FYS 4110: Non-relativistic quantum mechanics

Midterm Exam, Fall Semester 2004

The problem set is available from Friday October 15. The set consists of 2 problems written on 5 pages.

Deadline for returning solutions
is Friday October 22.

Return of solutions
The solutions can be returned either in written/printed form or as an e-mail attachment.
Written/printed solutions can be returned at Ekspedisjonskontoret in the Physics Building. Please add a copy that the lecturer can keep for evaluation at the final exam.
E-mailed solutions: Please send the solutions as one file, preferably in pdf format. E-mail address:j.m.leinaas@fys.uio.no.

Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Auditorium or Office 471); Monday morning available between 9 and 10 a.m.

Language
Solutions may be written in Norwegian or English, depending on your preference.

PROBLEMS

1 Particle encircling a magnetic flux
A particle with mass m and charge e moves freely on a circle of radius R. Through the circle passes a solenoid that carries a magnet flux Φ. We may consider the total flux to be confined to the solenoid so that the magnetic field vanishes on the circle where the particle moves.

In the following make use of the general expressions for the Hamiltonian of a particle in a magnetic field

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 \tag{1}$$

and for the probability current

$$\mathbf{j} = -\frac{i\hbar}{2m}(\psi^* \boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*) - \frac{e}{m}\mathbf{A}\psi^*\psi \tag{2}$$

Use the angle variable θ as coordinate for the particle on the circle.

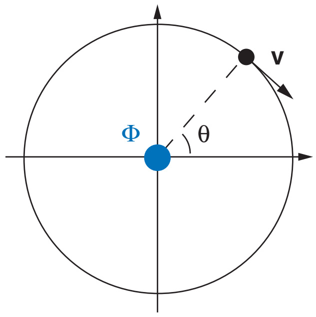

Fig. 1

a) Assume rotational invariance about the center of the circle and show that the vector potential on the circle takes the constant value $A = \Phi/2\pi R$ with direction along the circle. Explain why this vector potential has no influence on the motion of the particle when this is described by the classical equations of motion.

b) Express the Hamiltonian as an operator acting on the wave functions $\psi(\theta)$ for the particle on the circle. Find the energy eigenvalues and show that the energy spectrum varies periodically with the flux Φ. What is the flux period Φ_0? Plot the four lowest energies as functions of Φ in the interval from 0 to Φ_0. Characterize the ground state by its angular momentum in the same interval. What is special for the spectrum at $\Phi = \Phi_0/2$?

c) Find the probability current for a general wave function $\psi(\theta)$, and determine the value of the ground state current as a function of Φ. What is the maximum value of the ground state current and what value for the particle velocity does that correspond to.

d) Find the propagator $\mathcal{G}(\theta, t; 0, 0) = \langle \theta, t | 0, 0 \rangle$ expressed in terms of the Jacobi theta function for general Φ. Use the definition of the Jacobi theta function as given in problem 2.4 (Problem Set 2).

e) For the Lagrangian of a particle in a magnetic field the effect of the vector potential is to add a term proportional to the velocity

$$L = \frac{1}{2} m v^2 + e\mathbf{A} \cdot \mathbf{v} \tag{3}$$

Follow the path integral approach of problem 2.4 (Problem Set 2) to find the propagator by summing over all classical paths with the given initial and final conditions. Show in the same way as discussed there that the propagator derived in this way is equivalent to the one derived in d). (Use the properties listed in Problem 2.4 for the Jacobi theta function.)

2 Entangled photons

In this problem correlations between pairs of entangled photons are studied. The interesting degree of freedom is the polarization of each photon. For a single photon this means that the quantum state is a vector in a two-dimensional vector space spanned by the vectors $|H\rangle$ and $|V\rangle$, which correspond to linear polarization in the horizontal and vertical direction, respectively. A general polarization state is a linear combination of these two. As special cases we consider linearly polarized photons in a rotated direction,

$$|\theta\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle \tag{4}$$

and circularly polarized photons with right-handed and left-handed orientation, respectively,

$$|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)\ , \quad |L\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle) \tag{5}$$

The two-photon states, when only polarization is taken into account, are vectors in the tensor product space spanned by the four vectors,

$$\begin{aligned} |HH\rangle = |H\rangle \otimes |H\rangle\ , \quad |HV\rangle = |H\rangle \otimes |V\rangle\ , \\ |VH\rangle = |V\rangle \otimes |H\rangle\ , \quad |VV\rangle = |V\rangle \otimes |V\rangle\ , \end{aligned} \tag{6}$$

(Note that even if the photons are bosons there is no symmetry constraint on the two-photon states, since we assume that the two photons can be distinguished by their different direction of propagation.)

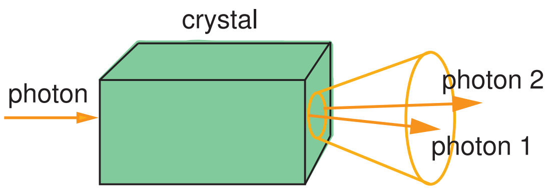

Fig. 2a

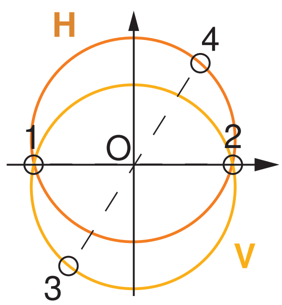

Fig. 2b

As a specific way to produce entangled photon pairs we consider the method of *parametric down conversion*, as described below and sketched in the Figure 2 and 3.

As illustrated in Fig. 2a a (weak) beam of photons enter a crystal, where each photon due to the non-linear interaction with the crystal is split into two photons. These appear with equal energy, half the energy of the incoming photon. The transverse momentum of the emerging photons is fixed so that their direction of propagation is limited to a cone, as indicated in the figure. The photons appear with constant probability around the cone. However, due to conservation of total transverse momentum, the two photons in each a pair are correlated so that they always are emitted at opposite sides of the cone.

There is furthermore a polarization effect, since photons with horizontal and vertical polarization (relative to the crystal planes) do not propagate in exactly the same way. As a consequence the cones corresponding to these two polarizations are slightly shifted. This is shown in the head-on view of Fig. 2b, where the cone corresponding to polarization H is slightly lifted relative to the cone corresponding to polarization V.

Two photons in a correlated pair will be located on opposite points of the central point O, like the pair of points 1 and 2 and the pair 3 and 4, and they always appear with orthogonal polarization. As shown by the figure this means that for most directions of the emitted photons the polarization of each photon is uniquely determined by its direction of propagation. For such a pair the two-photon state is a product state of the form $|HV\rangle$. As an illustration, the pair $3, 4$ of directions of the cone, as shown in Fig.2b, will be of this type.

However two directions are different since they lie on both cones. This is illustrated by the points 1 and 2 in Fig. 2b. A photon at one of these positions will be in a superposition of $|H\rangle$ and $|V\rangle$. Due to correlations between the photons a pair located at this position will be described by an entangled two-photon state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + e^{i\chi}|VH\rangle) \tag{7}$$

where the complex phase χ can be regulated in the experimental set up.

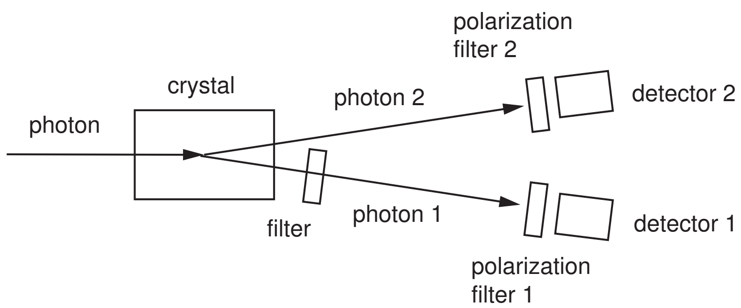

Fig. 3

We assume in the following that a filter close to the crystal will single out photons only in the directions 1 and 2. This is schematically shown in Fig. 3. To analyze correlations between the two photons in each pair, polarization filters are applied to photons in both directions as also shown in the figure. Those that pass the polarization filters are registered in the detectors and the registrations are paired by use of coincidence counters.

The polarization filters may be represented by operators that project on linearly polarized states along a (rotated) direction

$$\hat{P}(\theta) = |\theta\rangle\langle\theta| \tag{8}$$

In the following we examine the expected results of the polarization measurements by calculating the following expectation values

$$
\begin{aligned}
P_1(\theta_1) &\equiv \left\langle \hat{P}_1(\theta_1) \right\rangle && \text{photon 1} \\
P_2(\theta_1) &\equiv \left\langle \hat{P}_2(\theta_2) \right\rangle && \text{photon 2} \\
P_{12}(\theta_1, \theta_2) &\equiv \left\langle \hat{P}_1(\theta_1) \hat{P}_2(\theta_2) \right\rangle && \text{photon 1 and photon 2}
\end{aligned} \tag{9}
$$

a) Assume a series of N entangled photon pairs are used in an experiment. In this series n_1 photons are registered in detector 1, n_2 photons are registered in detector 2 and and n_{12} are registered at coincidence in the two detectors. What are the relations between the registered frequences n_1/N, etc. and the the expectation values P_1, P_2 and P_{12}?

b) For the general two-photon state of the form (7) find the density operator of the two-photon pair, and find the corresponding reduced density operators for photon 1 and photon 2.

We consider three different situations where the the incoming photon pairs are in the state (7) with I: $\chi = \pi$, II: $\chi = 0$ and III: $\chi = \pi/2$.

c) Consider an input state of the form I. Determine the detection probability P_1 of photon 1 as a function of the angle θ_1 of polarizer 1. Do the same with P_2 for photon 2. Determine next the probability P_{12} for detecting photons at both analyzers as a function of the angles θ_1 and θ_2. What do the results tell about correlations of the two photons?

d) Consider next a two-photon input state of the form II. Examine the same questions as in c). Are the results obtained rotationally invariant? Compare the cases b) and c).

e) Consider finally the case III. Find also in this case the expectation values P_1, P_2 and P_{12} as fuctions of the angles of the polarizers. Show that in this case there exists a mixed state, which is an *incoherent* mixture of $|HV\rangle$ and $|VH\rangle$, that has identical expectation values.

f) The Bell inequality, which is based on an assumed set of "hidden variables" as a source of the statistical distributions, can be written as a constraint on the function P_{12} in the following way (see Sect. 2.3.2 of the lecture notes),

$$
F(\theta_1, \theta_2, \theta_3) \equiv P_{12}(\theta_2, \theta_3) - |P_{12}(\theta_1, \theta_2) - P_{12}(\theta_1, \theta_3)| \geq 0 \tag{10}
$$

Examine the Bell inequality in the cases I, II and III for the special choice of angles $\theta_1 = 0$, $\theta_2 = \theta$ and $\theta_3 = 2\theta$ by plotting the function $F(0, \theta, 2\theta)$. Comment on which of the cases that show that the Bell inequality is not satisfied. Is there a relation between the conclusion for the case III and the results in e)?

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Midterm Exam, Fall Semester 2005

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Return of solutions
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Language
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PROBLEMS

1 Spin motion in an oscillating field.
We study in this problem first spin motion in a constant magnetic field, then the effect of including an additional, oscillating field. Results that are derived in Sect. 1.3.2 of the lecture notes may be used in the solution.

An electron with spin vector

$$\hat{\mathbf{S}} = (\hbar/2)\boldsymbol{\sigma} \tag{1}$$

and magnetic moment

$$\hat{\boldsymbol{\mu}} = \frac{e}{m}\hat{\mathbf{S}} \tag{2}$$

is situated in a constant magnetic field $\mathbf{B} = B_0\mathbf{k}$. The spin motion is assumed to be independent of the orbital motion of the electron.

a) The spin state is described by a time dependent density matrix

$$\rho(t) = \frac{1}{2}(\mathbf{1} + \mathbf{r}(t)\cdot\boldsymbol{\sigma}) \tag{3}$$

As initial condition for the motion we have $\mathbf{r} = \mathbf{r}_0$ for $t = 0$. Give a general expression for $\rho(t)$ in terms of the time evolution operator, and use this to determine the time dependent vector $\mathbf{r}(\mathbf{t})$.

b) Assume next the initial condition $\mathbf{r_0} = a\mathbf{k}$ ($\mathbf{k}$ is the unit vector in the z-direction). What are the allowed values of a? An oscillating field is turned on so that the total magnetic field is

$$\mathbf{B} = B_0\,\mathbf{k} + B_1(\cos\omega t\,\mathbf{i} + \sin\omega t\mathbf{j}). \tag{4}$$

Find also in this case the time evolution of $\mathbf{r}$.

c) Study the motion found in b) in the special case of resonance, $\omega = \omega_0 \equiv -\frac{eB_0}{m}$. Determine $\mathbf{r}(t)$ and make a qualitative description of the motion.

2 Charged particle in a strong magnetic field.

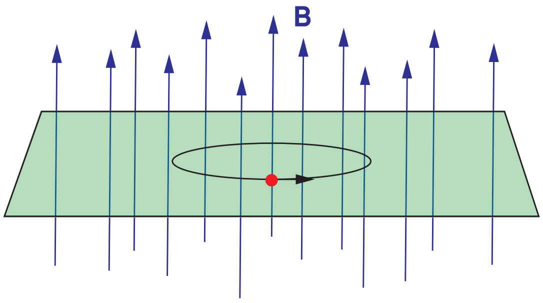

Figure 1:

We study in this problem a particle with electric charge e that moves in a strong magnetic field $\mathbf{B}$. The motion is assumed to be constrained to a plane (the x,y-plane) with the magnetic field orthogonal to the plane. The magnetic field is assumed to be constant over the plane, and eB is taken to be *negative*, with B as the z-component of $\mathbf{B}$. We assume in the following that the rotationally symmetric form of the vector potential is chosen, $\mathbf{A} = -(1/2)\mathbf{r} \times \mathbf{B}$. The relation between velocity and (canonical) momentum is $\mathbf{v} = (\mathbf{p} - e\mathbf{A})/m$, and the Hamiltonian has the standard form $H = (1/2m)(\mathbf{p} - e\mathbf{A})^2$.

We consider first the classical, non-relativistic form of the particle motion. Next we study the quantum description, where a set of coherent states is introduced for the particle in the degenerate ground state of the Hamiltonian. This description is particularly relevant for the study of the quantum Hall effect, where a 2-dimensional electron gas moves under the influence of a strong magnetic field.

a) Use Newton's second law for a charged particle in a magnetic field to show that, classically, the particle moves in a circular orbit with constant angular velocity $\omega = -eB/m$. Show, by use of the equation of motion, that generally the mechanical angular momentum $L_{mek} = m(xv_y - yv_x)$ is *not* a constant of motion, whereas $L = L_{mek} + (eB/2)r^2$ is conserved. (The last term can be viewed as an electromagnetic contribution.)

b) Consider the following vector, $\mathbf{R} = \mathbf{r} + (1/\omega)\mathbf{k}\times\mathbf{v}$, with $\mathbf{r}$ as the position and $\mathbf{v}$ as the velocity, $\mathbf{k}$ as the unit vector in the z-direction (orthogonal to the plane) and B as the z-component of the magnetic field. Show that $\mathbf{R}$ is a constant of motion, and give a physical interpretation of $\mathbf{R}$ and $\rho = (1/\omega)\mathbf{k}\times\mathbf{v}$ for the circular orbits? $\mathbf{R}$ is known as the guiding center coordinate.

c) In the quantum description, the position $\mathbf{r}$ and momentum $\mathbf{p}$ are, in the standard way replaced by operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ that satisfy the Heisenberg commutation relations. Show that the two components $\hat{X}$ and $\hat{Y}$ of the vector $\hat{\mathbf{R}}$, in the quantized form, do not commute. In what sense is the X,Y-plane similar to a two-dimensional phase space? Examine also commutators between the components $\hat{\rho}_x$ and $\hat{\rho}_y$ of $\hat{\boldsymbol{\rho}}$ in the same way.

d) Introduce dimensionless operators

$$\hat{a} = \frac{1}{\sqrt{2}l_B}(\hat{X} + i\hat{Y}) \;, \quad \hat{b} = \frac{1}{\sqrt{2}l_B}(\hat{\rho}_x - i\hat{\rho}_y) \tag{5}$$

where l_B is the so-called magnetic length, $l_B = \sqrt{\hbar/|eB|}$. Show that the set of operators $\{\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger\}$ satisfy the same commutation algebra as that of two independent harmonic oscillators. The corresponding set of harmonic oscillator states we denote by $|m, n\rangle$, where $\hat{a}^\dagger$ acts as a raising operator on the m quantum number and $\hat{b}^\dagger$ as a raising operator on the n quantum number.

e) Find expressions for the Hamiltonian $\hat{H}$ and angular momentum $\hat{L}$ in terms of the $\hat{a}$ and $\hat{b}$ operators. Show that $\hat{H}$ has an harmonic oscillator spectrum and find also the eigenvalues of $\hat{L}$ expressed in terms of m and n. In the following we assume the particle to be restricted to the degenerate ground state (the lowest Landau level). Show that this corresponds to the condition $n = 0$, while m is a free variable, so that the states $|m\rangle \equiv |m, 0\rangle, m = 0, 1, 2, ...$ form a complete set.

f) A coherent state in the Lowest Landau level can be defined by the equation

$$\hat{a}|z\rangle = z|z\rangle \;, \quad \hat{b}|z\rangle = 0 \tag{6}$$

Calculate the expectation values of the components of the position operator, $\hat{x}$ and $\hat{y}$ in the coherent state and show that it is peaked around the point $x = \sqrt{2}l_B \,\mathrm{Re}\, z, y = \sqrt{2}l_B \,\mathrm{Im}\, z$ in the x,y-plane. Use the coherent state representation for the $|m\rangle$ states, to demonstrate that the number of independent states in the lowest Landau level increases linearly with the available area in the x,y-plane. Find the density of states in the x,y-plane.

g) We assume that a weak, constant electric field E is introduced in the x-direction. Show that this effectively introduces the following Hamiltonian in the lowest Landau level,

$$H' = \frac{1}{2}\hbar\omega - \frac{l_b}{\sqrt{2}} eE(\hat{a} + \hat{a}^\dagger) \tag{7}$$

Also show that this Hamiltonian gives a time dependence to the coherent state $|z(t)\rangle$, corresponding to a drift with constant velocity in the y-direction. What is the drift velocity?

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Midterm Exam, Fall Semester 2006

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Deadline for returning solutions
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Questions concerning the problems
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Language
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For solving the problems, it may be useful to consult the relevant sections of the lecture notes.

