Solutions to problem set 1

1.1 Commutators and anti-commutators

We have

$$
\begin{aligned}
[A, BC] &= ABC - BAC + BAC - BCA = [A,B]\,C + B\,[A,C] \\
[A, BC] &= ABC + BAC - BAC - BCA = \{A,B\}\,C - B\,\{A,C\}
\end{aligned}
\tag{1}
$$

1.2 Trace and determinant

a) We have

$$A_{mn} = \langle m|\hat{A}|n\rangle \qquad A'_{mn} = \langle m'|\hat{A}|n'\rangle \tag{2}$$

and

$$|n\rangle = \sum_n U_{n'n}|n\rangle \qquad U_{n'n} = \langle n|n'\rangle \tag{3}$$

with U a unitary matrix. Then we get

$$A'_{mn} = \sum_{ij} \langle m|i\rangle\langle i|\hat{A}|j\rangle\langle j|n\rangle = (U^{-1}AU)_{mn} \tag{4}$$

From this we find

$$\mathrm{Tr}\,A' = \mathrm{Tr}(U^{-1}AU) = \mathrm{Tr}(UU^{-1}A) = \mathrm{Tr}\,A \tag{5}$$

where we use that $\mathrm{Tr}\,AB = \mathrm{Tr}\,BA$ and that $UU^{-1} = 1$ (U is unitary).

$$\det A' = \det(U^{-1}AU) = \det U^{-1}\det A\det U = \det A \tag{6}$$

since $\det(AB) = \det A \det B$ and $\det U^{-1} = (\det U)^{-1}$.

b) We write $\hat{A}$ in the basis of eigenstates, where it is diagonal with the eigenvalues $a_n, n = 1, 2, ...$ on the diagonal. Then the trace is just the sum of the diagonal elements, and the determinant the product.

$$\mathrm{Tr}\,\hat{A} = \sum_n a_n \quad \det\hat{A} = \prod_n a_n \tag{7}$$

c)

$$\det e^{\hat{A}} = \det \sum_n e^{a_n}|n\rangle\langle n| = \prod_n e^{a_n} = e^{\sum_n a_n} = e^{\mathrm{Tr}\,\hat{A}} \tag{8}$$

d) We decompose the states in a basis:

$$|\psi\rangle = \sum_n \psi_n |n\rangle \qquad |\phi\rangle = \sum_n \phi_n |n\rangle. \tag{9}$$

Then we get

$$\langle\psi|\phi\rangle = \sum_{m,n} \langle n|\psi_n^* \phi_m |m\rangle = \sum_{m,n} \psi_n^* \phi_m \langle n|m\rangle = \sum_n \psi_n^* \phi_n \tag{10}$$

$$\mathrm{Tr}(|\phi\rangle\langle\psi|) = \mathrm{Tr}\left(\sum_{m,n} \psi_n^* \phi_m |m\rangle\langle n|\right) = \sum_n \psi_n^* \phi_n = \langle\psi|\phi\rangle. \tag{11}$$

1.3 Dirac's delta function

a) We start from the definition of the delta function

$$f(x) = \int_{-\infty}^{\infty} dx' \, \delta(x - x') \, f(x') \tag{12}$$

and Fourier transform both sides:

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int dx e^{-ikx} \int dx' \, \delta(x - x') \, f(x') \\ &= \frac{1}{\sqrt{2\pi}} \int du \int dx' e^{-iku - ikx'} \delta(u) \, f(x') \\ &= \int du e^{-iku} \delta(u) \tilde{f}(k) \end{aligned} \tag{13}$$

where $u = x - x'$. Thus we must have that

$$\int du e^{-iku} \delta(u) = 1 \tag{14}$$

which means that the Fourier transform of the delta function is a constant

$$\tilde{\delta}(k) = \frac{1}{\sqrt{2\pi}} \tag{15}$$

Using the expression for the inverse Fourier transform we get that

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ikx} \tag{16}$$

b) Consider first the case where $g(x)$ has a single zero, $g(x_0) = 0$ and that $g'(x_0) > 0$. Change integration variable to $u = g(x)$ so that $dx = \frac{du}{g'(x)}$:

$$\int_{-\infty}^{\infty} dx\delta(g(x))f(x) = \int_{-\infty}^{\infty} \frac{du}{g'(x)}\delta(u)f(x) \\ = \frac{1}{g'(x_0)}f(x_0) = \int_{-\infty}^{\infty} \frac{dx}{g'(x_0)}\delta(x - x_0)f(x) \tag{17}$$

