Problem set 13

13.1 A radiation problem (Exam 2011)

We consider a one-dimensional problem where a two-level system (A) interacts with a scalar radiation field (B). The notation we use is similar to that in Problem 9.2. The Hamiltonian of the system we consider is

$$\hat{H} = \frac{1}{2}\hbar\omega_A\,\sigma_z + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + \kappa \sum_k \sqrt{\frac{\hbar}{2L\omega_k}}(\hat{a}_k\sigma_+ + \hat{a}_k^\dagger\sigma_-) = \hat{H}_0 + \hat{H}_{int} \tag{1}$$

The first term is the two-level Hamiltonian, with energy splitting $\hbar\omega_A$, the second one is the free field contribution, with $k = 2\pi n/L$ (n - integer) as the wave number of the photon. L is a (large) normalization length. The third term is the interaction term $\hat{H}_{int}$, with κ as an interaction parameter. The frequency parameter is $\omega_k = c|k|$.

a) A general state of the two-level system is characterized by a vector $\mathbf{r}$, with $r \leq 1$, and with the corresponding density matrix as

$$\rho_A = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \tag{2}$$

Consider first that the interaction term $\hat{H}_{int}$ is turned off, $\kappa = 0$, so that the time evolution operator of the two-level system is $\hat{\mathcal{U}}(t) = \exp(-\frac{i}{2}\omega_A\, t\,\sigma_z)$. Use this to determine the the density matrix $\rho_A(t)$ at time t, assuming that $\rho_A(0)$ is identical to the density matrix in (2), and show that the time evolution of $\mathbf{r}$ is a precession around the z-axis with angular velocity ω_A.

b) Assume next that $\kappa \neq 0$ and that initially the two-level system is in the excited "spin up state", while the scalar field is in the vacuum state. Thus, the initial state is $|+,0\rangle = |+\rangle \otimes |0\rangle$. It decays to the "spin down state" by emission of a field quantum. The final state we then write as $|-,1_k\rangle = |-\rangle \otimes |1_k\rangle$.

The occupation probability of the excited state $|+\rangle$ decays exponentially, $P_+(t) = \exp(-\gamma t)$, with a decay rate γ that to first order in the interaction, and in the limit $L \to \infty$, is given by

$$\gamma = \frac{L}{(2\pi\hbar)^2}\int dk\,|\langle -,1_k|\hat{H}_{int}|+,0\rangle|^2\,\delta(\omega_k - \omega_A) \tag{3}$$

Determine the decay rate γ, expressed in terms of the parameters of the problem.

As discussed in the lectures an approximate way to handle the decay is to introduce an imaginary contribution to the energy of the decaying state. Assuming a more general initial state, of the form

$$|\psi(0)\rangle = (\alpha|+\rangle + \beta|-\rangle) \otimes |0\rangle = \alpha|+,0\rangle + \beta|-,0\rangle \tag{4}$$

with α and β as unspecified coefficients, with $|\alpha|^2 + |\beta|^2 = 1$, we make the corresponding *ansatz* for the time evolved state

$$|\psi(t)\rangle = (e^{-\frac{i}{2}\omega_A t - \gamma t/2}\alpha|+\rangle + e^{\frac{i}{2}\omega_A t}\beta|-\rangle) \otimes |0\rangle + \sum_k c_k(t)|-,1_k\rangle \tag{5}$$

with $c_k(t)$ as decay parameters, which satify $c_k(0) = 0$.

c) Check what normalization of the state vector (5) means for the decay parameters, and determine the reduced density matrix matrix $\rho_A(t)$ of the two-level system.

d) Assume the same initial conditions as in b), $z(0) = 1$, $x(0) = y(0) = 0$ ($\alpha = 1$, $\beta = 0$). Determine the density matrix $\rho_A(t)$ and the corresponding time dependent vector $\mathbf{r}(t)$. Is the time evolution consistent with the expected exponential decay of the excited state of the two-level system? Give a brief description of the evolution of the entanglement between the two level system and the radiation field during the decay.

e) Choose another initial condition $x(0) = 1$, $y(0) = z(0) = 0$ ($\alpha = \beta = 1/\sqrt{2}$), and find also in this case the time evolution of the reduced density matrix and the components of the vector $\mathbf{r}(t)$. Sketch the time evolution of $\mathbf{r}(t)$ and compare qualitatively the motion with that in a) and d). Find $r(t)^2$ expressed as a function of γt, and sketch also this function. What does it show about the time evolution of the entanglement between the two subsystems A and B?

   Assume in this paragraph $\gamma << \omega_A$.

13.2 Exam 2015 Problem 3

13.3 Driven two-level system with damping

The Hamiltonian of an isolated two-level system is $H_0 = \frac{1}{2}\hbar\omega_0\sigma_z$. Let $|g\rangle$ be the ground state and $|e\rangle$ the excited state. The system is coupled to a radiation field, so that the excited state spontaneously will decay to the ground state, emitting a quantum of radiation (which could be photons, phonons or some other field excitation depending on the physical realization). This means that the density matrix ρ of the system will (to a good approximation) satisfy a Lindblad equation of the form

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] - \frac{1}{2}\gamma \left[\alpha^\dagger \alpha \rho + \rho \alpha^\dagger \alpha - 2\alpha\rho\alpha^\dagger\right] \tag{6}$$

where γ is the decay rate for the transition $|e\rangle \rightarrow |g\rangle$ and $\alpha = |g\rangle\langle e|$.

a) We parametrize the density matrix in the following way

$$\rho = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix}. \tag{7}$$

   Derive the equations for $\dot{p}_e$, $\dot{p}_g$ and $\dot{b}$ and check that they are consistent with the conservation of total probability, $p_e + p_g = 1$.

b) Find the solution of the Lindblad equation if the initial state is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$. Calculate the Bloch vector as a function of time and describe its motion in the Bloch sphere (Reminder: The density matrix can be expressed as $\rho = \frac{1}{2}\left(\mathbb{1} + \mathbf{r}\cdot\sigma\right)$ where $\mathbf{r}$ is the Bloch vector).

We excite the two-level system by an external wave, which we assume is described by adding a time dependent driving term to the Hamiltonian, so that it takes the form

$$H = \frac{1}{2}\hbar\omega_0\sigma_z + \frac{1}{2}\hbar\omega_1(\cos\omega t\sigma_x + \sin\omega t\sigma_y). \tag{8}$$

c) We want to study the system in a reference frame rotating around the z-axis with the frequency ω of the external wave. That is, we define the state in the rotating frame as $|\psi'\rangle = T(t)|\psi\rangle$ where $T(t)$ is a time dependent unitary transformation. Determine the form of $T(t)$ and derive the form of the Lindblad equation in the rotating frame.

d) Find the stationary solution of the Lindblad equation in the rotating frame. Describe the result in the limiting cases of small and large ω_1. What quantity should ω_1 be compared to for the limits to apply?