PROBLEMS

1 Spin coherent states

We consider a quantum spin $\hat{\mathbf{J}}$ which acts in a $2j+1$ dimensional vector space. In the standard way we introduce a set of basis vectors $|j,m\rangle$, where j is the quantum number of the total spin, so that $\hat{\mathbf{J}}^2 = j(j+1)\hbar^2\mathbb{1}$ and m is the quantum number of the z-component, $\hat{J}_z|j,m\rangle = m\hbar|j,m\rangle$. Thus, m runs from $-j$ to j and identifies the basis vectors, while j is a fixed number which characterizes the size of the total spin.

We remind about the following relations,

$$\begin{aligned} \hat{J}_z|j,m\rangle &= m\,\hbar\,|j,m\rangle \\ \hat{J}_-|j,m\rangle &= \sqrt{(j+m)(j-m+1)}\,\hbar\,|j,m-1\rangle \\ \hat{J}_+|j,m\rangle &= \sqrt{(j-m)(j+m+1)}\,\hbar\,|j,m+1\rangle \end{aligned} \tag{1}$$

where $\hat{J}_- = \hat{J}_x - i\hat{J}_y$ and $\hat{J}_+ = \hat{J}_x + i\hat{J}_y$.

The spin system has a certain similarity with a harmonic oscillator, in the sense that $\hat{J}_+$ and $\hat{J}_-$ are raising and lowering operators like $\hat{a}^\dagger$ and $\hat{a}$, and $\hat{J}_z$ like the harmonic oscillator Hamiltonian $\hat{H}_{ho}$ has a spectrum with constant separation between the levels, where these raising and lowering operators act. There are differences, in particular since the spectrum of $\hat{J}_z$ has a finite number of levels, whereas the number of levels of the harmonic oscillator is infinite. In spite of these differences, coherent states for the spin system can be introduced in a somewhat analogous way to that of the harmonic oscillator, but not precisely in the same way. In particular the coherent states cannot be defined as eigenstates of the lowering operator $\hat{J}_-$.

a) Since the m quantum number is limited by $m \leq j$ the lowering operator $\hat{J}_-$ has only one eigenvector, which is the lowest state $|j, -j\rangle$. How do you show this?

Instead of defining the coherent states as eigenvectors of the lowering operator, they are defined to be states of *minimum uncertainty* in the three components of the spin variable. To be more precise, such a state should minimize

$$(\Delta \mathbf{J})^2 = \left\langle \hat{\mathbf{J}}^2 \right\rangle - \left\langle \hat{\mathbf{J}} \right\rangle^2 \tag{2}$$

b) Assume a particular spin state points in the z-direction in the sense $\left\langle \hat{\mathbf{J}} \right\rangle = J\mathbf{k}$. Show that if this is a *minimum uncertainty state*, then $J = j\hbar$ or $J = -j\hbar$.

c) A *maximum spin state* is defined by

$$\mathbf{n} \cdot \hat{\mathbf{J}}\ |\mathbf{n}, j\rangle = j\hbar|\mathbf{n}, j\rangle \tag{3}$$

with $\mathbf{n}$ as a unit vector in an arbitrary direction. Based on the result of b), explain why such a maximum spin state is a minimum uncertainty state, with expectation value for the spin vector

$$\left\langle \hat{\mathbf{J}} \right\rangle \equiv \langle \mathbf{n}, j|\hat{\mathbf{J}}|\mathbf{n}, j\rangle = j\hbar\mathbf{n} \tag{4}$$

The above properties of $|\mathbf{n}, j\rangle$ justifies this to be *defined* as a coherent states of the spin system. The continuous set of all these states is specified by two variables, for example the polar angles θ, ϕ of the vector $\mathbf{n}$.

In the following we focus on the simplest case of spin half, $j = 1/2$. The spin coherent states in this case are

$$|\mathbf{n}\rangle \equiv |\mathbf{n}, \frac{1}{2}\rangle \tag{5}$$

corresponding to *spin up* along the $\mathbf{n}$ axis. For the particular case of spin along the z-axis we also introduce the notation $|0\rangle$ for the spin down state ($\mathbf{n} = -\mathbf{k}$) and $|1\rangle$ for the spin up state ($\mathbf{n} = \mathbf{k}$).

d) Show that for $j = 1/2$ the condition of minimum uncertainty is trivially statisfied, so that *any* state can be considered as a coherent state.

In order to bring the notation closer to that of the coherent states of the harmonic oscillator we represent the unit vector $\mathbf{n}$ by a complex number z in the following way

$$z = e^{-i\phi} \cot \frac{\theta}{2} \tag{6}$$

with ϕ and θ as the polar angles of $\mathbf{n}$. (This mapping from the unit sphere to the complex plane is referred to as a *stereographic projection*.) We further define $|z\rangle \equiv |\mathbf{n}\rangle$. With this definition $z = 0$ corresponds to the spin-down state $|0\rangle$ along the z-axis, and for general z we have

$$\sigma_{\mathbf{n}}|z\rangle = |z\rangle\ , \quad \sigma_{\mathbf{n}} = \boldsymbol{\sigma} \cdot \mathbf{n} \tag{7}$$

with $\mathbf{n}$ and z related by (6).

e) With $|k\rangle$, $k = 0, 1$ as the spin states along the z-axis, show that the transition function between these basis states and the coherent states $|z\rangle$ can be written as

$$\langle k|z\rangle = \frac{z^k}{\sqrt{1+|z|^2}} \,, \quad k = 0, 1 \tag{8}$$

(This corresponds to a particular choice of the complex phase of the coherent state. In the following we will make use of this choice of phase.)

We now introduce a coherent state representation by using the coherent states as basis vectors. For a general state $|\psi\rangle$ the wave function in the z-representation is then defined as

$$\psi(z) = \langle z|\psi\rangle \tag{9}$$

f) Determine for $|\psi\rangle = |z_0\rangle$ the square modulus of the wave function,

$$|\psi_{z_0}(z)|^2 \equiv |\langle z|z_0\rangle|^2 \tag{10}$$

g) Show that the spin coherent states satisfy a completeness relation of the form

$$\int \frac{d^2z}{2\pi} \frac{4}{(1+|z|^2)^2} |z\rangle\langle z| = \mathbb{1} \tag{11}$$

where d^2z denotes the standard area element in the two-dimensional plane, and demonstrate how this completeness relation can be used to reconstruct the abstract vector $|\psi\rangle$ of any spin state from the corresponding wave function $\psi(z)$.

2 Entanglement in a three-particle system.

Three spin-half particles, referred to as $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, are produced by a source $\mathcal{S}$ in a correlated quantum state. The corresponding state vector can be factorized in a spin state and a position state. We focus on the spin state, which has the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|uuu\rangle - |ddd\rangle) \tag{12}$$

where $|uuu\rangle = |u\rangle_{\mathcal{A}} \otimes |u\rangle_{\mathcal{B}} \otimes |u\rangle_{\mathcal{C}}$, is the tensor product of *spin up* along the z-axis for all the three particles, while $|ddd\rangle = |d\rangle_{\mathcal{A}} \otimes |d\rangle_{\mathcal{B}} \otimes |d\rangle_{\mathcal{C}}$ is the product state corresponding to *spin down* for all the three particles along the z-axis. The state (12) is often referred to as a GHZ state (Greenberger, Horne and Zeilinger). It corresponds to a so-called Bell-state for two particles, and can obviously be generalized to an arbitrary number of spin half particles.

a) What do we mean by saying that the spins of the three particles are in a *correlated* state, and what do we mean by calling the state *entangled*.

We first consider a division of the full system into two subsystems, $\mathcal{ABC} = \mathcal{A} + \mathcal{BC}$, where particle $\mathcal{A}$ is considered as one of the subsystems and the two other particles $\mathcal{B}$ and $\mathcal{C}$ as forming the other subsystem.

b) Find the reduced density matrices $\hat{\rho}_{\mathcal{A}}$ and $\hat{\rho}_{\mathcal{BC}}$ for the two subsystems. What are the corresponding values for the (von Neuman) entropies $S_{\mathcal{A}}$ and $S_{\mathcal{BC}}$. With the reference to these values why do we call the GHZ state of the full system $\mathcal{ABC}$ *maximally* entangled? If we consider only subsystem $\mathcal{BC}$ is there any entanglement between spin $\mathcal{B}$ and $\mathcal{C}$?

We introduce two new sets of spin basis vectors corresponding to quantized spin in the x- and y-directions. $|f\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)$ is spin up in the x-direction and $|b\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$ is spin down, while $|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle)$ is spin up in the y-direction and $|l\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|d\rangle)$ is spin down.

c) Rewrite the GHZ in two different ways, first by using basis vectors with quantized spin along the x-axis for all three particles, then by using vectors with quantized spin along the y-axis for spin $\mathcal{A}$ and $\mathcal{B}$, but quantized spin along the x-axis for spin $\mathcal{C}$.

d) Explain, by reference to the expressions for the GHZ state in c), how the three spin components of particle $\mathcal{A}$ can be determined by performing spin measurements on particle $\mathcal{B}$ and $\mathcal{C}$, while *not making any measurement* on particle $\mathcal{A}$. Specify in each of the three cases what directions should be chosen for spin measurements on $\mathcal{B}$ and $\mathcal{C}$ (not necessarily the same direction for both particles).

We assume now the three particle spins to be in the GHZ state, and *all the three particles to be far apart*. We follow the analysis of the EPR paradox and draw from point d) the conclusion that all three spin components of spin $\mathcal{A}$ represent *elements of reality* and can therefore be ascribed independent, measurable values $m_x^{\mathcal{A}}$, $m_y^{\mathcal{A}}$ and $m_z^{\mathcal{A}}$, even before any measurement is performed on the system. These values we have to consider as unknown, since they are not determined by the quantum state. Note that exactly the same argument can be used for the spin components of particle $\mathcal{B}$ and $\mathcal{C}$.

We follow up this analysis by simply assuming the particles to have sharp, although unknown values m_x, m_y and m_z, for all three spin components of each of the particles. For simplicity scale the spin variables m_x etc. to take values $+1$ and -1 for spin up and spin down. We confront this assumption with the predictions of quantum mechanics.

e) Show that the following composite spin operators, $\hat{\mathcal{O}}_1 = \sigma_x \otimes \sigma_y \otimes \sigma_y$, $\hat{\mathcal{O}}_2 = \sigma_y \otimes \sigma_x \otimes \sigma_y$ and $\hat{\mathcal{O}}_3 = \sigma_y \otimes \sigma_y \otimes \sigma_x$, have the GHZ state as an eigenstate. What are the corresponding eigenvalues? We also consider a fourth operator $\hat{\mathcal{O}}_4 = \sigma_x \otimes \sigma_x \otimes \sigma_x$. Show that $\hat{\mathcal{O}}_4 = -\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2\hat{\mathcal{O}}_3$ and therefore also has the GHZ state as eigenstate. What is the corresponding eigenvalue.

f) Assuming the spin components of all three particles to have well defined values, denoted m_x^A, m_y^A, etc., we interpret the eigenvalue equations for the operators $\hat{\mathcal{O}}_1$, $\hat{\mathcal{O}}_2$ and $\hat{\mathcal{O}}_3$ as giving similar equations for products of the m variables. Write these equations, which give constraints on the unknown values.

Write also the corresponding equation for operator $\hat{\mathcal{O}}_4$. Show that this equation is in conflict with the three first ones.

This demonstrates that the assumption that all spin components have well defined, although unknown values, by following EPR's arguments about *elements of reality*, leads to a direct conflict with predictions of quantum mechanics. In the case discussed here we note that this conflict is a consequence of the technical point that different components of the spin operator of a particle *anticommute* while the corresponding measurable values m_x, m_y and m_z necessarily have to *commute*.

FYS 4110: Non-relativistic quantum mechanics

Midterm Exam, Fall Semester 2007

The problem set consists of 2 problems written on 4 pages.
This set is available from Friday October 19.

Deadline for returning solutions
is Friday October 26.
Written/printed solutions should be returned to Ekspedisjonskontoret in the Physics Building.

Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Office: room 471, email: j.m.leinaas@fys.uio.no).
Language
Solutions may be written in Norwegian or English, depending on your preference.

For solving the problems, it may be useful to consult the relevant sections of the lecture notes.

PROBLEMS

1 Density operators

A density operator of a two-level system can be represented by a 2×2 (density) matrix in the form

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma}), \quad |\mathbf{r}| \leq 1 \tag{1}$$

where $\mathbb{1}$ is the 2×2 identity matrix, $\mathbf{r}$ is a vector in three dimensions and $\boldsymbol{\sigma}$ is a vector operator with the Pauli matrices as the Cartesian components. Geometrically the set of all density matrices form of a sphere in three dimensions, with the pure states $|\mathbf{r}| = 1$ as the surface of the sphere (the Bloch sphere), and the mixed states as the interior of the sphere.

a) The density operator can also be expressed in bra-ket formulation as

$$\hat{\rho} = \rho_{11}\,|+\rangle\langle+| \;+\; \rho_{12}\,|+\rangle\langle-| \;+\; \rho_{21}\,|-\rangle\langle+| \;+\; \rho_{22}\,|-\rangle\langle-| \tag{2}$$

where $|\pm\rangle$ is the state of the upper/lower level of the system, that is with $\sigma_z|\pm\rangle = \pm|\pm\rangle$. What are the coefficients $\rho_{ij},\ i,j = 1,2$, expressed in terms of the Cartesian components x, y, z of $\mathbf{r}$?

We consider in the following a composite system with two subsystems $\mathcal{A}$ and $\mathcal{B}$. These are both two-level systems so that the Hilbert space of the full system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is of dimension 4. A density matrix of the composite system can be written as

$$\hat{\rho} = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \sum_i a_i\,\sigma_i \otimes \mathbb{1} + \sum_j b_j\,\mathbb{1} \otimes \sigma_j + \sum_{ij} c_{ij}\sigma_i \otimes \sigma_j) \tag{3}$$

with a_i, b_j and c_{ij} as coefficients, and with the first factor in the tensor product corresponding to the $\mathcal{A}$ subsystem and the other to $\mathcal{B}$.

b) Find the reduced density matrices of subsystems $\mathcal{A}$ and $\mathcal{B}$ expressed in terms of the a, b and c coefficients. What condition should the a, b and c coefficients satisfy if the two subsystems should be completely uncorrelated?

We examine the four *Bell states* of the composite system,

$$\begin{aligned} |c,\pm\rangle &= \tfrac{1}{\sqrt{2}}\left(|++\rangle \pm |--\rangle\right) \\ |a,\pm\rangle &= \tfrac{1}{\sqrt{2}}(|+-\rangle \pm |-+\rangle) \end{aligned} \tag{4}$$

where $|ij\rangle = |i\rangle \otimes |j\rangle$, $i, j = \pm$, are tensor product states.

c) Give the expressions for the density operators of the four states, first in the bra-ket form, and then written in the form (3). What are the reduced density matrices of subsystems $\mathcal{A}$ and $\mathcal{B}$ for these four states? Give the entropy of the full system and the two subsystems in the four cases. Why do we call the Bell states *maximally entangled*?

d) We consider linear combinations of the form

$$\hat{\rho} = x\hat{\rho}_1 + (1-x)\hat{\rho}_2 \tag{5}$$

where $\hat{\rho}_1$ and ρ_2 represent two Bell states and x is a real parameter. Show that if we have $0 < x < 1$ the linear combination is a density operator. Why is that not the case if $x < 0$ or $x > 1$?

e) Choose a pair of Bell states and show that halfway between them ($x = 1/2$) the density matrix gets a particularly simple form. Show that it can be written in the form

$$\hat{\rho} = \frac{1}{8}\left[(\mathbb{1} + \mathbf{n}\cdot\boldsymbol{\sigma}) \otimes (\mathbb{1} + \mathbf{m}\cdot\boldsymbol{\sigma}) + (\mathbb{1} - \mathbf{n}\cdot\boldsymbol{\sigma}) \otimes (\mathbb{1} - \mathbf{m}\cdot\boldsymbol{\sigma})\right] \tag{6}$$

where $\mathbf{n}$ is a unit vector and $\mathbf{m} = \pm\mathbf{n}$. What does this expression show about entanglement between the two subsystems $\mathcal{A}$ and $\mathcal{B}$ for this particular state?

f) The Bell states define a subspace in the space of all 4×4 density matrices. Show that the density matrices in this subspace commute.

2 Jaynes-Cummings model

The Jaynes-Cummings model is a simplified model for the system of an atom interacting with the electromagnetic field. One assumes that only two of the atomic energy levels are involved in the interaction, so that the atom can be modelled as a two-level system. One further assumes that only one field mode is excited, so that the field can be modelled by an harmonic oscillator. In this oscillator model the different energy levels correspond to different numbers of photons in the excited field mode. The situation is most relevant for an atom in a reflecting cavity where a single mode of the field can be strongly excited.

We write the Hamiltonian of the model in the following way

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega a^\dagger a + i\hbar\lambda(a^\dagger\sigma_- - a\sigma_+) \equiv \hat{H}_0 + \hat{H}_1 \tag{7}$$

where the $\hat{H}_0$ includes the two first terms, which describe the non-interacting atom and photons, and $\hat{H}_1$ the third term, which describes interactions between the atoms and the photons. The expression $\hbar\omega_0$ in the first term gives the energy difference between the two atomic levels, while $\hbar\omega$ is the photon energy. (The zero point of the energy has been adjusted to absorb the ground state energy of the harmonic oscillator and to place the energies of the two-level system symmetrically about $E = 0$.) The Pauli matrices act as operators between the atomic levels, with $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$, and $a^\dagger$ and a are operators that create and destroys a photon. The interaction term thus has two parts, where the first part creates a photon while lowering the atomic energy and the other part destroys a photon while increasing the atomic energy. λ is a real valued parameter that determines the strength of the interaction. The simple form of the interaction $\hat{H}_1$ given here is valid in the *rotating wave approximation*. This gives a good approximation to the full interaction when the two frequencies ω_0 and ω are close in value.

The objective is to study the time evolution of this system, where the interaction term will induce oscillations between the atomic levels. These oscillations are called *Rabi oscillations* and are examined in a somewhat different way in Sect. 1.3.2 of the lecture notes. There the electromagnetic field was treated as an external time dependent perturbation, while we here include the field as a part of the full quantum system and describe it in terms of photons.