If instead $g'(x_0) < 0$ we have

$$\int_{-\infty}^{\infty} dx\delta(g(x))f(x) = \int_{\infty}^{-\infty} \frac{du}{g'(x)}\delta(u)f(x) = -\int_{-\infty}^{\infty} \frac{du}{g'(x)}\delta(u)f(x) \tag{18}$$

Both cases can then be written in the form

$$\delta(g(x)) = \frac{1}{|g'(x_0)|}\delta(x - x_0) \tag{19}$$

If the function $g(x)$ has several zeros, at the points $x = x_i$ we get one contribution from each zero, which gives the general formula

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i) \tag{20}$$

1.4 Position and momentum eigenstates

Consider the momentum eigenstate $|p\rangle$. In the coordinate representation it is given by the wave-function $\psi_p(x) = \langle x|p\rangle$ which is exactly the scalar product we have to find. The eigenvalue equation $\hat{p}|p\rangle = p|p\rangle$ is in the coordinate representation

$$\langle x|\hat{p}|p\rangle = p\langle x|p\rangle = p\psi_p(x) \tag{21}$$

The left hand side of this equation is the position basis representation of state that results from the action of the momentum operator on the state $|p\rangle$. This we know from introductory quantum mechanics to be (remember that $\psi_p(x) = \langle x|p\rangle$ is the position space representation of the momentum eigenstate)

$$\langle x|\hat{p}|p\rangle = -i\hbar\frac{d\psi_p(x)}{dx} \tag{22}$$

In fact, this equation is the proper meaning of the prescription $\hat{p} \to -i\hbar\frac{d}{dx}$ that is used in the position representation. Thus we get the differential equation

$$-i\hbar\frac{d\psi_p(x)}{dx} = p\psi_p(x) \tag{23}$$

with the solution

$$p(x) = Ae^{\frac{i}{\hbar}xp} \tag{24}$$

where A is an integration constant which we have to determine from the normalization condition

$$\langle p|p'\rangle = \int dx \langle p|x\rangle\langle x|p'\rangle = \int dx|A|^2 e^{\frac{i}{\hbar}x(p-p')} = 2\pi\hbar|A|^2\delta(p-p'). \tag{25}$$

So we have to choose $A = 1/\sqrt{2\pi\hbar}$ and get

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}xp} \tag{26}$$

1.5 Some operator expansions

We're given the following generally noncommuting operators:

$$\hat{F}(\lambda) = e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}, \quad \hat{G}(\lambda) = e^{\lambda\hat{A}}e^{\lambda\hat{B}}$$

a)

$$\begin{aligned} \frac{d\hat{F}}{d\lambda} &= \frac{de^{\lambda\hat{A}}}{d\lambda}\hat{B}e^{-\lambda\hat{A}} + \underbrace{e^{\lambda\hat{A}}\frac{d\hat{B}}{d\lambda}e^{-\lambda\hat{A}}}_{=0} + e^{\lambda\hat{A}}\hat{B}\frac{de^{-\lambda\hat{A}}}{d\lambda} \\ &= \hat{A}e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} - e^{\lambda\hat{A}}\hat{B}\hat{A}e^{-\lambda\hat{A}} \end{aligned}$$

Remembering that $e^{\lambda\hat{A}}$ can be written as the Taylor expansion evaluated for small λ, we know that $\left[e^{k\hat{A}}, \hat{A}\right] = 0$ since every operator commutes with itself.

$$\frac{d\hat{F}}{d\lambda} = \hat{A}e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} - e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}\hat{A} = \hat{A}\hat{F} - \hat{F}\hat{A} = \left[\hat{A}, \hat{F}\right] \tag{27}$$

