad \langle x|x'\rangle = \delta(x - x') \quad \int dx |x\rangle\langle x| = \mathbb{1} \quad (18)$$

$$\hat{p}|p\rangle = p|p\rangle \quad \langle p|p'\rangle = \delta(p - p') \quad \int dp |p\rangle\langle p| = \mathbb{1} \quad (19)$$

Furthermore, in the x-representation the momentum operator is given by  $\hat{p} = -i\hbar \frac{d}{dx}$ . Use these relations together with the Fourier expansion of the delta function to show that the scalar product of a momentum and a position state is given by

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}xp} \quad (20)$$

### 1.5 Some operator expansions

Assume  $\hat{A}$  and  $\hat{B}$  to be two operators, generally not commuting.

We define the following two composite operators:

$$\hat{F}(\lambda) = e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}, \quad \hat{G}(\lambda) = e^{\lambda\hat{A}}e^{\lambda\hat{B}} \quad (21)$$

a) Show the following relation

$$\frac{d\hat{F}}{d\lambda} = [\hat{A}, \hat{F}] \quad (22)$$

and use it to derive the expansion

$$\hat{F}(\lambda) = \hat{B} + \lambda [\hat{A}, \hat{B}] + \frac{\lambda^2}{2} [\hat{A}, [\hat{A}, \hat{B}]] + \dots \quad (23)$$

b) Show the following relation between  $\hat{G}(\lambda)$  and  $\hat{F}(\lambda)$ ,

$$\frac{d\hat{G}}{d\lambda} = (\hat{A} + \hat{F})\hat{G} \quad (24)$$

and use this to demonstrate the following expansion (Campbell-Baker-Hausdorff)

$$\hat{G}(\lambda) = e^{\lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}[\hat{A}, \hat{B}] + \dots} \quad (25)$$

by calculating the exponent on the right-hand side to second order in  $\lambda$ .

c) When  $[\hat{A}, \hat{B}]$  commutes with both  $\hat{A}$  and  $\hat{B}$  the expression (25) is exact without the higher order terms indicated by ... in (25). Verify this by use of (23) and (24), and by noting that the eigenvalues of  $\hat{G}$  satisfy a differential equation that can be integrated.

## 1.6 Spin operators and Pauli matrices

A spin half operator  $\hat{\mathbf{S}}$  is defined in the standard way as

$$\hat{\mathbf{S}} = \frac{\hbar}{2} \boldsymbol{\sigma} \quad (26)$$

where  $\boldsymbol{\sigma}$  is a vector with the three Pauli matrices  $(\sigma_1, \sigma_2, \sigma_3)$  (or equivalently written as  $(\sigma_x, \sigma_y, \sigma_z)$ ) as Cartesian components. We use the standard expressions for these 2x2 matrices, as given in the lecture notes. We also introduce the rotated Pauli matrix, defined by  $\sigma_{\mathbf{n}} = \mathbf{n} \cdot \boldsymbol{\sigma}$ , where  $\mathbf{n}$  is an unspecified three dimensional unit vector.

- a) Show that  $\sigma_{\mathbf{n}}$  has eigenvalues  $\pm 1$ , and the eigenstate (in matrix form) corresponding to the eigenvalue  $+1$  is (up to an arbitrary phase factor)

$$\Psi_{\mathbf{n}} = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} \quad (27)$$

with  $(\theta, \phi)$  as the polar angles of the unit vector  $\mathbf{n}$ . Also show the relation

$$\Psi_{\mathbf{n}}^\dagger \boldsymbol{\sigma} \Psi_{\mathbf{n}} = \left( \Psi_{\mathbf{n}}^\dagger \sigma_x \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_y \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_z \Psi_{\mathbf{n}} \right) = \mathbf{n} \quad (28)$$

- b) Show, by using the operator identity

$$e^{\lambda \hat{A}} \hat{B} e^{-\lambda \hat{A}} = \hat{B} + \lambda [\hat{A}, \hat{B}] + \frac{\lambda^2}{2} [\hat{A}, [\hat{A}, \hat{B}]] \dots,$$

the following relation

$$e^{-\frac{i}{2}\alpha\sigma_z} \sigma_x e^{\frac{i}{2}\alpha\sigma_z} = \cos \alpha \sigma_x + \sin \alpha \sigma_y \quad (29)$$

Explain why this shows that the unitary matrix

$$\hat{U} = e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = e^{-\frac{i}{\hbar}\alpha\mathbf{n}\cdot\hat{\mathbf{S}}} \quad (30)$$

induces a spin rotation of angle  $\alpha$  about the axis  $\mathbf{n}$ .

- c) Demonstrate, by expansion of the exponential function, the following identity

$$e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \cos \frac{\alpha}{2} \mathbb{1} - i \sin \frac{\alpha}{2} \sigma_{\mathbf{n}} \quad (31)$$

with  $\mathbb{1}$  as the 2x2 identity matrix.