We use the notation $|m,n\rangle$ for the eigenstates of the non-interacting Hamiltonian $\hat{H}_0$, with $m = \pm 1$ indicating the atomic state and $n = 0, 1, 2, ...$ indicating the number of photons (which is here the level number of the harmonic oscillator).

a) Show that the interaction Hamiltonian $\hat{H}_1$ couples the unperturbed levels only in pairs that differ by one photon. We define such a pair of states as $|1\rangle \equiv |1, n-1\rangle$ and $|2\rangle \equiv |-1, n\rangle$. Show that the Hamiltonian in the subspace spanned by this pair of states can be written as a 2x2 matrix of the form

$$H = \frac{1}{2}\hbar \begin{pmatrix} \Delta & -ig \\ ig & -\Delta \end{pmatrix} + \epsilon \mathbb{1} \tag{8}$$

with $\mathbb{1}$ as the 2×2 identity matrix, and find the expressions for Δ, g and ϵ.

b) Solve the eigenvalue problem for this 2x2 matrix Hamiltonian and find the energy eigenvalues and the eigenvectors in matrix form. To simplify expressions it may be convenient to write the matrix elements in terms of new parameters Ω and θ defined by

$$\Delta = \Omega\cos\theta\,, \quad g = \Omega\sin\theta\,, \tag{9}$$

c) In matrix form a general time dependent state can be written as

$$\psi(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \tag{10}$$

Find the time dependent coefficients $c_1(t)$ and $c_2(t)$ expressed in terms of Ω and θ for the initial condition $c_1 = 0$, $c_2 = 1$ at time $t = 0$. Show that $|c_1(t)|^2 = \sin^2\theta \sin^2\frac{\Omega t}{2}$.

d) Give a qualitative description of the result for the excitation of the atom, and make a comparison with the result of Sect. 1.3.2 of the lecture notes. How does the field strength B_1 in the lecture notes relate to the photon number n in the present case?

The system consisting of the atom and the photons can be considered as a composite quantum system, where the atom is subsystem $\mathcal{A}$ and the electromagnetic field (the photons) defines subsystem $\mathcal{B}$. The Hilbert space of the full system is then a tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The eigenstates of $\hat{H}_0$ referred to above are special cases of tensor product states,

$$\begin{aligned} |1\rangle &= |1, n-1\rangle = |+1\rangle_A \otimes |n-1\rangle_B \\ |2\rangle &= |-1, n\rangle \;\;\, = |-1\rangle_A \otimes |n\rangle_B \end{aligned} \qquad (11)$$

e) Write the time dependent state (10) as a "ket" vector expanded in the above product states, and give the expression for the corresponding density operator in the bra-ket form. (Write the expressions in terms of $c_1(t)$ and $c_2(t)$ without using the solutions for these.)

f) Show that the reduced density matrix of the atom (subsystem $\mathcal{A}$) can be written as a 2x2 matrix that depends only on $|c_1|^2$ and $|c_2|^2$. Find the corresponding von Neumann entropy expressed in terms of θ and Ω and plot this as a time dependent function for several different values of $\sin\theta$ with Ω fixed. What do these plots show about variations in the entanglement between the atom and the electromagnetic field?

FYS 4110: Non-relativistic quantum mechanics

Midterm Exam, Fall Semester 2008

The problem set is available from Friday October 17. The set consists of 2 problems written on 4 pages. For solving the problems, it may be useful to consult the relevant sections of the lecture notes.

Deadline for returning solutions
Friday October 24. *Written/printed* solutions should be returned to Ekspedisjonskontoret in the Physics Building.

Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Office: room 471, email: j.m.leinaas@fys.uio.no) or the assistant Mads Stormo Nilsson (Office: room 340V, email: m.s.nilsson@fys.uio.no).

Language
Solutions may be written in Norwegian or English, depending on your preference.

PROBLEMS

1 Spin splitting in positronium

Positronium is a bound system of an electron and a positron. The two particles have the same mass m and charges of opposite signs $\pm e$, with e denoting the electron charge. The energy spectrum of the bound system is similar to that of a hydrogen atom, but the energy scale is different since the *reduced mass* of the two-particle system has about half the value in positronium compared to hydrogen. Positronium has a finite life time since the electron and the positron will eventually annihilate.

The ground state of positronium is degenerate due to the spin degrees of freedom of the two particles. We distinguish between *para-positronium*, which is a spin *singlet* state with total spin $S = 0$, and *ortho-positronium* which is a *triplet* state with total spin $S = 1$. Para-positronium has a life time of 125 picoseconds while the life time of ortho-positronium is about 140 nanoseconds.

The interaction between the magnetic moments of the two particles give rise to a (hyperfine) splitting of the ground state energy, so that the singlet state has a slightly lower energy than the triplet state. In the following we make the simplifying assumption that this effect can be studied in the four-dimensional spin space of the two particles. This means that we assume no coupling between the spin and orbital coordinates of the particles so that the wave function of the orbital motion is the same for all the spin states and can therefore be neglected.

We denote in the following the *spin up* state vector of the z-component of the spin for any of the two particles as $|+\rangle$ and the *spin down* state by $|-\rangle$. The four dimensional space of spin states has the tensor product form $\mathcal{H} = \mathcal{H}_e \otimes \mathcal{H}_p$, with $\mathcal{H}_e$ as the two-dimensional spin space of the electron and $\mathcal{H}_p$ as the spin space of the positron. The full space is spanned by the four product states

$$
\begin{aligned}
|++\rangle &= |+\rangle \otimes |+\rangle\,, \quad |+-\rangle = |+\rangle \otimes |-\rangle\,, \\
|-+\rangle &= |-\rangle \otimes |+\rangle\,, \quad |--\rangle = |-\rangle \otimes |-\rangle\,,
\end{aligned} \tag{1}
$$

where we assume the first factor in the tensor product to describe the *electron* spin. In the four-dimensional spin space the spin operators of the electron and the positron have the following forms,

$$
\begin{aligned}
\hat{\mathbf{S}}_e &= \frac{\hbar}{2}\boldsymbol{\sigma}_e \otimes \mathbb{1}_p \equiv \frac{\hbar}{2}\boldsymbol{\Sigma}_e \\
\hat{\mathbf{S}}_p &= \frac{\hbar}{2}\mathbb{1}_e \otimes \boldsymbol{\sigma}_p \equiv \frac{\hbar}{2}\boldsymbol{\Sigma}_p
\end{aligned} \tag{2}
$$

with $\mathbb{1}_e$ as the identity operator in the two-dimensional spin space of the electron,$\mathbb{1}_p$ as the identity operator in the spin space of the positron, and $\boldsymbol{\sigma}_e$ and $\boldsymbol{\sigma}_p$ as the Pauli matrices acting in the two-dimensional spin spaces of the electron and the positron respectively.

a) Show that in the product basis we have

$$
\langle ij|\boldsymbol{\Sigma}_e \cdot \boldsymbol{\Sigma}_p|kl\rangle = \langle i|\boldsymbol{\sigma}_e|k\rangle \cdot \langle j|\boldsymbol{\sigma}_p|l\rangle \tag{3}
$$

b) Find the operator product $\hat{\mathbf{S}}_e \cdot \hat{\mathbf{S}}_p$ expressed as a 4×4 matrix in the product basis. (In the matrix representation list the basis vectors in the order $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$.)

We now introduce another basis, the *spin basis* with the four vectors

$$
|0,0\rangle \;=\; \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \tag{4}
$$

and

$$
\begin{aligned}
|1,1\rangle &= |++\rangle \\
|1,0\rangle &= \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\
|1,-1\rangle &= |--\rangle
\end{aligned} \tag{5}
$$

c) Show that $\hat{\mathbf{S}}_e \cdot \hat{\mathbf{S}}_p$ is a diagonal matrix in the new basis.

The total (intrinsic) spin of the two particles is $\hat{\mathbf{S}} = \hat{\mathbf{S}}_e + \hat{\mathbf{S}}_p$. Show that the new basis vectors are eigenstates of $\hat{\mathbf{S}}^2$ and $\hat{S}_z$ and find the eigenvalues. Check that the result for the eigenvalues is consistent with (4) being the singlet state and (5) being the triplet state.

The Hamiltonian in the spin space can be written in the form

$$
H_0 = E_0\mathbb{1} + \kappa\hat{\mathbf{S}}_e \cdot \hat{\mathbf{S}}_p \tag{6}
$$

where E_0 is the ground state energy with spin effects excluded, $\mathbb{1}$ is the identity operator in the four-dimensional spin space and κ is a positive constant determined by the magnetic moments of the particles.

A magnetic field $\mathbf{B}$ is turned on in the z direction. This leads to a splitting of the spin energy states, referred to as the *Zeeman effect*. The form of the modified Hamiltonian is

$$\hat{H} = E_0 \mathbb{1} + \kappa \hat{\mathbf{S}}_e \cdot \hat{\mathbf{S}}_p + \lambda \hbar (\hat{S}_{e\,z} - \hat{S}_{p\,z}) \tag{7}$$

with λ as a parameter proportional to B.

d) Write the Hamilton H as a 4×4 matrix in the spin basis.

e) Solve the eigenvalue problem for the Hamiltonian (7) and find the energies expressed in terms of the parameters E_0, κ and λ. Plot the energies as functions of $x \equiv \lambda/\kappa$ for fixed E_0 and κ.

Two of the energy eigenstates are mixtures of $|0,0\rangle$ and $|1,0\rangle$. We write these two states as

$$\begin{aligned} |A\rangle &= a\,|+-\rangle + b\,|-+\rangle \\ |B\rangle &= -b^*\,|+-\rangle + a^*\,|-+\rangle \end{aligned} \tag{8}$$

where a and b are functions of x, with $|a|^2 + |b|^2 = 1$.

f) Give the expressions for the corresponding density operators $\hat{\rho}_A$ and $\hat{\rho}_B$, and for the *reduced density operators* $\hat{\rho}_{Ae}, \hat{\rho}_{Ap}$ and $\hat{\rho}_{Be}, \hat{\rho}_{Bp}$ of the electron and positron subsystems. The degree of entanglement in the system is given by the *von Neumann entropy* of the reduced density operators. Show that the degree of entanglement is the same for $|A\rangle$ and $|B\rangle$ and can be expressed as a function of $|a|^2$.

g) Determine the function $|a(x)|^2$ from the eigenvalue problem in e) and use this to make a plot of the degree of entanglement in the system as a function of x for the two states $|A\rangle$ and $|B\rangle$. Are these *maximum entanglement states* for any value of x?

2 Driven harmonic oscillator

The Hamiltonian of a one-dimensional harmonic oscillator is given by the expression

$$\hat{H} = \frac{1}{2m}(\hat{p}^2 + m^2\omega_0^2\hat{x}^2) = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{9}$$

with the *raising* and *lowering* operators defined by

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega_0}}(m\omega_0\hat{x} + i\hat{p})\ , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega_0}}(m\omega_0\hat{x} - i\hat{p}) \tag{10}$$

The time evolution operator is

$$\hat{\mathcal{U}}_0(t) = e^{-\frac{i}{\hbar}t\hat{H}_0} \tag{11}$$

The coherent states of the oscillator are defined as eigenvectors of the lowering operator

$$\hat{a}|z\rangle = z|z\rangle \tag{12}$$

The general coherent state $|z\rangle$ is related to the ground state of the oscillator $|0\rangle$ by

$$|z\rangle = \hat{\mathcal{D}}(z)|0\rangle = e^{-z^*z}e^{z\hat{a}^\dagger}|0\rangle \tag{13}$$

where the unitary shift operator is given by

$$\hat{\mathcal{D}}(z) = e^{z\hat{a}^\dagger - z^*\hat{a}} \tag{14}$$

a) Show that for a general operator $\hat{A}$ we have the relation

$$\hat{\mathcal{U}} e^{\hat{A}} \hat{\mathcal{U}}^{-1} = e^{\hat{\mathcal{U}}\hat{A}\hat{\mathcal{U}}^{-1}} \tag{15}$$

and use that to calculate the operator $\hat{\mathcal{U}}_0(t)\hat{\mathcal{D}}(z)\hat{\mathcal{U}}_0(t)^\dagger$. Make use of the result to determine the time dependent state vector $|\psi(t)\rangle$, when this initially is a coherent state $|\psi(0)\rangle = |z_0\rangle$. Show that $|\psi(t)\rangle$ at later times t is also a coherent state.

We next assume the harmonic oscillator to be under influence of a time dependent external potential, so that the hamiltonian now is

$$\hat{H} = \hat{H}_0 + \hat{W}(x,t) \tag{16}$$

In the following we assume the external potential to have the specific form

$$\hat{W}(x,t) = A\hat{x}\sin\omega t \tag{17}$$

with A as a constant and ω as the oscillation frequency of the external potential.

b) Find the Heisenberg equation of motion for $\hat{x}$ and $\hat{p}$ and show that they correspond to the equation of motion of a *driven* harmonic oscillator, that is subject to the periodic force $f(t) = -A\sin\omega t$.

c) Give the definition of the time evolution operator $\hat{\mathcal{U}}_I(t)$ in the *interaction picture* and show that it satisfies an equation of the form

$$i\hbar\frac{d}{dt}\hat{\mathcal{U}}_I(t) = \hat{H}_I(t)\hat{\mathcal{U}}_I(t) \tag{18}$$

Assume $\hat{W}$ is treated as the interaction. Show that $\hat{H}_I(t)$ then is a linear function of $\hat{a}$ and $\hat{a}^\dagger$,

$$\hat{H}_I(t) = \theta(t)^*\,\hat{a} + \theta(t)\,\hat{a}^\dagger \tag{19}$$

and determine the function $\theta(t)$.

d) Show that the equation (18) has a solution of the form

$$\hat{\mathcal{U}}_I(t) = e^{\xi(t)\hat{a}^\dagger - \xi^*(t)\hat{a}}\, e^{i\phi(t)} \tag{20}$$

with $\xi(t)$ as a complex and $\phi(t)$ as a real function of time. What are the equations that these two functions should satisfy?

e) Use the expressions for $\hat{\mathcal{U}}_0(t)$ and $\hat{\mathcal{U}}_I(t)$ to find the time dependent state vector $|\psi(t)\rangle$ in the Schrödinger picture, with the same initial condition as in a). Show that also in this case it describes a time dependent coherent state, of the form $|\psi(t)\rangle = e^{i\gamma(t)}|z(t)\rangle$. Find $z(t)$ expressed in terms of z_0, $\xi(t)$ and ω_0.

f) Determine the function $\xi(t)$ and find an explicit expression for $z(t)$. The corresponding real coordinate is $x(t) = \sqrt{2\hbar/m\omega_0}\,\mathrm{Re}\,z(t)$. Does this coordinate satisfy the classical equation of motion of the driven harmonic oscillator?

FYS 4110: Ikke-relativistisk kvantemekanikk

Midttermineksamen, høsten 2009

Oppgavesettet er tilgjengelig fra fredag 9. oktober. Settet består av 2 oppgaver på 4 sider.

Frist for innlevering av besvarelser
Fredag 16. oktober. *Trykte/håndskrevne* besvarelser innleveres på ekspedisjonskontoret ved Fysisk institutt.

Spørsmål angående oppgavene
Spørsmål kan rettes til foreleser, Jon Magne Leinaas (rom 471Ø) eller assistent på kurset, Mads Stormo Nilsson (rom 340V).

Språk
Oppgavene er skrevet på engelsk, siden alt skriftlig materiell i kurset har vært på engelsk, men besvarelser kan skrives på norsk eller engelsk etter eget ønske.

OPPGAVER

1 Harmonic oscillator coupled to spin

We consider the motion of an electron in a one dimensional harmonic oscillator potential, with the electron spin being under the influence of a space-dependent magnetic field. The Hamiltonian has the form

$$\hat{H} = \frac{1}{2m}(\hat{p}^2 + m^2\omega^2\hat{x}^2) - \frac{e}{m}\mathbf{B}\cdot\mathbf{S} \tag{1}$$

where $\mathbf{B}$ is an x-dependent magnetic field

$$\mathbf{B}(x) = B_0\,\mathbf{k} + B_1\,\frac{x}{d}\,\mathbf{i} \tag{2}$$

with B_0 and B_1 as constants, d as a parameter with dimension of length, $\mathbf{k}$ as a unit vector orthogonal to the direction of motion and $\mathbf{i}$ as a unit vector in the direction of motion.

We introduce the raising operator $\hat{a}^\dagger$ and the lowering operator $\hat{a}$ of the harmonic oscillator, and further the raising and lowering operators of the z-component of the spin , $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. The Hamiltonian can then be split in two parts

$$\hat{H} = \hat{H}_0 + \hat{H}_1 \tag{3}$$

with

$$\hat{H}_0 = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \frac{1}{2}\hbar\omega_0\,\sigma_z \tag{4}$$

and

$$\hat{H}_1 = \frac{1}{2}\hbar\lambda\,(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_- + \hat{a}\sigma_- + \hat{a}^\dagger\sigma_+) \tag{5}$$

where $\omega_0 = -eB_0/m$ and $\lambda = -(eB_1/md)\sqrt{\hbar/(2m\omega)}$.

We shall further make the assumption that the $\omega \approx \omega_0$, so that $|\omega - \omega_0| << \omega + \omega_0$ and shall also assume that the spin energy contribution from $\hat{H}_1$ is smaller than the spin energy of $\hat{H}_0$, in the sense $\lambda \lesssim \omega$. This allows us to introduce a simplification, by neglecting the $\hat{a}\sigma_-$ and the $\hat{a}^\dagger\sigma_+$ terms in H_1, so it takes the form

$$\hat{H}_1 = \frac{1}{2}\hbar\lambda\left(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-\right) \tag{6}$$

In the following we shall use this simplified form for $\hat{H}_1$.

a) We write the eigenvalue equation of $\hat{H}_0$ as

$$\hat{H}_0|n,m\rangle = E^0_{nm}|n,m\rangle \tag{7}$$

with $n = 0, 1, 2, ...$ as the energy quantum number of the harmonic oscillator and $m = \pm 1/2$ as the quantum number of $\hat{S}_z$. What is the expression for the energies $E^0_{n,m}$? Can you give a reason (qualitative) why the terms left out in $\hat{H}_1$ (Eq.(6)) may be considered as less important than the ones that are kept?

b) Show that $\hat{H}_1$ couple the states $|n,m\rangle$ only in pairs, and show in particular that it gives a coupling between $|0,+\frac{1}{2}\rangle$ and $|1,-\frac{1}{2}\rangle$, but with no coupling to other states $|n,m\rangle$. Write the full energy eigenvector equation as a matrix equation for these two states and determine the energies, expressed in terms of ω, ω_0 and λ.