Then expanding $\hat{F}$ for small λ, we get:

$$\begin{aligned} \hat{F}(\lambda) &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\hat{F}^{(n)}(0) \\ &= \hat{F}(0) + \lambda\hat{F}'(0) + \frac{\lambda^2}{2}\hat{F}''(0) + \cdots \\ &= \hat{F}(0) + \lambda\left[\hat{A}, \hat{F}(0)\right] + \frac{\lambda^2}{2}\left[\hat{A}, \hat{F}'(0)\right] + \cdots \\ &= \hat{F}(0) + \lambda\left[\hat{A}, \hat{F}(0)\right] + \frac{\lambda^2}{2}\left[\hat{A}, \left[\hat{A}, \hat{F}(0)\right]\right] + \cdots \\ &= \hat{B} + \lambda\left[\hat{A}, \hat{B}\right] + \frac{\lambda^2}{2}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \cdots \quad \because \hat{F}(0) = \hat{B} \end{aligned} \tag{28}$$

b)

$$\begin{aligned} \frac{d\hat{G}}{d\lambda} &= \frac{de^{\lambda\hat{A}}}{d\lambda}e^{\lambda\hat{B}} + e^{\lambda\hat{A}}\frac{de^{\lambda\hat{B}}}{d\lambda} \\ &= \hat{A}e^{\lambda\hat{A}}e^{\lambda\hat{B}} + e^{\lambda\hat{A}}\hat{B}e^{\lambda\hat{B}} \\ &= \hat{A}e^{\lambda\hat{A}}e^{\lambda\hat{B}} + e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}e^{\lambda\hat{A}}e^{\lambda\hat{B}} \\ &= \hat{A}\hat{G} + \hat{F}\hat{G} \\ &= \left(\hat{A} + \hat{F}\right)\hat{G} \end{aligned} \tag{29}$$

Then, we need to show the relation

$$\hat{G}\left(\lambda\right) = e^{\lambda\hat{A}+\lambda\hat{B}+\frac{\lambda^2}{2}\left[\hat{A},\hat{B}\right]+\cdots} \tag{30}$$

We're told to compute the exponential to second order in λ. If we then treat $\lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}\left[A, B\right]$ as a single variable, we can easily see that we can expand in the following way for small λ:

$$e^{\lambda\hat{A}+\lambda\hat{B}+\frac{\lambda^2}{2}\left[\hat{A},\hat{B}\right]} = \sum_{n=0}^{\infty}\frac{\left(\lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}\left[A, B\right]\right)^n}{n!} = 1+\lambda\hat{A}+\lambda\hat{B}+\frac{\lambda^2}{2}\left[\hat{A},\hat{B}\right]+\frac{1}{2}\left(\lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}\left[\hat{A},\hat{B}\right]\right)^2+\cdots$$

Neglecting terms that are of $\mathcal{O}\left(\lambda^3\right)$, we get:

$$\begin{aligned} e^{1+\lambda\hat{A}+\lambda\hat{B}+\frac{\lambda^2}{2}\left[\hat{A},\hat{B}\right]} &= 1 + \lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}\left[\hat{A},\hat{B}\right] + \frac{1}{2}\left(\lambda\hat{A} + \lambda\hat{B}\right)^2 + \cdots \\ &= 1 + \lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}\left(\left[\hat{A},\hat{B}\right] + \left(\hat{A} + \hat{B}\right)^2\right) + \cdots \end{aligned} \tag{31}$$

If we now go on to expand the left hand side of (30) to second order in λ for small λ, we get:

$$\hat{G}\left(\lambda\right) = \hat{G}(0) + \lambda\hat{G}'(0) + \frac{\lambda^2}{2}\hat{G}''(0) + \cdots$$

Calculating the second order term:

$$\frac{d^2\hat{G}}{d\lambda^2} = \frac{d}{d\lambda}\left(\hat{A} + \hat{F}\right)\hat{G} = \left(\underbrace{\frac{d\hat{A}}{d\lambda}}_{=0} + \frac{d\hat{F}}{d\lambda}\right)\hat{G} + \left(\hat{A} + \hat{F}\right)\frac{d\hat{G}}{d\lambda} = \left[\hat{A},\hat{F}\right]\hat{G} + \left(\hat{A} + \hat{F}\right)^2\hat{G}$$

Inserting this back yields:

$$\begin{aligned} \hat{G}\left(\lambda\right) &= \hat{G}(0) + \lambda\left(\hat{A} + \hat{F}(0)\right)\hat{G}(0) + \frac{\lambda^2}{2}\left(\left[\hat{A},\hat{F}(0)\right]\hat{G}(0) + \left(\hat{A} + \hat{F}(0)\right)^2\hat{G}(0)\right) + \cdots \\ &= 1 + \lambda\left(\hat{A} + \hat{B}\right) + \frac{\lambda^2}{2}\left(\left[\hat{A},\hat{B}\right] + \left(\hat{A} + \hat{B}\right)^2\right) + \cdots \end{aligned} \tag{32}$$

