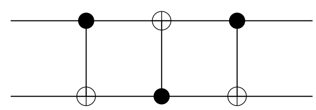
Problem set 10

10.1 SWAP gate

a) A useful quantum gate is the SWAP gate which interchanges the state of two qubits. That is, if the input state is $|\psi\rangle \otimes |\phi\rangle$ the output will be $SWAP|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$. Show that the following quantum circuit on two qubits gives a decomposition of the SWAP gate using three CNOT gates.



b) Write the matrix for the SWAP gate. Describe which basis you use.

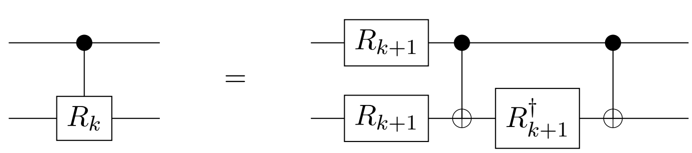
c) Generalize the above circuit to implement the SWAP gate on two n-qubit registers using only CNOT gates. How many CNOT gates do you need?

10.2 Quantum circuit for controlled R_k

a) In the quantum Fourier transformation, we needed to perform a controlled R_k operation. The one-qubit operator

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

is then performed on the target qubit if the control qubit is in the state $|1\rangle$. When the control qubit is in the state $|0\rangle$ no operation is performed on the target qubit. We know that all two-qubit operators can be decomposed in single qubit operators and controlled NOT (CNOT) operations. Show that the following quantum circuit is one such decompostion for the controlled R_k operation



b) We consider now general controlled U operations, with U a one-qubit operator. This means that the operation U is performed on the target qubit if the control qubit is in the state $|1\rangle$. When the control qubit is in the state $|0\rangle$ no operation is performed on the target qubit. In both cases, the control qubit is not changed. If this was a classical system, this would be all the possibilities, but in a quantum system, one can have a control qubit that is in a superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ of the two basis states. In general, the two qubits will be entangled by this operation, so no definite quantum state can be ascribed to any of them. However, a special situation arises if the initial state of the target qubit is an eigenstate of U. Draw a quantum circuit desribing this situation. Show that in this case, the two qubits are not entangled by the operation. Show also that in this case, it is the target qubit that is not changed, while the state of the control qubit is changed. Find the final state of the control qubit in terms of the eigenvalues of U.

c) This result is surprising if we only are used to the classical world, and deserves an explanation. Explain in words why the target is not changed while the state of the control does change.

10.3 Midterm exam 2017, Problem 2 a) to d) (or more if you have time)