A simplification of expressions, here and in the following, may be suggested by introducing the abbreviations $\Delta\omega = \omega - \omega_0$ and $\Omega = \sqrt{\Delta\omega^2 + \lambda^2}$.

c) The two energy eigenstates $|\psi_\pm\rangle$, in the subspace spanned by $|0,+\frac{1}{2}\rangle$ and $|1,-\frac{1}{2}\rangle$, can be written as

$$|\psi_+\rangle = \cos\beta\,|0,+\frac{1}{2}\rangle - \sin\beta\,|1,-\frac{1}{2}\rangle\,, \quad |\psi_-\rangle = \sin\beta\,|0,+\frac{1}{2}\rangle + \cos\beta\,|1,-\frac{1}{2}\rangle \tag{8}$$

Find the coefficients $\cos\beta$ and $\sin\beta$ expressed in terms of the parameters ω, ω_0 and λ (or $\Delta\omega$ and Ω) of the Hamiltonian.

d) Consider next the time evolution of a state

$$|\psi(t)\rangle = c_1(t)|0,+\frac{1}{2}\rangle + c_2(t)|1,-\frac{1}{2}\rangle \tag{9}$$

with initial condition $c_1(0) = 1$ and $c_2(0) = 0$. Determine the time-dependent coefficients $c_1(t)$ and $c_2(t)$.

e) Find and expression for $|c_2(t)|^2 = 1 - |c_1(t)|^2$ and show that it changes periodically with t. What is the period? Determine the maximum value as a function of the parameters of the Hamiltonian and plot it as a function of ω_0 with ω and λ fixed. What is the condition for resonance, where $|c_2(t)|^2$ has its largest maximum value?

f) The density operator can be written as

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| \tag{10}$$

Give the expression for $\hat{\rho}(t)$ in terms of the basis vectors $|n, m\rangle$. Find the corresponding expression for the reduced density operators $\hat{\rho}_s(t)$, which involves only the spin degrees of freedom, and $\hat{\rho}_p(t)$ which involves only the position degrees of freedom.

g) For the time dependent state $|\psi(t)\rangle$ determine the expectation values

$$\begin{aligned} \langle \boldsymbol{\sigma}(t) \rangle &= \langle \psi(t) | \, \boldsymbol{\sigma} \, | \psi(t) \rangle \\ \langle x(t) \rangle &= \langle \psi(t) | \, \hat{x} \, | \psi(t) \rangle \\ \langle x\boldsymbol{\sigma}(t) \rangle &= \langle \psi(t) | \, \hat{x}\boldsymbol{\sigma} \, | \psi(t) \rangle \end{aligned} \tag{11}$$

Give a qualitative explanation of the results based on the classical understanding of motion of the particle in the harmonic oscillator potential and of the electron spin in the magnetic field.

2 Squeezed coherent states

Coherent states, as defined in the lecture notes, play an important role in quantum optics, in particular in the description of laser light. There are some related states, called *squeezed states* that are also important. They are closely related to the coherent states and are like these *minimal uncertainty* states. They are referred to as squeezed states, since the uncertainty in one direction in phase space is (at a given time) *reduced* relative to that of the coherent state, while the uncertainty in the conjugate direction is *increased*. These uncertainties will then typically oscillate in time.

In this problem we study some of the properties of squeezed states for a one-dimensional harmonic oscillator, with Hamiltonian of the standard form

$$\hat{H} = \frac{1}{2m}(\hat{p}^2 + m^2\omega^2\hat{x}^2) = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{12}$$

The *squeeze operator* is introduced by the following expression

$$S_\lambda = e^{\frac{1}{2}[\lambda^*\hat{a}^2 - \lambda\,\hat{a}^{\dagger\,2}]} \tag{13}$$

with λ as a complex *squeezing parameter*. We define the squeezed coherent states as

$$|z, \lambda\rangle = S_\lambda |z\rangle \tag{14}$$

with $|z\rangle$ as a coherent state, and introduce squeezed raising and lowering operators in the following way

$$\hat{b}_\lambda = S_\lambda \hat{a} S_\lambda^\dagger \,, \quad \hat{b}_\lambda^\dagger = S\hat{a}^\dagger S^\dagger \tag{15}$$

a) Show that S_λ is a unitary operator, demonstrate that the following expressions are correct

$$\hat{b}_\lambda = \cosh|\lambda|\,\hat{a} + \frac{\lambda}{|\lambda|}\sinh|\lambda|\,\hat{a}^\dagger \tag{16}$$

$$\hat{b}_\lambda^\dagger = \cosh|\lambda|\,\hat{a}^\dagger + \frac{\lambda^*}{|\lambda|}\sinh|\lambda|\,\hat{a} \tag{17}$$

and show that $\hat{b}_\lambda$ and $\hat{b}_\lambda^\dagger$ satisfy the same commutation relations as $\hat{a}$ and $\hat{a}^\dagger$.

b) Show that the squeezed coherent states $|z, \lambda\rangle$ are eigenvectors of $\hat{b}_\lambda$. What are the eigenvalues?

c) Show that if the squeezing parameter λ is real the squeezing operator scales the the position and momentum operators in resiprocal ways,

$$S_\lambda \, \hat{x} \, S_\lambda^\dagger = d \, \hat{x} \,, \quad \hat{S}_\lambda \, \hat{p} \, S_\lambda^\dagger = \frac{1}{d} \, \hat{p} \tag{18}$$

and determine the scale factor d. Show that the squeezed states $|z, \lambda\rangle$ (with λ real) satisfy the same minimal uncertainty relation as the coherent states $|z\rangle$.

d) A squeezed ground state (with $z = 0$) can be expanded in the energy eigenstates as

$$|0, \lambda\rangle = \sum_{n=0}^{\infty} c_n |2n\rangle \tag{19}$$

Explain, based on the definition of $|z, \lambda\rangle$, why only eigenstates corresponding to even values $2n$ appear in the expansion. By use of the results of a) and b) show that the coefficients c_n satisfy a recursion relation and use this to determine the coefficients.

e) For three different values $\lambda = 0.5, 1.0$ and 1.5, plot the coefficient $|c_n|^2$ as function of the discrete variable n. Give a comment on how the distribution over energy levels changes with λ.

f) Assume that the harmonic oscillator is initially, for $t = 0$, in the squeezed coherent state $|\psi(0)\rangle = |z_0, \lambda_0\rangle$. Show that during the time evolution it will remain a squeezed state of the form $|\psi(t)\rangle = e^{i\alpha(t)}|z(t), \lambda(t)\rangle$, with $\alpha(t)$ as an unspecified complex phase. Determine $z(t)$ and $\lambda(t)$.

g) In the case $z_0 = 0$ evaluate for λ_0 real and positive, the following time dependent mean values, $\langle \hat{x} \rangle$, $\langle \hat{p} \rangle$, and variances Δx^2, Δp^2. Make a plot of the time dependence of Δx^2 and Δp^2 for two values $\lambda_0 = 0.1$ and $\lambda_0 = 0.5$. Give a qualitative description of the results shown by the plot.

Språklig kommentar

Sqeezed state kan gjerne på norsk oversettes med *sammenpresset* eller *presset tilstand*.

FYS 4110: Ikke-relativistisk kvantemekanikk

Midttermineksamen, høsten 2010

Oppgavesettet er tilgjengelig fra fredag 15. oktober. Settet består av 2 oppgaver på 4 sider.

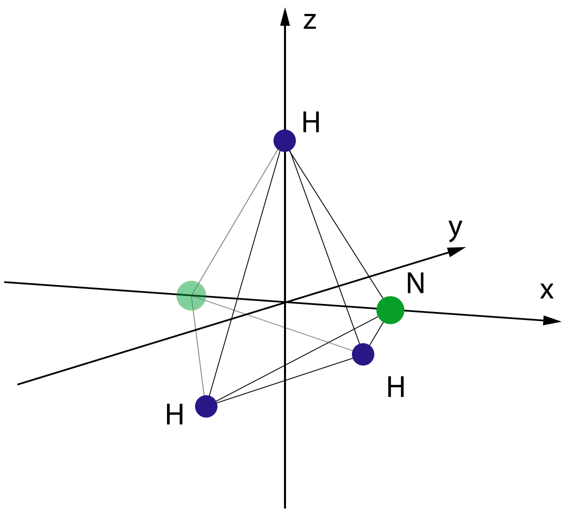
Frist for innlevering av besvarelser
Fredag 22. oktober. *Trykte/håndskrevne* besvarelser innleveres på ekspedisjonskontoret ved Fysisk institutt.

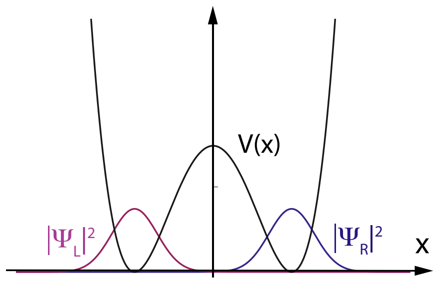
Spørsmål angående oppgavene
Spørsmål kan rettes til foreleser, Jon Magne Leinaas (rom 471Ø) eller assistent på kurset, Marianne Rypestøl (rom 457Ø).

Språk
Oppgavene er skrevet på engelsk, siden alt skriftlig materiell i kurset har vært på engelsk, men besvarelser kan skrives på norsk eller engelsk etter eget ønske.

OPPGAVER

1 Oscillations in ammonia molecules





The ammonia molecule has the chemical formula NH_3, which means that it is composed of three hydrogen and one nitrogen atoms. The left part of the figure shows the spatial structure of the molecule, where the hydrogen atoms define a planar, equilateral triangle (in the yz-plane) and the nitrogen atom is located on the orthogonal symmetry axis (x-axis) at some distance from the plane of the hydrogen atoms.

With the plane of the hydrogen atoms being fixed, there are, however, two possible positions of the nitrogen atom that are equally favored with respect to potential energy. They are located symmetrically about the plane, as indicated in the figure.

In the quantum description we associate two different state vectors $|\psi_R\rangle$ and $|\psi_L\rangle$ with these two positions of the nitrogen atom. In the right part of the figure the situation is pictured with the potential energy and the two wave functions shown as functions of the position of the nitrogen atom along the symmetry axis. The potential has the form of a double well with two degenerate ground state positions. With $\hat{H}$ as the Hamiltonian we write this degeneracy as

$$\langle\psi_L|\hat{H}|\psi_L\rangle = \langle\psi_R|\hat{H}|\psi_R\rangle \equiv E_0 \tag{1}$$

There is however a correction to this picture. Even though there is potential barrier between the two equilibrium positions, there is a small probability for quantum tunneling from one position to the other. This is represented by a non-vanishing matrix element

$$\langle\psi_L|\hat{H}|\psi_R\rangle \equiv \lambda \tag{2}$$

were we may assume λ to be real and positive. The value of this matrix element is very small, which means that the corresponding transition time from one minimum of the potential to the other is very long, but the result is that if the nitrogen atom initially is in one of the wells it will oscillate back and forth between the two minima at a low frequency (compared to other atomic frequencies).

The true ground state is however a stationary state, which to a good approximation is a superposition of the states $|\psi_L\rangle$ and $|\psi_R\rangle$ associated with the two minima. In the following we restrict the description to the two-dimensional Hilbert space spanned by these two vectors.

a) Write the Hamiltonian as a 2×2 matrix and find the energy eigenvalues $E_0^{\pm}$ and eigenstates $|\psi_0^{\pm}\rangle$, when the λ terms are included. Express the ground state $|\psi_0^{-}\rangle$ and the excited state $|\psi_0^{+}\rangle$ as linear combinations of $|\psi_L\rangle$ and $|\psi_R\rangle$ and describe briefly with words the characteristics of the two energy eigenstates.

The ammonia molecule has an electric dipole moment which arises from the tendency of the nitrogen atom to attract an electron from the hydrogen atoms. The dipole moment is directed along the symmetry axis in the opposite direction of the nitrogen atom. We assume now that the ammonia molecule is located in a constant electric field $\mathcal{E}$ directed along the x-axis. The field introduces a new term $\hat{H}_d$ in the Hamiltonian with matrix elements

$$\begin{aligned}
\langle\psi_L|\hat{H}_d|\psi_L\rangle &= -\langle\psi_R|\hat{H}_d|\psi_R\rangle \equiv \Delta \\
\langle\psi_L|\hat{H}_d|\psi_R\rangle &= \langle\psi_R|\hat{H}_d|\psi_L\rangle = 0
\end{aligned} \tag{3}$$

with $\Delta = \mathcal{E}\, d$, and with d as the electric dipole moment of the molecule.

b) Determine the new energy eigenvalues $E_{\pm}$ with this additional term in the Hamiltonian, and make a plot that shows how the two energy levels change with variable Δ from a large negative to a large positive value (from $\Delta << -\lambda$ to $\Delta >> \lambda$). Choose $E_{\pm}/\lambda$ and Δ/λ as variables.

c) Determine eigenvectors $|\psi_{\pm}\rangle$ expressed in terms of $|\psi_L\rangle$ and $|\psi_R\rangle$ and plot, as functions of Δ, the overlaps $|\langle\psi_L|\psi_{\pm}\rangle|^2$ between the energy eigenvectors and $|\psi_L\rangle$.

The situation we have here is sometimes referred to as an *avoided crossing* between the two energy levels. Give a brief qualitative description of the crossing based on the plotted curves.

We next assume the electric field to vary periodically with time, $\mathcal{E} = \mathcal{E}_0 \cos\omega t$, and correspondingly, $\Delta = \Delta_0 \cos\omega t$, with Δ_0 as a positive constant.

d) Show that in the $\{|\psi_0^{\pm}\rangle\}$ basis the Hamilton can be expressed as

$$\hat{H} = E_0 \mathbb{1} + \lambda\sigma_z + \Delta_0 \cos\omega t \sigma_x \tag{4}$$

with σ_x and σ_z as standard Pauli matrices.

The last term in (6) can be written as

$$\Delta_0 \cos \omega t \sigma_x = \frac{1}{2}\Delta_0(e^{i\omega t}\sigma_- + e^{-i\omega t}\sigma_+) + \frac{1}{2}\Delta_0(e^{-i\omega t}\sigma_- + e^{+i\omega t}\sigma_+) \tag{5}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ can be viewed as raising and lowering operators in the spectrum of the two-level Hamiltonian. Assuming ω to be positive the first term will usually give the most important contribution to the Hamiltonian. This motivates the so-called *rotating wave approximation*, where the last term in (5) is ommitted. In the following apply this approximation with the Hamiltonian given by

$$\hat{H} = E_0\mathbb{1} + \lambda\sigma_z + \frac{1}{2}\Delta_0(e^{i\omega t}\sigma_- + e^{-i\omega t}\sigma_+) \tag{6}$$

e) Show that this has the same form as the spin Hamiltonian in a rotating magnetic field, discussed in Sect.1.3.2 of the lecture notes. Outline the method used to find the time evolution operator and give the expressions for the Rabi frequency Ω and resonance frequency ω_0 in terms of the parameters λ and Δ_0. It may be convenient here to re-define the zero-point of the energy so that $E_0 = 0$. Comment on why the value of E_0 is not important. It is sufficient to refer to results from the lecture notes without a detailed derivation.

f) Initially, at time $t = 0$, the system is in the left shifted state $|\psi_L\rangle$. Determine the time dependence of the overlap of the time evolved state $|\psi(t)\rangle$ with the right shifted state, $\langle\psi_R|\psi(t)\rangle$.

g) Assuming that the strength of the oscillating field is given by $\Delta_0 = 2\lambda$, examine numerically the time dependent function $|\langle\psi_R|\psi(t)\rangle|^2$ by making a plot over several periods of this function, for two different values of the frequency, 1) at resonance, $\omega = \omega_0$ and 2) off resonance with $\omega = \omega_0/10$. Use the dimensional variable $\tau = 2\pi\lambda t$ as time coordinate. Make a (qualitative) discussion of what the curves show and compare with the related curve in the case when the electric field is turned off, $\Delta_0 = 0$.

2 Two coupled spin-half systems

Two particles have spins that are decoupled from the motion of the particles, with a spin Hamiltonian of the form

$$H = \omega(\hat{S}_{1z} + \hat{S}_{2z}) + 2\frac{\alpha}{\hbar}\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 \tag{7}$$

The first term is due to an external magnetic field, which acts on the magnetic moments of the two spins, and the second term is due to a spin-spin interaction between the two particles. Both particles have spin half, and expressed in terms of the Pauli matrices of the two spin systems, the total spin operators have the tensor product form

$$\hat{\boldsymbol{S}}_1 = \frac{\hbar}{2}\boldsymbol{\sigma} \otimes \mathbb{1}\,, \quad \hat{\boldsymbol{S}}_2 = \frac{\hbar}{2}\mathbb{1} \otimes \boldsymbol{\sigma} \tag{8}$$

We make use of the following notation for the eigenvectors of the Pauli matrix σ_z,

$$\sigma_z|\pm\rangle = \pm|\pm\rangle \tag{9}$$

and remind you about the form of the eigenvectors $|s, m\rangle$ of $\hat{\boldsymbol{S}}^2$ (with quantum number s) and $\hat{S}_z$ (with quantum number m), where $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$ is the total spin,

$$\begin{aligned}
|1,1\rangle &= |++\rangle \\
|1,0\rangle &= \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\
|1,-1\rangle &= |--\rangle \\
|0,0\rangle &= \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)
\end{aligned} \tag{10}$$

with $|++\rangle = |+\rangle \otimes |+\rangle$ etc.

a) Show that the spin states in (10) are eigenstates of the Hamiltonian and find the corresponding eigenvalues.

b) At the initial time $t = 0$ the spin system is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |+-\rangle) \tag{11}$$

Find the corresponding expression for the density operator $\hat{\rho}(0)$. Characterize the state as being pure or mixed and being uncorrelated, separable or entangled. The reduced density operators of the two spin systems can be written as

$$\hat{\rho}_1(0) = \frac{1}{2}(\mathbb{1} + \mathbf{r}_1(0) \cdot \boldsymbol{\sigma}), \quad \hat{\rho}_2(0) = \frac{1}{2}(\mathbb{1} + \mathbf{r}_2(0) \cdot \boldsymbol{\sigma}) \tag{12}$$

Find the vectors $\mathbf{r}_1(0)$ and $\mathbf{r}_2(0)$.

c) Determine the time-dependent state $|\psi(t)\rangle$ and the corresponding density operator $\hat{\rho}(t)$ expanded in the product basis of the spin states $|\pm\rangle$.

d) Find the reduced density operator $\hat{\rho}_1(t)$ of the first spin, and the corresponding time dependent vector $\mathbf{r}_1(t)$. Give a qualitative description of the time evolution of this vector when $\omega >> \alpha$.

e) Show that the entanglement entropy S of the composite spin system can be expressed as a function of $r_1 = |\mathbf{r}_1|$ and use the expression to make a plot of S as a function of αt. What is the maximal value of S during the time evolution? Compare with the maximally allowed value for S in this system. (Specify whether you use base-2 logarithm or natural logarithm.)

f) The sum $\mathbf{r}(t) = \mathbf{r}_1(t) + \mathbf{r}_2(t)$ satisfies a simple equation of motion. Find this equation and characterize the motion.

FYS 4110: Non-relativistic quantum mechanics

Midterm Exam, Fall Semester 2011

The problem set is available from Friday October 14. The set consists of 2 problems written on 4 pages.

Deadline for returning solutions
Friday October 21. *Written/printed* solutions should be returned to Ekspedisjonskontoret in the Physics Building before closing time.