Comparing (31) and (32), we see they are the same. Q.E.D

c) If $\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]=\left[\hat{B},\left[\hat{A},\hat{B}\right]\right]=0$, then from (28), we get $\hat{F}(\lambda)=1+\lambda\left[\hat{A},\hat{B}\right]$. Using this and (29), we get

$$\frac{d\hat{G}}{d\lambda} = \left(\hat{A}+\hat{B}+\lambda\left[\hat{A},\hat{B}\right]\right)\hat{G}$$

Using $\hat{G}(0)=1$, and the fact that the operators $\hat{A}+\hat{B}+\lambda\left[\hat{A},\hat{B}\right]$ for different λ commute we have that

$$\hat{G}(\lambda)=e^{\lambda\left(\hat{A}+\hat{B}\right)+\frac{\lambda^2}{2}\left[\hat{A},\hat{B}\right]}$$

Q.E.D

1.6 Spin operators and Pauli matrices

a) We set $\mathbf{n}=(n_1,n_2,n_3)$, then

$$\mathbf{n}\cdot\sigma=n_1\sigma_1+n_2\sigma_2+n_3\sigma_3=\begin{pmatrix} n_3 & n_1-in_2 \\ n_1+in_2 & -n_3 \end{pmatrix} \tag{33}$$

The eigenvalue equation is:

$$\det\left(\sigma_{\mathbf{n}}-\mathbb{1}\right)=0 \quad\Rightarrow\quad \det\begin{pmatrix} n_3-\lambda & n_1-in_2 \\ n_1+in_2 & -n_3-\lambda \end{pmatrix}=0$$

$$\begin{aligned} -\left(n_3-\lambda\right)\left(n_3+\lambda\right)-\left|n_1+in_2\right|^2 &= 0 \\ \Rightarrow -n_3^2+\lambda^2-\left(n_1^2+n_2^2\right) &= 0 \\ \Rightarrow \lambda^2 &= n_1^2+n_2^2+n_3^2 \\ \Rightarrow \lambda &= \pm\left|\mathbf{n}\right|^2=\pm1 \end{aligned}$$

The corresponding eigenstate equation for $\lambda=1$ is:

$$\sigma_{\mathbf{n}}\Psi_{\mathbf{n}}=\Psi_{\mathbf{n}}$$

By switching to spherical coordinates, we get $\mathbf{n}=(n_1,n_2,n_3)=(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)$. Inserting this into the matrix (33), the equation gets the form:

$$\begin{pmatrix} \cos\theta & (\cos\phi-i\sin\phi)\sin\theta \\ (\cos\phi+i\sin\phi)\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Which corresponds to these equations:

$$\begin{aligned} \psi_1\cos\theta+\psi_2 e^{-i\phi}\sin\theta &= \psi_1 \\ \psi_1 e^{i\phi}\sin\theta-\psi_2\cos\theta &= \psi_2 \end{aligned}$$

We need to solve one of them, I'm choosing the second, where I get the relation:

$$\frac{e^{i\phi}\sin\theta}{1+\cos\theta}\psi_1 = \psi_2$$

Then:

$$\frac{e^{i\phi}\sin\theta}{1+\cos\theta} = \frac{e^{i\phi}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}} = \frac{e^{i\phi}\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

This gives us the relation:

$$e^{i\phi}\sin\frac{\theta}{2}\psi_1 = \psi_2\cos\frac{\theta}{2}$$

This is true for $\psi_1 = \cos\frac{\theta}{2}$ and $\psi_2 = e^{i\phi}\sin\frac{\theta}{2}$, which defines the eigenvectors for $\lambda = 1$:

$$\Psi_{\mathbf{n}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

For $\Psi_{\mathbf{n}}^\dagger \sigma \Psi_{\mathbf{n}} = \mathbf{n}$, we get:

$$\Psi_{\mathbf{n}}^\dagger \sigma \Psi_{\mathbf{n}} = \left(\Psi_{\mathbf{n}}^\dagger \sigma_1 \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_2 \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_3 \Psi_{\mathbf{n}}\right)$$