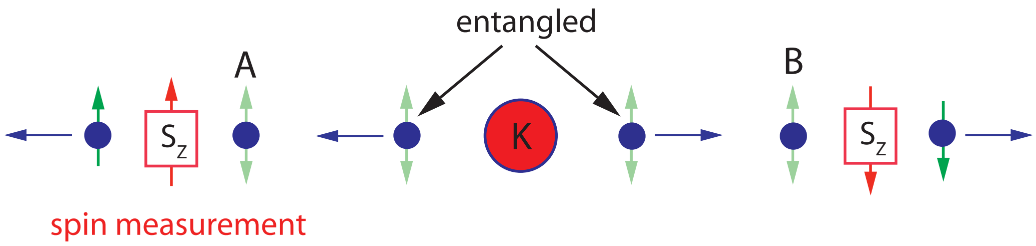
Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Office: room Ø471) or the assistant Marianne Rypestøl (Office: Ø457).

Language
Solutions may be written in Norwegian or English, depending on your preference.

PROBLEMS

1 Entanglement and Bell inequalities

We consider an experimental situation, similar to the one discussed in the lecture notes, where pairs of spin 1/2 particles are initially prepared in a correlated spin state, and then are separated in space while keeping the spin state unchanged. When far apart spin measurements are performed on the particles in each pair, and the results are registered and compared. The situation is illustrated in the figure, where a series of entangled pairs are created in a source K, and where measurements of the z-components of the spin are performed on both particles (A and B). When the spins in the z-directions are strictly anticorrelated, the result *spin up* (*spin down*) for particle A is always followed by the result *spin up* (*spin down*) for particle B.



We consider the situation where three different sets of measurements are performed, with different spin states,

$$\begin{aligned} \text{I}: &\qquad \hat{\rho}_1 = |\psi_a\rangle\langle\psi_a|, \quad |\psi_a\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \\ \text{II}: &\qquad \hat{\rho}_2 = |\psi_s\rangle\langle\psi_s|, \quad |\psi_s\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\ \text{III}: &\qquad \hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2) \end{aligned} \tag{1}$$

The notation is $|+-\rangle = |+\rangle \otimes |-\rangle$, where $|\pm\rangle$ are spin states of a single particle, with S_z quantized. The first factor in the tensor product refers to particle A and the second one to particle B. Note that all three states are strictly anticorrelated with respect to the z-component of the spin of the two particles. The purpose of the (hypothetical) experiment is to examine correlation functions that are relevant for the Bell inequalities, as already discussed for case I in the lecture notes, to see if the three states show different behavior. This involves performing the spin measurements also for rotated directions of the spin axes.

a) Of the three density operators only $\hat{\rho}_1$ is rotationally invariant. Demonstrate this by calculating the expectation value of $\mathbf{S}^2$ for the three cases, where $\mathbf{S} = (\hbar/2)(\boldsymbol{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{\sigma})$ is the spin vector of the full system, and comment on the results.

b) What are the reduced density operators $\hat{\rho}_A$ and $\hat{\rho}_B$ in the three cases? Determine the von Neumann entropy S of the full system, as well as the entropies S_A and S_B of the subsystems. Check if the classical restriction on the entropies $S \geq \max\{S_A, S_B\}$ is satisfied in any of the cases. In each of the cases examine if the states are entangled or separable, and give, if possible, a numerical measure of the degree of entanglement.

We assume the direction of the two measurement devices can be rotated so they measure spin components of the form

$$S_\theta = \cos\theta S_z + \sin\theta S_x \tag{2}$$

where the angle θ can be chosen independently for A and B. The state $|\theta\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle$ is the *spin up* vector in the rotated direction and the operator $\hat{P}(\theta) = |\theta\rangle\langle\theta|$ projects on the corresponding spin vector.

c) Show that the given expression for $|\theta\rangle$, as claimed above, is the *spin up* state of S_θ. Determine the expectation value $P_A(\theta) = \left\langle \hat{P}(\theta) \right\rangle_A$, for particle A, in the three cases I, II and III. Comment on the result.

d) Determine, for the three cases, the joint probability distribution $P(\theta, \theta') = \left\langle \hat{P}(\theta) \otimes \hat{P}(\theta') \right\rangle$, with the two angles θ and θ' as independent variables.

The Bell inequality, according to the *hidden variable* analysis described in the lecture notes, gives a constraint on the possible classical correlations of the two spins. In the present case the inequality can be written as

$$F(\theta, \theta') \equiv P(0, \theta') - |P(\theta, 0) - P(\theta, \theta')| \geq 0 \tag{3}$$

where one of the angles is set to 0 since we, for the states we consider, will only have strict anticorrelation for spin measurements along the z-axis. (For details see the derivation in the lecture notes.)

e) Make plots of the function $F(\theta, 0.5\,\theta)$ for the three cases I, II and III, with θ varying in the interval $0 < \theta < 2\pi$. Check in all cases whether the inequality (3) is satisfied or broken, and compare the results with what is known from point b) concerning entanglement between the two particles.

In addition to these plots, examine the functions for other choices $\theta' = \lambda\,\theta$ with $\lambda \neq 0.5$ to see if the results are not changed. Alternatively make a 3D plot of the two-variable function $F(\theta, \theta')$ and check whether the conclusion concerning the Bell inequality holds in the full parameter space. State the conclusions, but it is not needed to include the additional plots in the written/ printed solutions.

f) Assume an experimental series is performed, with the two angles fixed. The number of pairs registered with *spin up* (in the chosen direction) for both spins A and B is n_{++}, and the number with *spin down* for both spins is n_{--}. Similarly n_{+-} is the number of pairs registered with *spin up* for A and *spin down* for B, n_{-+} is the number of pairs registered with *spin down* for A and *spin up* for B. The total number of pairs in the series is N.

We refer to the experimental results corresponding to $P_A(\theta)$, $P_B(\theta')$, and $P(\theta,\theta')$ as $P^A_{exp}(\theta)$, $P^B_{exp}(\theta')$, and $P_{exp}(\theta,\theta')$. What are these quantities expressed in terms of the numbers $\{n_{ij}, i,j=\pm\}$ and N?

2 Rabi oscillations in a composite quantum system

An atom interacts with the electromagnetic field within a small reflecting cavity. Only one of the cavity modes of the field has a frequency that matches energy differences between the lowest energy levels of the atom. The interaction can therefore be described by a simplified model, where only two atomic levels are included, denoted $|g\rangle$ (ground state) and $|e\rangle$ (excited state), and only one field mode, with energy levels $|n\rangle$, where n is the photon number of this mode. The model Hamiltonian is

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\lambda(\hat{a}^\dagger\sigma_- - \hat{a}\sigma_+) \equiv \hat{H}_0 + \hat{H}_1 \tag{4}$$

where the $\hat{H}_0$ includes the two first terms, which describe the non-interacting atom and photons, and $\hat{H}_1$ the third term, which describes interactions between the atoms and the photons. $\hbar\omega_0$ is then the energy difference between the two atomic levels, $\hbar\omega$ is the photon energy, and $\lambda\hbar$ is an interaction energy. The model is known as the Jaynes-Cummings model, and it has precisely the form of a two-level system interacting with a harmonic oscillator. The Pauli matrices act between the two atomic levels, with σ_z as the standard diagonal matrix, and with $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ as matrices that raise or lower the atomic energy. The raising and lowering operators of the harmonic oscillator, $\hat{a}$ and $\hat{a}^\dagger$, have the physical interpretation as photon creation and destruction operators. The model is based on the *rotating wave approximation*, where terms of the form $\hat{a}^\dagger\sigma_+$ and $\hat{a}\sigma_-$ are suppressed since they are unimportant close to resonance, where $\omega_0 \approx \omega$. An unimportant constant energy contribution to $\hat{H}$ has also been subtracted.

a) Show that the interaction Hamiltonian $\hat{H}_1$ couples the unperturbed levels only in pairs that differ by one photon. We define such a pair of states as $|n1\rangle \equiv |g\rangle\otimes|n\rangle = |g,n\rangle$ and $|n2\rangle \equiv |e\rangle\otimes|n-1\rangle = |e,n-1\rangle$ for $n \geq 1$. Show that the Hamiltonian in the subspace spanned by this pair of states can be written as a 2x2 matrix of the form

$$H_n = \frac{1}{2}\hbar\begin{pmatrix} \Delta & i\omega_n \\ -i\omega_n & -\Delta \end{pmatrix} + \epsilon_n\mathbb{1} \tag{5}$$

with $\mathbb{1}$ as the 2×2 identity matrix, and find the expressions for Δ, ω_n and ϵ_n. Assume $|n1\rangle$ to correspond to the upper matrix elements of H_n and $|n2\rangle$ to the lower ones, with the corresponding matrix elements of a state vector referred to as c_{n1} and c_{n2}.

The state $|g,0\rangle = |g\rangle\otimes|0\rangle$ seems not to have any partner. What happens to this state under time evolution?

In the following we assume the resonance condition $\omega = \omega_0$ to be satisfied.

b) Solve the eigenvalue problem for this 2x2 matrix Hamiltonian, and find the two energy eigenvalues $E_n^\pm$ and the corresponding eigenvectors $\phi_n^\pm$ in matrix form. For a general, time dependent state $\psi_n(t)$ find the coefficients $c_{n1}(t)$ and $c_{n2}(t)$ expressed in terms of the coefficients $c_{n1}(0)$ and $c_{n2}(0)$ at the initial time $t=0$.

c) A general state, with all n-components included, can be written as

$$|\psi\rangle = \sum_{n=0}^{\infty}\sum_{i=1}^{2} c_{ni}|ni\rangle \tag{6}$$

What are the corresponding expressions for the matrix elements of the density matrix, $\rho_{ni,n'j}$, Find the expressions also for the reduced matrix elements ρ_{ij} of the atom. What is the physical interpretation of the diagonal terms ρ_{11} and ρ_{22}?

Consider two different initial conditions for a state vector at $t = 0$:

I $|\psi(0)\rangle = |e\rangle \otimes |m-1\rangle$, where m is a specific, but at this point unspecified photon number.

II $|\psi(0)\rangle = |e\rangle \otimes |\alpha\rangle$ with $|\alpha\rangle$ a coherent state, defined by

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} |n\rangle \tag{7}$$

where α is a complex number. Write in both cases the expressions for the reduced density matrix elements $\rho_{ij}(0)$ of the atom.

d) Find the matrix elements of the time dependent, reduced density matrix of the atom $\rho_{ij}(t)$, for both initial conditions I and II.

e) Make plots of $\rho_{11}(t)$ as function of t, for both cases I and II. Make the following choice for the parameters, $\alpha = 4$ and $m = 16$. Use λt as time variable on the horizontal axis. Make a short time plot, $0 < \lambda t < 5$, of both cases in the same diagram. Make also a long time plot for case II, for example with $0 < \lambda t < 100$. Comment on the results and compare with the case discussed in the lecture notes, where the (electro)magnetic field is treated classically.

FYS 4110: Non-relativistic quantum mechanics

Midterm Exam, Fall Semester 2012

The problem set is available from Friday October 19.

Deadline for returning solutions
Friday October 26. *Written/printed* solutions should be returned to Ekspedisjonskontoret in the Physics Building before closing time.

Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Office: room Ø471) or the assistant Marianne Rypestøl (Office: Ø457).

Language
Solutions may be written in Norwegian or English, depending on your preference.

The problem set consists of 2 problems written on 4 pages.

PROBLEMS

1 A three-spin problem

We consider a system consisting of three electrons. They all sit at fixed positions, with their spins as free variables. A constant magnetic field is pointing in the z-direction and there is a spin-spin interaction between the particles, so that the Hamiltonian takes the form

$$H = a(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 + \hat{\mathbf{S}}_2 \cdot \hat{\mathbf{S}}_3 + \hat{\mathbf{S}}_3 \cdot \hat{\mathbf{S}}_1) + b(\hat{S}_{1z} + \hat{S}_{2z} + \hat{S}_{3z}) \tag{1}$$

with a and b as positive constants, and with the subscripts 1, 2, and 3, referring to each of the three particles. (Note that in these expressions the tensor product form of the operators are not specified explicitly.)

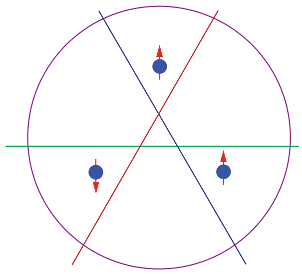


Figure 1: The three-spin-half system. Each of the straight lines shows a division of the full system into two parts, where one part contains a single spin and the othe part contains two spins.

a) The total spin we denote as $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2 + \hat{\mathbf{S}}_3$. Show that the Hamiltonian can be expressed in terms of $\hat{\mathbf{S}}^2$ and $\hat{S}_z$, and give the expression. Use the rule for composition of quantum spins to show that the (spin) Hilbert space consists of three orthogonal subspaces, characterized by spins $1/2$, $1/2$ and $3/2$ respectively.

b) For certain values of a and b the ground state of the system is doubly degenerate, with the states of this subspace having spin quantum numbers $s = 1/2$ (for $\hat{\mathbf{S}}^2$) and $m = -1/2$ (for $\hat{S}_z$). What is the restriction on a and b when this is the case? We assume in the following this condition to be satisfied.

c) Show that the following three states all lie in the two-dimensional, degenerate subspace of the ground state,

$$
\begin{aligned}
|\psi_a\rangle &= \frac{1}{\sqrt{2}}(|-+-\rangle - |--+\rangle) \\
|\psi_b\rangle &= \frac{1}{\sqrt{2}}(|--+\rangle - |+--\rangle) \\
|\psi_c\rangle &= \frac{1}{\sqrt{2}}(|+--\rangle - |-+-\rangle)
\end{aligned} \tag{2}
$$

where $|-+-\rangle = |-\rangle_1 \otimes |+\rangle_2 \otimes |-\rangle_3$, with the first factor of the tensor product is the *spin down* state of particle 1, the second factor is a *spin up* state of particle 2 and the third factor is a *spin down* state of particle 3, all with spins quantized in the z-direction. (Similar expressions are valid for all the other three-particle states in the above expressions.)

d) The three-particle system can be considered as a bipartite system, with particle 1 defining one subsystem and particles 2 and 3 defining the other part. We write this partition of the system symbolically as $123 = 1 + (23)$. Two other partitions of this type are possible, namely $123 = 2 + (13)$ and $3 + (12)$ (see figure). Determine the reduced density operators of the two subsystems, for all three partitions, in the case of state $|\psi_a\rangle$. Determine the entanglement entropy in the three cases and show that the state is maximally entangled with respect to two of the divisions of the system, but is unentangled with respect to the last one. Comment on the situation for the two other states $|\psi_b\rangle$ and $|\psi_c\rangle$. In what sense is the entanglement in these states a *two-particle* entanglement?

e) We seek new states, in the same subspace, where the entanglement is distributed evenly between all the three particles. For this purpose consider the following two state vectors,

$$
\begin{aligned}
|\psi_I\rangle &= \frac{1}{\sqrt{3}}((|+--\rangle + e^{2\pi i/3}|-+-\rangle + e^{-2\pi i/3}|--+\rangle) \\
|\psi_{II}\rangle &= \frac{1}{\sqrt{3}}((|+--\rangle + e^{-2\pi i/3}|-+-\rangle + e^{2\pi i/3}|--+\rangle)
\end{aligned} \tag{3}
$$

Show that these vectors are orthogonal and span the two-dimensional subspace of the degenerate ground state.

f) Show that the entanglement entropy of the states I and II is the same for all three partitions of the systems in two parts, as described above, and determine the value. Is the entanglement entropy larger, smaller or equal to the average entanglement entropy of the states (2), when this is averaged over the three partitions of the system.

g) A measurement of the observable $\hat{S}_{1z}$ is made on particle 1, with the system in the $|\psi_I\rangle$state. Determine in both cases, when the result is *spin up* and when it is *spin down*, what the entanglement of the subsystem (23) is.

2 Charged particle in a strong magnetic field.

We study in this problem the motion of electrons in a strong, homogenous magnetic field $\mathbf{B}$, with the electrons constrained to a plane orthogonal to the magnetic field. We choose a coordinate system, with $\mathbf{B}$ pointing along the z-axis, and the electrons thus moving in the x,y-plane. The electrons are assumed to be fully spin polarized along the magnetic field, and we can therefore ommit the spin variable in the description.

The magnetic field is described by the following vector potential , $\hat{\mathbf{A}} = -(1/2)\hat{\mathbf{r}} \times \mathbf{B}$. The relation between velocity and (canonical) momentum is $\hat{\mathbf{v}} = (\hat{\mathbf{p}} - e\hat{\mathbf{A}})/m$, and the Hamiltonian has the standard form $\hat{H} = (1/2m)(\hat{\mathbf{p}} - e\hat{\mathbf{A}})^2$. With $\mathbf{B} = B\mathbf{k}$ assume in the following eB to be positive.

a) Show that the angular momentum, written as

$$\hat{L} = (\hat{\mathbf{r}} \times \hat{\mathbf{p}})_z \tag{4}$$

is a conserved quantity. (The label z means the z-component of the vector product.)

b) It is convenient to introduce combinations of the position and velocity in the following way,

$$\hat{\mathbf{R}} = \hat{\mathbf{r}} + \hat{\boldsymbol{\eta}}\,, \quad \hat{\boldsymbol{\eta}} = (1/\omega)\hat{\mathbf{v}} \times \mathbf{k} \tag{5}$$

with ω as the cyclotron frequency, $\omega = eB/m$. We refer to the components of the vector $\hat{\mathbf{R}}$ as $\hat{X}$ and $\hat{Y}$, and the components of $\hat{\boldsymbol{\eta}}$ as $\hat{\eta}_x$ and $\hat{\eta}_y$, and introduce the dimensionless operators

$$\hat{a} = \frac{1}{\sqrt{2}\,l_B}(\hat{X} - i\hat{Y})\,, \quad \hat{b} = \frac{1}{\sqrt{2}\,l_B}(\hat{\eta}_x + i\hat{\eta}_y) \tag{6}$$

where l_B is the so-called magnetic length, $l_B = \sqrt{\hbar/eB}$. Show that the set of operators $\{\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger\}$ satisfies the same commutation algebra as that of two independent harmonic oscillators.

c) Find the form of the Hamiltonian $\hat{H}$ and the angular momentum $\hat{L}$ expressed in terms of $\{\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger\}$. Show that the energy levels, commonly known as *Landau levels*, are equally spaced and that the vectors of the lowest level are defined by $\hat{b}|\psi\rangle = 0$. Further show that the corresponding subspace is spanned by angular momentum states $|n\rangle$, $n = 0, 1, 2, ...$, defined by $\hat{a}|0\rangle = 0, \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. What is the angular momentum of these states?