$$\begin{aligned}
\Psi_{\mathbf{n}}^\dagger \sigma_1 \Psi_{\mathbf{n}} &= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\
&= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \\
&= e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\
&= \frac{1}{2}\sin\theta\left(e^{i\phi}+e^{-i\phi}\right) \\
&= \cos\phi\sin\theta = n_1
\end{aligned}$$

$$\begin{aligned}
\Psi_{\mathbf{n}}^\dagger \sigma_2 \Psi_{\mathbf{n}} &= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\
&= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} -ie^{i\phi}\sin\frac{\theta}{2} \\ i\cos\frac{\theta}{2} \end{pmatrix} \\
&= i\left(-e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) \\
&= \frac{1}{2}i\sin\theta\left(e^{-i\phi}-e^{i\phi}\right) \\
&= -i^2\sin\theta\sin\phi = n_2
\end{aligned}$$

$$\begin{aligned}
\Psi_{\mathbf{n}}^\dagger \sigma_3 \Psi_{\mathbf{n}} &= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\
&= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} \cos\frac{\theta}{2} \\ -e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\
&= \cos^2\frac{\theta}{2} - e^{-i\phi}e^{i\phi}\sin^2\frac{\theta}{2} \\
&= \cos\theta = n_3
\end{aligned}$$

Thus:

$$\Psi_{\mathbf{n}}^{\dagger}\sigma\Psi_{\mathbf{n}} = \left(\Psi_{\mathbf{n}}^{\dagger}\sigma_1\Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^{\dagger}\sigma_2\Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^{\dagger}\sigma_3\Psi_{\mathbf{n}}\right) = (n_1, n_2, n_3) = \mathbf{n}$$

b)

$$e^{-\frac{i}{2}\alpha\sigma_z}\sigma_x e^{\frac{i}{2}\alpha\sigma_z}$$

We have:

$$e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} = \hat{B} + \lambda\left[\hat{A}, \hat{B}\right] + \frac{\lambda^2}{2}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \cdots \tag{34}$$

We choose $\hat{A} = -\frac{i}{2}\sigma_z$, we get our expression on the same form as in (34):

$$\begin{aligned} e^{-\frac{i}{2}\alpha\sigma_z}\sigma_x e^{\frac{i}{2}\alpha\sigma_z} &= \sigma_x + \alpha\left[-\frac{i}{2}\sigma_z, \sigma_x\right] + \frac{\alpha^2}{2}\left[-\frac{i}{2}\sigma_z, \left[-\frac{i}{2}\sigma_z, \sigma_x\right]\right] + \frac{\alpha^3}{6}\left[-\frac{i}{2}\sigma_z, \left[-\frac{i}{2}\sigma_z, \left[-\frac{i}{2}\sigma_z, \sigma_x\right]\right]\right] + \cdots \\ &= \sigma_x + \alpha\left(-\frac{i}{2}\right)[\sigma_z, \sigma_x] + \frac{\alpha^2}{2}\left(-\frac{i}{2}\right)^2[\sigma_z, [\sigma_z, \sigma_x]] + \frac{\alpha^3}{6}\left(-\frac{i}{2}\right)^3[\sigma_z, [\sigma_z, [\sigma_z, \sigma_x]]] + \cdots \end{aligned}$$

Then we know that the commutators are such that $[\sigma_x, \sigma_y] = 2i\epsilon_{xyz}\sigma_z$ where $\epsilon_{xyz} = 1$ and any odd number of permutations returns -1, and any even number of permutations return 1. Thus: $[\sigma_z, \sigma_x] = 2i\sigma_y$, $[\sigma_z, \sigma_y] = -2i\sigma_x$, and