In the following we restrict the Hilbert space to the lowest Landau level. A coherent state in the lowest Landau level is defined by the equation,

$$\hat{a}|z\rangle = z|z\rangle \tag{7}$$

and we remind you about the scalar products, derived in the lecture notes,

$$\langle z|z'\rangle = e^{-\frac{1}{2}(|z|^2+|z'|^2)+z'z^*}\,, \quad \langle n|z\rangle = \frac{z^n}{\sqrt{n!}}e^{-\frac{1}{2}|z|^2} \tag{8}$$

Assume two electrons, which we label by 1 and 2, are present in the system. Since the electrons are fermions, all states are antisymmetric with respect to permuation of the electron labels. In particular a two-particle coherent state, with the particles being located symmetrically about the origin, gets the following form after antisymmetrization,

$$|Z,-Z\rangle_a = N(Z)(|Z,-Z\rangle - |-Z,Z\rangle) \tag{9}$$

with $|Z_1,Z_2\rangle = |Z_1\rangle \otimes |Z_2\rangle$ and $N(Z)$ as a normalization factor. (We use here capital letters in the definition of the state in order to avoid confusion whith the complex coordinates used below.) Similarly the observables are all symmetric in the particle labels, and the lowering operators of the two electrons, $\hat{a}_1$ and $\hat{a}_2$, for the same reason, do not separately represent observables, but the symmetric combinations of them do.

d) Show that the antisymmetrized two-particle coherent state (9) is an eigenstate of the two symmetric operators $\hat{a}_1 + \hat{a}_2$ and $\hat{a}_1\hat{a}_2$. Determine the normalization factor $N(Z)$, and find expressions for the density operator $\hat{\rho}$ of the two-particle state and of the reduced density operators $\hat{\rho}_1$ and $\hat{\rho}_2$.

e) Determine the reduced density matrix of particle 1 in the coherent state representation, $\rho_1(z,z')$, and plot the one-particle density, defined as $\rho(z) = 2\rho_1(z,z)$, for three different (real) values of particle coordinate Z, $Z = 2.0, 1, 0$ and 0.1. Make a 3D plot, or alternatively, a contour plot, with the real and imaginary parts of the coordinate z as variables. Comment on the results.

f) Show that $\hat{\rho}_1$ has the two states $|\pm Z\rangle$ as eigenstates and determine the corresponding eigenvalues. Use this to determine the entanglement of the two particles. What is this entanglement due to.

g) Assume N electrons occupy the lowest angular momentum states $n = 0, 1, 2, ..., (N-1)$. Show that the reduced density operator of particle 1 then gets the form

$$\hat{\rho}_1 = \frac{1}{N}\sum_{n=0}^{N-1} |n\rangle\langle n| \tag{10}$$

Find the corresponding expression for the one-particle density, $\rho(z) = N\rho_1(z,z)$, and make a 3D plot (or contour plot) of this as a function of the variable z, for $N = 10$. In what sense is this the most densely populated N-particle state, localized around the origin?

h) Make also a plot of the one-particle density for $N = 2$ and compare with the density plot of the antsymmetrized two-particle coherent state for $Z = 0.1$. Show that the state (9) in fact coincides with the $N = 2$ state (10) in the limit $Z \to 0$.

FYS 4110: Ikke-relativistisk kvantemekanikk

Midttermineksamen, høsten 2013

Oppgavesettet er tilgjengelig fra fredag 18. oktober. Settet består av 2 oppgaver på 3 sider. I tillegg finnes på side 4 en kort engelsk-norsk ordliste.

Frist for innlevering av besvarelser

Fredag 25. oktober. Besvarelser innleveres på ekspedisjonskontoret ved Fysisk institutt.

Spørsmål angående oppgavene

Spørsmål kan rettes til foreleser, Jon Magne Leinaas (rom 471Ø) eller assistent på kurset, Ola Liabøtrø (rom 469Ø).

Språk

Oppgavene er skrevet på engelsk, siden alt skriftlig materiell i kurset har vært på engelsk, men besvarelser kan skrives på norsk eller engelsk etter eget ønske.

OPPGAVER

1 Density operators of a composite system

A density operator of a two-level system can be represented by a 2×2 matrix in the form

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma}), \quad |\mathbf{r}| \leq 1 \tag{1}$$

where $\mathbb{1}$ is the 2×2 identity matrix, $\mathbf{r}$ is a vector in three dimensions and $\boldsymbol{\sigma}$ is a vector operator with the Pauli matrices as the Cartesian components. We consider in the following two such systems, denoted $\mathcal{A}$ and $\mathcal{B}$, which together form a composite system with a four-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

A density matrix of the composite system can generally be written as

$$\hat{\rho} = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{b} \cdot \boldsymbol{\sigma} + \sum_{ij} c_{ij} \sigma_i \otimes \sigma_j) \tag{2}$$

where the vector components a_i, b_j of $\mathbf{a}$ and $\mathbf{b}$, and the coefficients c_{ij}, are all real-valued. The first factor in the tensor products corresponds to subsystem $\mathcal{A}$ and the second factor to $\mathcal{B}$.

a) Find an expression for $\hat{\rho}^2$ of the same form as (2). Find also the reduced density matrices $\hat{\rho}_A$ and $\hat{\rho}_B$, and $\hat{\rho}_A^2$ and $\hat{\rho}_B^2$. (As a reminder, the Pauli matrices satisfy the product rule, $\sigma_i \sigma_j = \delta_{ij} + i \sum_k \epsilon_{ijk} \sigma_k$.)

b) A necessary condition for $\hat{\rho}$ to be a density matrix is $\mathrm{Tr}\, \hat{\rho}^2 \leq 1$, with equality when $\hat{\rho}$ is a pure state. Show this from the general conditions satisfied by density matrices, and find the corresponding inequalities for the a, b and c coefficients.

c) What relation should the coefficients a, b and c satisfy if the state of the composite system should be a tensor product state? Show that the condition for $\hat{\rho}$ being pure and maximally entangled is that $\mathbf{a}$ and $\mathbf{b}$ vanish, and that the coefficients c_{ij} satisfy the two conditions

$$\sum_{ij} c_{ij}^2 = 3, \quad \frac{1}{2} \sum_{klmn} \epsilon_{kmi} \epsilon_{lnj} c_{kl} c_{mn} = -c_{ij} \tag{3}$$

d) Two Bell states are defined by

$$|B1\rangle = \frac{1}{\sqrt{2}}\left(|++\rangle + |--\rangle\right) , \quad |B2\rangle = \frac{1}{\sqrt{2}}\left(|++\rangle - |--\rangle\right) \tag{4}$$

with $|++\rangle = |+\rangle \otimes |+\rangle$ as the tensor product of two state vectors, both with quantized spins along the positive z-axis, and $|--\rangle$ similarly with both spins quantized along the negative z-axis.

Write the corresponding density matrices $\hat{\rho}_{B1}$ and $\hat{\rho}_{B2}$ in the form (2) and check that the conditions found in c), for pure states with maximal entanglement, are satisfied.

e) Consider a time dependent state vector

$$|\psi_1(t)\rangle = \cos\omega t\,|B1\rangle + \sin\omega t\,|B2\rangle \tag{5}$$

Find the time dependent density operator $\hat{\rho}_1(t)$ corresponding to $|\psi_1(t)\rangle$, written in the form (2). Determine the entanglement entropy of the two-spin system and plot it as a function of ωt.

f) Consider another time dependent state, described by the density operator

$$\hat{\rho}_2(t) = \cos^2\omega t\,\hat{\rho}_{B1} + \sin^2\omega t\,\hat{\rho}_{B2} \tag{6}$$

with $\hat{\rho}_{B1}$ and $\hat{\rho}_{B2}$ as the density operators corresponding to the two Bell states. Determine and plot the (von Neumann) entropy of the full system. What is in this case the entropy of the subsystems? (Note that $\hat{\rho}_2(t)$ has $|B1\rangle$ and $|B2\rangle$ as eigenstates).

g) For $\omega t = \pi/4$ both states $\hat{\rho}_1$ and $\hat{\rho}_2$ are a separable states. Show this and give the expressions for the density operators.

2 Atom-photon interactions

An atom is trapped inside a small reflecting cavity. The energy difference between the ground state and the first excited state is $\Delta E = \hbar\omega$, with ω matching the frequency of one of the modes of the electromagnetic field in the cavity. This gives a strong coupling between the atomic states and this field mode, while the couplings to the other cavity modes are weak and can be neglected.

The composite system, the atom plus the resonant cavity mode, is described by the following effective Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\lambda(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) - i\gamma\hbar a^\dagger a \tag{7}$$

where the Pauli matrices act between the two atomic levels, with σ_z being diagonal in the energy basis, and $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ being matrices that raise or lower the atomic energy. $\hat{a}^\dagger$ and $\hat{a}$ are the photon creation and destruction operators, λ is an interaction parameter and γ is a decay parameter. The decay is due to the process where the photon escapes through the cavity walls. Both λ and γ are real-valued parameters, and we assume $\omega > \lambda > \gamma$.

We characterize the relevant states of the composite system as $|g,0\rangle$, $|g,1\rangle$ and $|e,0\rangle$, where g refers to the atomic ground state, e to the excited state, and 0 and 1 refers to the absence or presence of a photon in the cavity mode.

a) Show that in the two-dimensional subspace spanned by the vectors $|g,1\rangle$ and $|e,0\rangle$ the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar(\omega - i\gamma)\mathbb{1} + \frac{1}{2}\hbar\begin{pmatrix} i\gamma & \lambda \\ \lambda & -i\gamma \end{pmatrix} \tag{8}$$

where $|e, 0\rangle$ corresponds to the upper row of the matrix and $|g, 1\rangle$ to the lower one, and $\mathbb{1}$ is the identity matrix.

We define the time evolution operator in the usual way as

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{\hbar}\hat{H}t} \tag{9}$$

with the expression being valid for $t \geq 0$. The Hamiltonian (8) is non-hermitian due to the decay of the cavity field, and therefore the time evolution operator is non-unitary. However, we shall below see how to compensate for this.

b) Show that the time evolution operator can be written as

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{2}(\omega - i\gamma)t}(\cos(\Omega t)\mathbb{1} - i\sin(\Omega t)\,\frac{\mathbf{\Omega}}{\Omega}\cdot\boldsymbol{\sigma}) \tag{10}$$

where $\mathbf{\Omega}$ is a complex vector, with $\Omega^2 \equiv \mathbf{\Omega}^2$ being real and positive. Determine $\mathbf{\Omega}$ and Ω. (Note that $\mathbf{\Omega}^2$ contains no complex conjugation, and should therefore not be confused with $|\mathbf{\Omega}|^2$.) The Pauli matrix $\boldsymbol{\sigma}$ in (10) refers to the 2×2 matrix formulation (8) of $\hat{H}$.

c) Assume the system initially to be in the state $|\psi(0)\rangle = |e, 0\rangle$. Determine the time evolution of the state vector, $|\psi(t)\rangle$.

There is one important defect with the description of the time evolution discussed so far. Since the time evolution operator is non-unitary, the norm of of the state vector $|\psi(t)\rangle$ is not preserved, but decays with time. Something seems thus to be missing in the description, and we shall now correct for that. Let us for this purpose add a contribution to the density operator $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$, to give the full density operator of the atom-photon system in the cavity as

$$\hat{\rho}_{cav}(t) = \hat{\rho}(t) + f(t)|g, 0\rangle\langle g, 0| \tag{11}$$

with the function $f(t)$ defined so that the norm of $\hat{\rho}_{cav}(t)$ is conserved with value 1.

d) Determine function $f(t)$, and comment on in what sense the addition of the last term in (11) is reasonable, when considering the physical process described by the Hamiltonian (8).

e) Determine and plot, in a common diagram, the time dependent occupation probabilities of the two atomic levels, as well as the probability for one photon to be present in the cavity. Use in the plot $\tau = \lambda t$ as dimensionless time parameter, $\gamma/\lambda = 0.1$ as numerical value for the dimensionless decay parameter, and make the plot for a the interval $0 < \tau < 50$.

The transmission of the photon through the walls implies that the atom-photon system in the cavity, which we now consider as one subsystem, is coupled to the electromagnetic field outside the cavity, which we consider as a second subsystem. We make the assumption that the total system, consisting of the two subsystems, is all the time in a pure, but entangled, quantum state.

f) Show that the density operator $\hat{\rho}_{cav}(t)$ of the atom-photon system has two non-vanishing eigenvalues, given by $f(t)$ and $1 - f(t)$, and use this to determine the entanglement entropy of the two subsystems. Make a plot of the time-dependent entanglement entropy in the same time interval as the first plot.

ORDLISTE

engelsk	norsk
density operator	tetthetsoperator
pure state	ren tilstand
mixed state	blandet tilstand
entanglement	sammenfiltring
subsystem	delsystem
field mode	feltmode
interaction	vekselvirkning
cavity	kavitet, hulrom

FYS 4110 Non-relativistic quantum mechanics

Midterm Exam, Fall Semester 2014

The problem set is available from Monday morning, October 13.

Deadline for returning solutions
Monday October 20, at 10:00. *Written/printed* solutions should be returned to Ekspedisjonskontoret in the Physics Building.

Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Office: room 471Ø), or the assistant Ola Liabøtrø (room 469Ø).

Language
Solutions may be written in Norwegian or English, depending on your preference.
A short English-Norwegian dictionary is included on the last page.

The problem set consists of 2 problems written on 5 pages.

PROBLEMS

1 Spin splitting in positronium

Positronium is a bound system of an electron and a positron. The two particles have the same mass m and charges of opposite signs $\pm e$, with e denoting the electron charge. The energy spectrum of the bound system is similar to that of a hydrogen atom, but the energy scale is different since the *reduced mass* of the two-particle system has about half the value in positronium compared to hydrogen. Positronium has a finite life time since the electron and the positron will eventually annihilate.

The ground state of positronium is degenerate due to the spin degrees of freedom of the two particles. We distinguish between *para-positronium*, which is a spin *singlet* state with total spin $S = 0$, and *ortho-positronium* which is a *triplet* state with total spin $S = 1$. Para-positronium has a life time of 125 picoseconds while the life time of ortho-positronium is about 140 nanoseconds.

The interaction between the magnetic moments of the two particles give rise to a (hyperfine) splitting of the ground state energy, so that the singlet state has a slightly lower energy than the triplet state. In the following we make the simplifying assumption that this effect can be studied in the four-dimensional spin space of the two particles. This means that we assume no coupling between the spin and orbital coordinates of the particles so that the wave function of the orbital motion is the same for all the spin states and can therefore be neglected.

We denote in the following the *spin up* state vector of the z-component of the spin for any of the two particles as $|+\rangle$ and the *spin down* state by $|-\rangle$. The four dimensional space of spin states has the tensor product form $\mathcal{H} = \mathcal{H}_e \otimes \mathcal{H}_p$, with $\mathcal{H}_e$ as the two-dimensional spin space of the electron and $\mathcal{H}_p$ as the spin space of the positron. The full space is spanned by the four product states

$$\begin{aligned} |++\rangle &= |+\rangle \otimes |+\rangle\,, \quad |+-\rangle = |+\rangle \otimes |-\rangle\,, \\ |-+\rangle &= |-\rangle \otimes |+\rangle\,, \quad |--\rangle = |-\rangle \otimes |-\rangle\,, \end{aligned} \tag{1}$$

where we assume the first factor in the tensor product to describe the *electron* spin. In the four-dimensional spin space the spin operators of the electron and the positron have the following forms,

$$\hat{\mathbf{S}}_e = \frac{\hbar}{2}\boldsymbol{\sigma}_e \otimes \mathbb{1}_p \equiv \frac{\hbar}{2}\boldsymbol{\Sigma}_e$$
$$\hat{\mathbf{S}}_p = \frac{\hbar}{2}\mathbb{1}_e \otimes \boldsymbol{\sigma}_p \equiv \frac{\hbar}{2}\boldsymbol{\Sigma}_p \tag{2}$$

with $\mathbb{1}_e$ as the identity operator in the two-dimensional spin space of the electron, $\mathbb{1}_p$ as the identity operator in the spin space of the positron, and $\boldsymbol{\sigma}_e$ and $\boldsymbol{\sigma}_p$ as the Pauli matrices acting in the two-dimensional spin spaces of the electron and the positron respectively.

a) Show that in the product basis we have

$$\langle ij|\boldsymbol{\Sigma}_e \cdot \boldsymbol{\Sigma}_p|kl\rangle = \langle i|\boldsymbol{\sigma}_e|k\rangle \cdot \langle j|\boldsymbol{\sigma}_p|l\rangle \tag{3}$$

b) Find the operator product $\hat{\mathbf{S}}_e \cdot \hat{\mathbf{S}}_p$ expressed as a 4×4 matrix in the product basis. (In the matrix representation list the basis vectors in the order $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$.)

We now introduce another basis, the *spin basis* with the four vectors

$$|0,\ 0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \tag{4}$$

and

$$|1,\ 1\rangle = |++\rangle$$
$$|1,\ 0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$
$$|1,-1\rangle = |--\rangle \tag{5}$$

c) Show that $\hat{\mathbf{S}}_e \cdot \hat{\mathbf{S}}_p$ is a diagonal matrix in the new basis.

The total (intrinsic) spin of the two particles is $\hat{\mathbf{S}} = \hat{\mathbf{S}}_e + \hat{\mathbf{S}}_p$. Show that the new basis vectors are eigenstates of $\hat{\mathbf{S}}^2$ and $\hat{S}_z$ and find the eigenvalues. Check that the result for the eigenvalues is consistent with (4) being the singlet state and (5) being the triplet state.

The Hamiltonian in the spin space can be written in the form

$$H_0 = E_0\mathbb{1} + \kappa\hat{\mathbf{S}}_e \cdot \hat{\mathbf{S}}_p \tag{6}$$

where E_0 is the ground state energy with spin effects excluded, $\mathbb{1}$ is the identity operator in the four-dimensional spin space and κ is a positive constant determined by the magnetic moments of the particles.

A magnetic field $\mathbf{B}$ is turned on in the z direction. This leads to a splitting of the spin energy states, referred to as the *Zeeman effect*. The form of the modified Hamiltonian is

$$\hat{H} = E_0\mathbb{1} + \kappa\hat{\mathbf{S}}_e \cdot \hat{\mathbf{S}}_p + \lambda\hbar(\hat{S}_{e\,z} - \hat{S}_{p\,z}) \tag{7}$$

with λ as a parameter proportional to B.

d) Write the Hamilton H as a 4×4 matrix in the spin basis.

e) Find the energy eigenvalues of the Hamiltonian (7) expressed in terms of the parameters E_0, κ and λ. Plot the energies as functions of $x \equiv \lambda/\kappa$ for fixed E_0 and κ.