$$\begin{aligned} e^{-\frac{i}{2}\alpha\sigma_z}\sigma_x e^{\frac{i}{2}\alpha\sigma_z} &= \sigma_x + \alpha\left(-\frac{i}{2}\right)2i\sigma_y + \frac{\alpha^2}{2}\left(-\frac{i}{2}\right)^2(2i)[\sigma_z, \sigma_y] \\ &\quad + \frac{\alpha^3}{6}\left(-\frac{i}{2}\right)^3(2i)[\sigma_z, [\sigma_z, \sigma_y]] + \cdots \qquad (35) \\ &= \sigma_x + \alpha\left(-\frac{i}{2}\right)2i\sigma_y - \frac{\alpha^2}{2}\left(-\frac{i}{2}\right)^2(2i)^2\sigma_x - \frac{\alpha^3}{6}\left(-\frac{i}{2}\right)^3(2i)^2[\sigma_z, \sigma_x] + \cdots \\ &= \sigma_x + \alpha\sigma_y - \frac{\alpha^2}{2}\sigma_x - \frac{\alpha^3}{6}\sigma_y + \cdots \qquad (36) \\ &= \left(1 - \frac{\alpha^2}{2} + \cdots\right)\sigma_x + \left(\alpha - \frac{\alpha^3}{6} + \cdots\right)\sigma_y \end{aligned}$$

The series continues in the familiar pattern of $\cos\alpha$ and $\sin\alpha$ due to the pattern in (35), which leaves us:

$$e^{-\frac{i}{2}\alpha\sigma_z}\sigma_x e^{\frac{i}{2}\alpha\sigma_z} = \cos\alpha\sigma_x + \sin\alpha\sigma_y \tag{37}$$

The unitary matrix $\hat{U} = e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}}$, when transforming an operator/matrix, causes the transformations:

$$\sigma \to \hat{U}\sigma\hat{U}^{\dagger} = e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}}\sigma e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}}$$

As we saw in (37), if $\mathbf{n} = (0, 0, 1) = z$, we got a rotation about the z axis. If we now imagine a different orthonormal coordinate system with unit vectors $\mathbf{n} = \mathbf{n}_z, \mathbf{n}_y, \mathbf{n}_x$, then the transformation $\hat{U}\sigma_{\mathbf{n}_x}\hat{U}^{\dagger}$ would look like:

$$\sigma_{\mathbf{n}_x} \to e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}}\sigma e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \cos\alpha\sigma_{\mathbf{n}_x} + \sin\alpha\sigma_{\mathbf{n}_y}$$

and rotates the spin basis around the axis $\mathbf{n}$.

c)

$$e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \sum_{k=0}^{\infty}\frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^k}{k!}$$

Then we need to know what the different powers of $\sigma_{\mathbf{n}}$ are:

$$\begin{aligned}\sigma_{\mathbf{n}}^2 &= \begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}^2 = \begin{pmatrix}\cos^2\theta + e^{-i\phi}e^{i\phi}\sin^2\theta & \cos\theta e^{-i\phi}\sin\theta - \cos\theta e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta\cos\theta - \cos\theta e^{i\phi}\sin\theta & e^{i\phi}\sin\theta e^{-i\phi}\sin\theta + \cos^2\theta\end{pmatrix} \\ &= \begin{pmatrix}\cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta\end{pmatrix} = \mathbb{1}\end{aligned}$$

Thus, we see that $\sigma_{\mathbf{n}}$ follows the standard Pauli matrix identites: $\sigma_{\mathbf{n}}^{2k} = \mathbb{1}$ and $\sigma_{\mathbf{n}}^{2k+1} = \sigma_{\mathbf{n}}$ for $k = 0, 1, \ldots$ This let's us rewrite our series as:

$$e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \sum_{k=0}^{\infty}\frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^k}{k!} = \sum_{k=0}^{\infty}\frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^{2k+1}}{(2k+1)!}$$

Using that $(-i)^{2k} = (-1)^k$, $(-i)^{2k+1} = -i^{2k+1} = -i^{2k}i = (-1)^{k+1}i$, and the identities for the Pauli matrices, we get:

$$\begin{aligned}e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} &= \mathbb{1}\underbrace{\sum_{k=0}^{\infty}\frac{\left(\frac{\alpha}{2}\right)^{2k}(-1)^k}{(2k)!}}_{=\cos\frac{\alpha}{2}} - i\sigma_{\mathbf{n}}\underbrace{\sum_{k=0}^{\infty}\frac{\left(\frac{\alpha}{2}\right)^{2k+1}(-1)^k}{(2k+1)!}}_{=\sin\frac{\alpha}{2}} \\ e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} &= \cos\frac{\alpha}{2}\mathbb{1} - i\sin\frac{\alpha}{2}\sigma_{\mathbf{n}}\end{aligned}$$