Two of the energy eigenstates are mixtures of $|0,0\rangle$ and $|1,0\rangle$. We write these two states as

$$\begin{aligned} |a\rangle &= \alpha\,|+-\rangle + \beta\,|-+\rangle \\ |b\rangle &= -\beta^*\,|+-\rangle + \alpha^*\,|-+\rangle \end{aligned} \tag{8}$$

where α and β are functions of x, with $|\alpha|^2 + |\beta|^2 = 1$.

f) Give the expressions for the corresponding density operators $\hat{\rho}_a$ and $\hat{\rho}_b$, and for the *reduced density operators* $\hat{\rho}_{ae}, \hat{\rho}_{ap}$ and $\hat{\rho}_{be}, \hat{\rho}_{bp}$ of the electron and positron subsystems. The degree of entanglement in the system is given by the *von Neumann entropy* of the reduced density operators. Show that the entanglement entropy is the same for $|a\rangle$ and $|b\rangle$ and can be expressed as a function of $|\alpha|^2$.

g) Determine the function $|\alpha(x)|^2$ from the eigenvalue problem in e) and use this to make a plot of the degree of entanglement in the system as a function of x for the two states $|a\rangle$ and $|b\rangle$. Are these *maximum entanglement states* for any value of x?

2 Spin coherent states

We consider a quantum spin $\hat{\mathbf{J}}$ which acts in a $2j+1$ dimensional vector space. In the standard way we introduce a set of basis vectors $|j,m\rangle$, where j is the quantum number of the total spin, so that $\hat{\mathbf{J}}^2 = j(j+1)\hbar^2\mathbb{1}$ and m is the quantum number of the z-component, $\hat{J}_z|j,m\rangle = m\hbar|j,m\rangle$. Thus, m runs from $-j$ to j and identifies the basis vectors, while j is a fixed number which characterizes the size of the total spin.

We remind you about the following relations,

$$\begin{aligned} \hat{J}_z|j,m\rangle &= m\,\hbar\,|j,m\rangle \\ \hat{J}_-|j,m\rangle &= \sqrt{(j+m)(j-m+1)}\;\hbar\,|j,m-1\rangle \\ \hat{J}_+|j,m\rangle &= \sqrt{(j-m)(j+m+1)}\;\hbar\,|j,m+1\rangle \end{aligned} \tag{9}$$

where $\hat{J}_- = \hat{J}_x - i\hat{J}_y$ and $\hat{J}_+ = \hat{J}_x + i\hat{J}_y$.

The spin system has a certain similarity with a harmonic oscillator, in the sense that $\hat{J}_+$ and $\hat{J}_-$ are raising and lowering operators like $\hat{a}^\dagger$ and $\hat{a}$, and $\hat{J}_z$ like the harmonic oscillator Hamiltonian $\hat{H}_{ho}$ has a spectrum with constant separation between the levels, where these raising and lowering operators act. There are differences, in particular since the spectrum of $\hat{J}_z$ has a finite number of levels, whereas the number of levels of the harmonic oscillator is infinite. In spite of these differences, coherent states for the spin system can be introduced in an analogous way to that of the harmonic oscillator, but not precisely in the same way. In particular the coherent states cannot generally be defined as eigenstates of the lowering operator $\hat{J}_-$.

a) Since the m quantum number has an upper limit, $m \leq j$, the lowering operator $\hat{J}_-$ has only one eigenvector, which is the lowest state $|j,-j\rangle$. Similarly $\hat{J}_+$ has $|j,j\rangle$ as the only eigenvector. Show this.

Instead of defining the general coherent states as eigenvectors of the lowering operator, we introduce them directly as *minimum uncertainty* states for the three components of the spin variable. To be more precise, such a state should minimize

$$(\Delta\mathbf{J})^2 = \left\langle \hat{\mathbf{J}}^2 \right\rangle - \left\langle \hat{\mathbf{J}} \right\rangle^2 \tag{10}$$

b) Show that $|j,-j\rangle$ and $|j,j\rangle$ are minimum uncertainty states, and that such states more general satisfy the eigenvalue equation

$$\mathbf{n}\cdot\hat{\mathbf{J}}\;|j,\mathbf{n}\rangle = j\hbar|j,\mathbf{n}\rangle \tag{11}$$

with $\mathbf{n}$ as a unit vector, and with the eigenvectors labeled by $\mathbf{n}$. What is the value of $(\Delta \mathbf{J})^2$ for these states?

c) Show that for $j = 1/2$ the condition of minimum uncertainty is trivially satisfied, so that *any* (pure) quantum state can be considered as a coherent state.

The spin coherent states can thus be associated with points on a sphere, identified by the unit vector $\mathbf{n}$. Only for the lowest value $j = 1/2$ this set of states is identical to the full set of quantum states, while for larger j they form a subset. However, for all j the spin coherent states define a complete set of states, which can be used to define a coherent state representation for the spin Hilbert space.

In order to bring the notation closer to that of the coherent states of the harmonic oscillator we represent the unit vector $\mathbf{n}$ by a complex number z in the following way

$$z = e^{-i\phi} \cot \frac{\theta}{2} \tag{12}$$

with ϕ and θ as the polar angles of $\mathbf{n}$. (This mapping from the unit sphere to the complex plane is referred to as a *stereographic projection*.) We further introduce the notation $|z\rangle \equiv |\mathbf{n}\rangle$. With this definition the spin-down state $m = -j$ corresponds to $z = 0$, while the spin up state $(m = j)$ is mapped to $z = \infty$.

For simplicity we restrict the discussion in the following to $j = 1/2$. In this case we have

$$\sigma_{\mathbf{n}}|z\rangle = |z\rangle \ , \quad \sigma_{\mathbf{n}} = \boldsymbol{\sigma} \cdot \mathbf{n} \tag{13}$$

with $\mathbf{n}$ and z related by (12).

d) With $|m\rangle$, $m = \pm 1/2$ as the spin states along the z-axis, show that the transition function between these basis states and the coherent states $|z\rangle$ can be written as

$$\langle m|z\rangle = \frac{z^{m+1/2}}{\sqrt{1+|z|^2}} \ , \quad m = \pm 1/2 \tag{14}$$

(This corresponds to a particular choice of the complex phase of the coherent state. In the following we will make use of this choice of phase.)

We now introduce a coherent state representation by using the coherent states as basis vectors. For a general state $|\psi\rangle$ the wave function in the z-representation is then defined as

$$\psi(z) = \langle z|\psi\rangle \tag{15}$$

e) Determine for $|\psi\rangle = |z_0\rangle$ the square modulus of the wave function,

$$|\psi_{z_0}(z)|^2 \equiv |\langle z|z_0\rangle|^2 \tag{16}$$

and make plot of this as a function of $r = |z|$ for the case $z_0 = 0$. Compare with a similar plot for the harmonic oscillator coherent states.

f) Show that the spin coherent states satisfy a completeness relation of the form

$$\int \frac{d^2z}{\pi} \frac{2}{(1+|z|^2)^2} |z\rangle\langle z| = \mathbb{1} \tag{17}$$

where d^2z denotes the standard area element in the two-dimensional plane, and demonstrate how this completeness relation can be used to reconstruct the abstract vector $|\psi\rangle$ of any spin state from the corresponding wave function $\psi(z)$.

g) Assume the Hamiltonian to have the form

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z \tag{18}$$

with $\hat{U}(t)$ as the corresponding time evolution operator. Show that the time evolution of a coherent state has the form

$$\hat{U}(t)|z_0\rangle = e^{i\alpha(t)}|z(t)\rangle \tag{19}$$

and determine the time dependent variable $z(t)$ and the complex phase $\alpha(t)$.

ORDLISTE

engelsk	**norsk**
density operator	tetthetsoperator
entanglement	sammenfiltring
interaction	vekselvirkning
minimum uncertainty state	minimalusikkerhets-tilstand
orbital motion	banebevegelse
spin coherent state	spinn-koherent tilstand
subsystem	delsystem

FYS 4110/9110 Modern Quantum Mechanics

Midterm Exam, Fall Semester 2015

Return of solutions
The problem set is available from Monday morning, October 19.
Written/printed solutions should be returned to Ekspedisjonskontoret in the Physics Building before Monday October 26, at 12:00.
Use candidate numbers rather than full names.

Language
Note: The problem set is available also in Norwegian.
Solutions may be written in Norwegian or English depending on your preference.

Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Office: room 471Ø), or the assistant Ola Liabøtrø (room 469Ø).

The problem set consists of 2 problems written on 4 pages.

PROBLEMS

1 A three-spin problem

We consider a system consisting of three electrons. They all sit at fixed positions, with their spins as free variables.

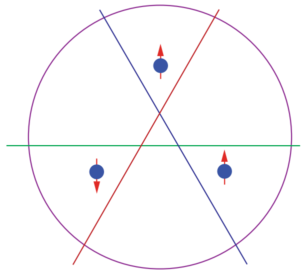


Figure 1: The three-spin-half system. Each of the straight lines shows a division of the full system into two parts, where one part contains a single spin and the othe part contains two spins.

a) The total spin we write as $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2 + \hat{\mathbf{S}}_3$. Use the rule for composition of quantum spins to show that the (spin) Hilbert space consists of three orthogonal subspaces, characterized by spin values $s = 1/2$, $1/2$ and $3/2$ respectively, with $\hat{\mathbf{S}}^2 = s(s+1)\hbar^2$.

b) We consider the following three states of the spin system

$$|\psi_n\rangle = \frac{1}{\sqrt{3}}((|udd\rangle + e^{2\pi in/3}|dud\rangle + e^{-2\pi in/3}|ddu\rangle) , \quad n = 0, \pm 1 \tag{1}$$

where $|u\rangle$ is a spin up state along the z-axis and $|v\rangle$ is a spin down state along the same axis. We use the notation $|udd\rangle = |u\rangle \otimes |d\rangle \otimes |d\rangle$, with the first factor in the tensor product referring to particle 1, the second one to particle 2, and the last one to particle 3. Show that the vectors (1) are orthogonal and have well defined values for the the total spin operators $\mathbf{S}^2$ and S_z. Determine these values.

c) The three-particle system can be considered as a bipartite system, with particle 1 defining one subsystem and particles 2 and 3 defining the other part. We write this partition of the system symbolically as $123 = 1 + (23)$. With this partition what is the corresponding entanglement entropy of the system in the three cases $n = 0, \pm 1$? Compare with the maximum possible entanglement entropy in the bipartite system. With the two other partitions, $123 = 2 + (13)$ and $123 = 3 + (12)$, is there any difference in the entanglement?

d) A measurement of the observable $\hat{S}_{1z}$ is made on particle 1, with the system in one of the states $|\psi_n\rangle$. If the result is *spin up*, what is the entanglement of 2 and 3 in subsystem (23), after the measurement? If the result instead is *spin down*, what is then the entanglement?

e) Consider next the state

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|uuu\rangle - |ddd\rangle) \tag{2}$$

Determine the entanglement entropy of this state with respect to any of the partitions defined in c), and compare with the result found for the states $|\psi_n\rangle$.

We introduce state vectors for spin up and down in the x-direction by

$$|f\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)\,, \quad |b\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle) \tag{3}$$

and for up and down in the y-direction by

$$|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle)\,, \quad |l\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|d\rangle) \tag{4}$$

f) Rewrite the state vector (2) in two different ways, first by using the spin basis (3) for all three spins and next by using spin basis (4) for spin 1 and 2 and basis (3) for spin 3. Use the expressions to show that all spin components of particle 1, S_{1x}, S_{1y} and S_{1z}, can be determined by performing spin measurements on particles 2 and 3, while *not making any measurement* on particle 1. Specify in each case which measurement that should be performed on particle 2 and 3.

2 Entanglement and Bell inequalities

We consider an experimental situation, similar to the one discussed in the lecture notes, where pairs of spin 1/2 particles are initially prepared in a correlated spin state, and then are separated in space while keeping the spin state unchanged. When far apart spin measurements are performed on the particles in each pair, and the results are registered and compared.

The situation is illustrated in the figure, where a series of entangled pairs are created in a source K, and where measurements of the z-components of the spin are performed on both particles (A and B). When the spins in the z-directions are strictly anticorrelated, the result *spin up* (*spin down*) for particle A is always followed by the result *spin down* (*spin up*) for particle B.

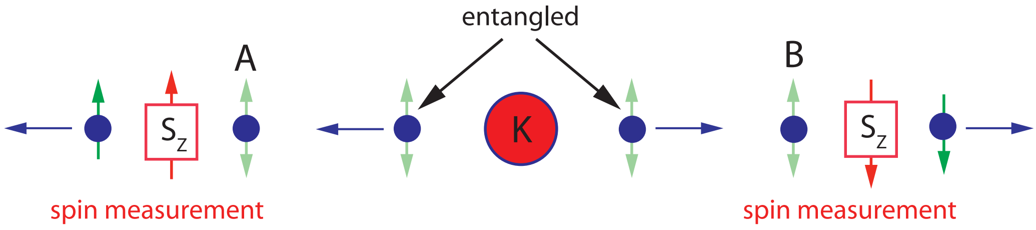


Figure 2: EPR experiment with correlated spins

We consider the situation where three different sets of measurements are performed, with different spin states,

$$\begin{aligned}
\text{I}: \qquad & \hat{\rho}_1 = |\psi_a\rangle\langle\psi_a|, \quad |\psi_a\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \\
\text{II}: \qquad & \hat{\rho}_2 = |\psi_s\rangle\langle\psi_s|, \quad |\psi_s\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\
\text{III}: \qquad & \hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2)
\end{aligned} \tag{5}$$

The notation is $|+-\rangle = |+\rangle \otimes |-\rangle$, where $|\pm\rangle$ are spin states of a single particle, with S_z quantized. The first factor in the tensor product refers to particle A and the second one to particle B. Note that all three states are strictly anticorrelated with respect to the z-component of the spin of the two particles. The purpose of the (hypothetical) experiment is to examine correlation functions that are relevant for the Bell inequalities, as already discussed for case I in the lecture notes, to see if the three states show different behavior. This involves performing the spin measurements also for rotated directions of the spin axes.

a) Of the three density operators only $\hat{\rho}_1$ is rotationally invariant. Demonstrate this by calculating the expectation value of $\mathbf{S}^2$ for the three cases, where $\mathbf{S} = (\hbar/2)(\boldsymbol{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{\sigma})$ is the spin vector of the full system, and comment on the results.

b) What are the reduced density operators $\hat{\rho}_A$ and $\hat{\rho}_B$ in the three cases? Determine the von Neumann entropy S of the full system, as well as the entropies S_A and S_B of the subsystems. Check if the classical restriction on the entropies $S \geq \max\{S_A, S_B\}$ is satisfied in any of the cases. In each of the cases examine if the states are entangled or separable, and give, if possible, a numerical measure of the degree of entanglement.

We assume the direction of the two measurement devices can be rotated so they measure spin components of the form

$$S_\theta = \cos\theta S_z + \sin\theta S_x \tag{6}$$

where the angle θ can be chosen independently for A and B. The state $|\theta\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle$ is the *spin up* vector in the rotated direction and the operator $\hat{P}(\theta) = |\theta\rangle\langle\theta|$ projects on the corresponding spin vector.

c) Show that the given expression for $|\theta\rangle$, as claimed above, is the *spin up* state of S_θ. Determine the expectation value $P_A(\theta) = \left\langle \hat{P}(\theta) \right\rangle_A$, for particle A, in the three cases I, II and III. Comment on the result.

d) Determine, for the three cases, the joint probability distribution $P(\theta,\theta') = \left\langle \hat{P}(\theta) \otimes \hat{P}(\theta') \right\rangle$, with the two angles θ and θ' as independent variables.

The Bell inequality, according to the *hidden variable* analysis described in the lecture notes, gives a constraint on the possible classical correlations of the two spins. In the present case the inequality can be written as

$$F(\theta,\theta') \equiv P(0,\theta') - |P(\theta,0) - P(\theta,\theta')| \geq 0 \tag{7}$$

where one of the angles is set to 0 since we, for the states we consider, will only have strict anticorrelation for spin measurements along the z-axis. (For details see the derivation in the lecture notes.)

e) Make plots of the function $F(\theta, 0.5\,\theta)$ for the three cases I, II and III, with θ varying in the interval $0 < \theta < 2\pi$. Check in all cases whether the inequality (7) is satisfied or broken, and compare the results with what is known from point b) concerning entanglement between the two particles.

In addition to these plots, examine the functions for other choices $\theta' = \lambda\,\theta$ with $\lambda \neq 0.5$ to see if the results are not changed. Alternatively make a 3D plot of the two-variable function $F(\theta,\theta')$ and check whether the conclusion concerning the Bell inequality holds in the full parameter space.

f) Assume an experimental series is performed, with the two angles fixed. The number of pairs registered with *spin up* (in the chosen directions) for both spins A and B is n_{++}, and the number with *spin down* for both spins is n_{--}. Similarly n_{+-} is the number of pairs registered with *spin up* for A and *spin down* for B, n_{-+} is the number of pairs registered with *spin down* for A and *spin up* for B. The total number of pairs in the series is N.

We refer to the experimental results corresponding to $P_A(\theta)$, $P_B(\theta')$, and $P(\theta,\theta')$ as $P^A_{exp}(\theta)$, $P^B_{exp}(\theta')$, and $P_{exp}(\theta,\theta')$. What are these quantities expressed in terms of the numbers $\{n_{ij}, i, j = \pm\}$ and N?

FYS 4110/9110 Modern Quantum Mechanics

Midterm Exam, Fall Semester 2016

Return of solutions
The problem set is available from Monday morning, 17 October.
Written/printed solutions should be returned to Ekspedisjonskontoret in the Physics Building before Monday, 24 October, at 12:00.
Use candidate numbers rather than full names.

Language
Note: The problem set is available also in Norwegian.
Solutions may be written in Norwegian or English depending on your preference.

Questions concerning the problems
Please ask Jon Magne Leinaas (room Ø471), or Paul Bätzing (room V316, or on the Piazza page).

The problem set consists of 2 problems written on 5 pages.

PROBLEMS

1 Entangled photons
In this problem correlations between pairs of entangled photons are studied. The interesting degree of freedom is the photon polarization. For a single photon the polarization corresponds to a quantum state vector in a two-dimensional Hilbert space spanned by the vectors $|H\rangle$ and $|V\rangle$. These vectors correspond to linear polarization in the horizontal and vertical direction, respectively. A general polarization state is a linear combination of these two. As special cases we consider linearly polarized photons in rotated directions,

$$|\theta\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle \tag{1}$$

The two-photon states, when only polarization is taken into account, are vectors in the tensor product space spanned by the four vectors,

$$\begin{aligned} |HH\rangle &= |H\rangle \otimes |H\rangle\,, \quad |HV\rangle = |H\rangle \otimes |V\rangle\,, \\ |VH\rangle &= |V\rangle \otimes |H\rangle\,, \quad |VV\rangle = |V\rangle \otimes |V\rangle\,, \end{aligned} \tag{2}$$

(Note that even if the photons are bosons there is no symmetry constraint on the two-photon states, since we assume that the two photons can be distinguished by their different direction of propagation.)

As a specific way to produce entangled photon pairs we consider the method of *parametric down conversion*, as outline below and sketched in Figs. 2 and 3. As illustrated in Fig. 2a a beam of photons enters a crystal, where single photons, due to the non-linear interaction with the crystal, are split into pairs of photons, which carry half the energy of the incoming photon. The transverse momentum of the emerging photons is fixed so that their direction of propagation is limited to a cone, as indicated in the figure. The photons appear with constant probability around the cone. However, due to conservation of total transverse momentum, the two photons in each a pair are correlated so that they always are emitted at opposite sides of the cone.

There is furthermore a polarization effect, since photons with horizontal and vertical polarization (relative to the crystal planes) do not propagate in exactly the same way. As a consequence the

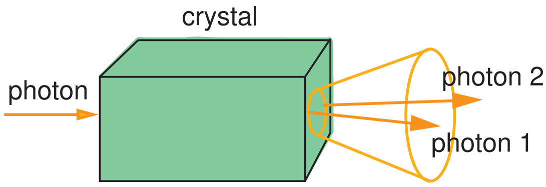


Fig. 2a

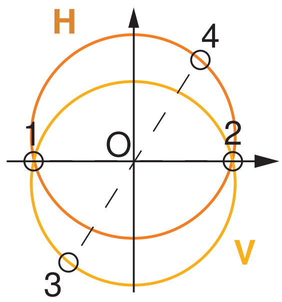


Fig. 2b

cones corresponding to these two polarizations are slightly shifted. This is shown in the head-on view of Fig. 2b, where the cone corresponding to polarization H is slightly lifted relative to the cone corresponding to polarization V.

Two photons in a correlated pair will be located on opposite points of the central point O, like the pair of points 1 and 2 and the pair 3 and 4, and they always appear with orthogonal polarization. As shown by the figure this means that for most directions of the emitted photons the polarization of each photon is uniquely determined by its direction of propagation. For such a pair the two-photon state is a product state of the form $|HV\rangle = |H\rangle \otimes |V\rangle$. As an illustration, the pair $3, 4$ of directions of the cone, as shown in Fig.2b, will be of this type.

However two directions are unique since they lie on both cones. This is illustrated by the points 1 and 2 in Fig. 2b. A photon at one of these positions will be in a superposition of $|H\rangle$ and $|V\rangle$. Due to correlations between the photons a pair located at these positions will be described by an entangled two-photon state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + e^{i\chi}|VH\rangle) \tag{3}$$

where the complex phase χ can be regulated in the experimental set up.

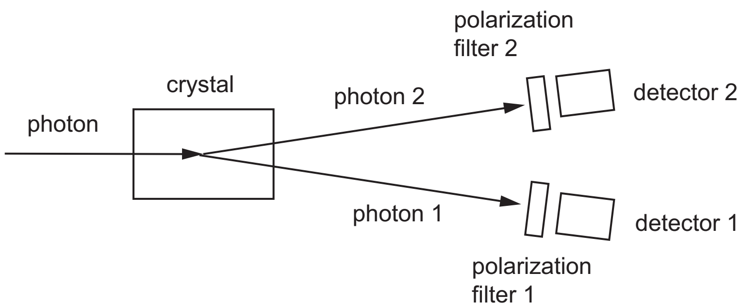


Fig. 3

The experimental set up is schematically shown in Fig. 3. It is assumed that only pairs of entangled photons are filtered out in the beams that reach the two detectors. To analyze correlations between the two photons, polarization filters are applied to photons in both directions, as shown in the figure.

Those that pass the polarization filters are registered in the detectors and the registrations are paired by use of coincidence counters. We assume idealized conditions, by disregarding experimental errors.

The polarization filters may be represented by operators that project on linearly polarized states along rotated directions

$$\hat{P}(\theta) = |\theta\rangle\langle\theta|\,, \quad |\theta\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle \tag{4}$$

In the following we examine the expected results of the polarization measurements by calculating the following expectation values

$$\begin{aligned} P_1(\theta_1) &\equiv \left\langle \hat{P}_1(\theta_1) \right\rangle && \text{photon 1} \\ P_2(\theta_2) &\equiv \left\langle \hat{P}_2(\theta_2) \right\rangle && \text{photon 2} \\ P_{12}(\theta_1, \theta_2) &\equiv \left\langle \hat{P}_1(\theta_1) \otimes \hat{P}_2(\theta_2) \right\rangle && \text{photon 1 and photon 2} \end{aligned} \tag{5}$$

a) Assume that the photon beam produces N entangled photon pairs in a given time interval. In this time interval n_1 photons are registered in detector 1, n_2 photons are registered in detector 2 and and n_{12} are registered at coincidence in the two detectors. What are the relations between the frequences n_1/N, etc. and the the expectation values P_1, P_2 and P_{12}?

b) For the general two-photon state of the form (3) find the density operator of the two-photon pair, and the corresponding reduced density operators for photon 1 and photon 2. Characterize the degree of entanglement of the two photons.

We consider now three different situations where the the entangled photon pairs are produced in the states (3) with I: $\chi = \pi$, II: $\chi = 0$ and III: $\chi = \pi/2$.

c) For all the three cases I, II and III, determine $P_1(\theta_1)$, $P_2(\theta_2)$, and $P_{12}(\theta_1, \theta_2)$.

d) Show that there exists a separable state, in the form of a probabilistic mixture of $|HV\rangle$ and $|VH\rangle$, which has identical expectation values to those in case III.

e) The Bell inequality, which is based on an assumed set of "hidden variables" as a source of the statistical distributions, can be written as a constraint on the function P_{12} in the following way (see Sect. 2.3.2 of the lecture notes),

$$F(\theta_1, \theta_2, \theta_3) \equiv P_{12}(\theta_2, \theta_3) - |P_{12}(\theta_1, \theta_2) - P_{12}(\theta_1, \theta_3)| \geq 0 \tag{6}$$

Examine the Bell inequality in the cases I, II and III for the special choice of angles $\theta_1 = 0$, $\theta_2 = \theta$ and $\theta_3 = 2\theta$ by plotting $F(0, \theta, 2\theta)$ as a function of θ. Based on the plots comment on whether the Bell inequality is satisfied or not and show in particular that in case III Bell's inequality is *not* broken. Is there a relation between this conclusion for case III and the results in d)?

For entangled photons one would expect that one will be able to detect breaking of Bell's inequality. However, in case III, this seems not to bee the case. A possible explanation may be that this is due to the restriction of the two analyzers to *linear* polarization. To investigate this one of the analyzer is changed with the new polarization states and projection operators

$$\hat{P}(\theta_\phi) = |\theta_\phi\rangle\langle\theta_\phi|\,, \quad |\theta_\phi\rangle = \frac{1}{\sqrt{2}}(e^{i\phi}\cos\theta|H\rangle + e^{-i\phi}\sin\theta|V\rangle) \tag{7}$$

where we restrict ϕ to the two cases $\phi = \pi/4$ and $\phi = -\pi/4$, which correspond to circular polarization, either left-handed or right-handed.

f) Consider a similar experimental set up as before, with detector 1 having an unchanged filter, which projects on states of the form (4), while detector 2 now is projecting on the new states (7). We distinguish between the two cases A: $\phi = \pi/4$ and B: $\phi = -\pi/4$. Determine also in these two cases the joint probability $P_{12}(\theta_1, \theta_2)$ and show that in these cases Bell's inequality is broken. Make a comparison with the earlier cases I-III.

2 Atom-photon interactions in a microcavity

An atom is trapped inside a small reflecting cavity. The energy difference between the ground state and the first excited state is $\Delta E = \hbar\omega$, with ω matching the frequency of one of the modes of the electromagnetic field in the cavity. This gives a strong coupling between the atomic states and this field mode, while the couplings to the other cavity modes are weak and can be neglected.

The composite system, the atom plus the resonant cavity mode, is described by the following effective Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\lambda(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) - i\gamma\hbar a^\dagger a \tag{8}$$

where the Pauli matrices act between the two atomic levels, with σ_z being diagonal in the energy basis, and $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ being matrices that raise or lower the atomic energy. $\hat{a}^\dagger$ and $\hat{a}$ are the photon creation and destruction operators, λ is an interaction parameter and γ is a decay parameter. The decay is due to the process where the photon escapes through the cavity walls. Both λ and γ are real-valued parameters, and we assume $\omega > \lambda > \gamma$.

We characterize the relevant states of the composite system as $|g,0\rangle$, $|g,1\rangle$ and $|e,0\rangle$, where g refers to the atomic ground state, e to the excited state, and 0 and 1 refers to the absence or presence of a photon in the cavity mode.

a) Show that in the two-dimensional subspace spanned by the vectors $|g,1\rangle$ and $|e,0\rangle$ the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar(\omega - i\gamma)\mathbb{1} + \frac{1}{2}\hbar\begin{pmatrix} i\gamma & \lambda \\ \lambda & -i\gamma \end{pmatrix} \tag{9}$$

where $|e,0\rangle$ corresponds to the upper row of the matrix and $|g,1\rangle$ to the lower one, and $\mathbb{1}$ is the identity matrix.

We define the time evolution operator in the usual way as

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{\hbar}\hat{H}t} \tag{10}$$

with the expression being valid for $t \geq 0$. The Hamiltonian (9) is non-hermitian due to the decay of the cavity field, and therefore the time evolution operator is non-unitary. However, we shall below see how to compensate for this.

b) Show that the time evolution operator can be written as

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{2}(\omega - i\gamma)t}(\cos(\Omega t)\mathbb{1} - i\sin(\Omega t)\,\frac{\boldsymbol{\Omega}}{\Omega}\cdot\boldsymbol{\sigma}) \tag{11}$$

where $\boldsymbol{\Omega}$ is a complex vector, with $\Omega^2 \equiv \boldsymbol{\Omega}^2$ being real and positive. Determine $\boldsymbol{\Omega}$ and Ω. (Note that $\boldsymbol{\Omega}^2$ contains no complex conjugation, and should therefore not be confused with $|\boldsymbol{\Omega}|^2$.) The Pauli matrix $\boldsymbol{\sigma}$ in (11) refers to the 2×2 matrix formulation (9) of $\hat{H}$.

c) Assume the system initially to be in the state $|\psi(0)\rangle = |e, 0\rangle$. Determine the time evolution of the state vector, $|\psi(t)\rangle$.

There is one important defect with the description of the time evolution discussed so far. Since the time evolution operator is non-unitary, the norm of of the state vector $|\psi(t)\rangle$ is not preserved, but decays with time. Something seems thus to be missing in the description, and we shall now correct for that. Let us for this purpose add a contribution to the density operator $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$, to give the full density operator of the atom-photon system in the cavity as

$$\hat{\rho}_{cav}(t) = \hat{\rho}(t) + f(t)|g, 0\rangle\langle g, 0| \tag{12}$$

with the function $f(t)$ defined so that the norm of $\hat{\rho}_{cav}(t)$ is conserved with value 1.

d) Determine function $f(t)$, and comment on in what sense the addition of the last term in (12) is reasonable, when considering the physical process described by the Hamiltonian (9).

e) Determine and plot, in a common diagram, the time dependent occupation probabilities of the two atomic levels, as well as the probability for one photon to be present in the cavity. Use in the plot $\tau = \lambda t$ as dimensionless time parameter, $\gamma/\lambda = 0.1$ as numerical value for the dimensionless decay parameter, and make the plot for a the interval $0 < \tau < 50$.

The transmission of the photon through the walls implies that the atom-photon system in the cavity, which we now consider as one subsystem, is coupled to the electromagnetic field outside the cavity, which we consider as a second subsystem. We make the assumption that the total system, consisting of the two subsystems, is all the time in a pure, but entangled, quantum state.

f) Show that the density operator $\hat{\rho}_{cav}(t)$ of the atom-photon system has two non-vanishing eigenvalues, given by $f(t)$ and $1 - f(t)$, and use this to determine the entanglement entropy of the two subsystems. Make a plot of the time-dependent entanglement entropy in the same time interval as the first plot.

FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2017

Return of solutions:
The problem set is available from Monday morning, 23 October.
Written/printed solutions should be returned to Ekspedisjonskontoret in the Physics Building before Monday, 30 October, at 12:00.
Use candidate numbers rather than full names.

Language:

Solutions may be written in Norwegian or English depending on your preference.

Questions concerning the problems:

Please ask Joakim Bergli (room V405, or on the Piazza page).

The problem set consists of 2 problems written on 5 pages.

Problem 1: Entanglement in the Jaynes Cummings model

We have in the lectures discussed Rabi oscillations of a Two Level System (TLS) driven by an external oscillating field. In this case the field is treated as a classical quantity with a given time dependence which is not affected by the dynamics of the TLS. We have also studied the Jaynes-Cummings model which is an extension of the Rabi problem to a quantized field (in a cavity, so that emitted photons are not lost, but return and can be reabsorbed). The two models gave to some extent similar results, and in this problem you are going to extend the comparison between the two models beyond what was discussed in the lecture notes or the lectures.

a) We begin by recalling the main features that we have derived. Describe the solution of Rabi problem. Sketch the derivation of these results. You do not have to repeat the full calculations, but give sufficient information so that a person familiar with the concepts will recall the arguments even if considerable time has passed since she studied it.

b) Do the same for the Jaynes-Cummings model. In particular, if we assume that the TLS is initially in the ground state and that there are $n+1$ photons in the cavity, what is the probability to find the TLS in the excited state as a function of time? Show that by a suitable mapping of the parameters, one can identify this with the solution of the Rabi problem.

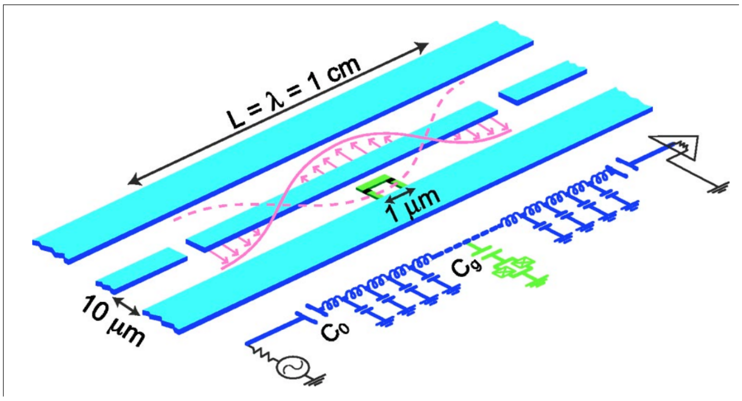
c) If we study the situation in more detail, we will see that there are differences between the two models. Assume that the initial state of the TLS is the ground state and that there are $n+1$ photons in the cavity. Find the reduced density matrix of the TLS as a function of time. Find the entanglement entropy as a function of time. What is the maximal entanglement for given parameters and when is the state maximally entangled?

d) Find the Bloch vector for the state both for the Rabi problem and the Jaynes-Cummings mode. Draw the motion of the Bloch vector in the Bloch sphere and compare the two. Describe the differences between the two models.

e) We usually think that quantum physics should approach classical in the limit where the energy of the system is much larger than the level spacing, which in this case means in the limit $n \to \infty$ where the number of photons is large. Consider you results in this limit, and discuss to what extent we have a reasonable classical limit in this case. Do you have any ideas for what could be changed to make the behaviour more classical-like in certain limit? No calculations are expected to answer this point.

Problem 2: Manipulation and readout of a superconducting qubit in a cavity

In this problem we are going to study a superconducting qubit placed inside a microwave cavity.



The cavity is a 1D transmission line resonator, which consists of a full-wave section of superconducting coplanar waveguide. A Cooper-pair box qubit is placed between the superconducting lines and is capacitively coupled to the center trace at a maximum of the voltage standing wave, yielding a strong electric dipole interaction between the qubit and a single photon in the cavity. Further details can be found in A. Blais *et al.*, Phys. Rev. A **69**, 062320 (2004).

The system is described by the usual Jaynes-Cummings model.

$$H = \hbar\omega_r(a^\dagger a + \frac{1}{2}) + \frac{\hbar\Omega}{2}\sigma^z + \hbar g(a^\dagger\sigma^- + a\sigma^+)$$

where ω_r is the frequency of the cavity mode, $\hbar\Omega$ is the energy splitting of the qubit and g is the interaction strength. Let $|\uparrow\rangle$ and $|\downarrow\rangle$ represent the qubit ground and excited states, and $|n\rangle$ be the state of the cavity with n photons. In the noninteracting case, the eigenstates of the system are then of the form $|\uparrow, n\rangle$ and $|\downarrow, n\rangle$.

a) Find the energy eigenvales and the eigenstates of the Hamiltonian.

b) Consider in particular the case when the detuning $\Delta = \Omega - \omega_r \gg g$ and show that to second order in g, the level separation is independent of n, but depends on the state of the qubit. Find the level separation for the two qubit states $|\uparrow\rangle$ and $|\downarrow\rangle$.

c) Photons can be added to the cavity by sending external microwaves in at the end of the cavity. They will interact with the mode in the cavity by capacitive coupling through the two gaps in the central conductor in the figure. If the frequency of the external microwaves is resonant with a transition between eigenstates of the system, the coupling will be efficient, and this will result in transmission of microwaves through the system. Otherwise, most of the microwave photons will be reflected, and transmission will be small. Explain how this can be used to read out the qubit state, and specify which frequency you would use to have good discrimination between the qubit states.

d) We can obtain the same result for the state-dependent energy shift of the cavity states by a different method which will be useful in the following. Consider the unitary transform

$$U = e^{\frac{g}{\Delta}(a\sigma^+ - a^\dagger\sigma^-)} \tag{1}$$

Show that to second order in g, the transformed Hamiltonian is (here we have omitted some constant terms, which can be removed by a shift in the zero of energy).

$$UHU^\dagger \approx \hbar\left(\omega_r + \frac{g^2}{\Delta}\sigma^z\right)a^\dagger a + \frac{\hbar}{2}\left(\Omega + \frac{g^2}{\Delta}\right)\sigma^z \tag{2}$$

Compare with the result in b) and confirm that the resulting frequency shift is the same.

e) We can also use microwaves to manipulate the qubit. This is described by adding a term

$$H_{\mu w} = \hbar\epsilon(t)\left(a^\dagger e^{-i\omega_{\mu w}t} + ae^{i\omega_{\mu w}t}\right)$$

where $\omega_{\mu w}$ is the microwave frequency and $\epsilon(t)$ is the amplitude. It is time dependent to indicate that the microwaves will be turned on and off, and with possibly varying amplitude to achieve the desired manipulation of the qubit state. Show that to first order in g the transformed Hamiltonian is

$$UH_{\mu w}U^\dagger \approx \hbar\epsilon(t)\left(a^\dagger e^{-i\omega_{\mu w}t} + ae^{i\omega_{\mu w}t}\right) + \frac{\hbar g\epsilon(t)}{\Delta}\left(\sigma^+ e^{-i\omega_{\mu w}t} + \sigma^- e^{i\omega_{\mu w}t}\right) \tag{3}$$

We do not include second order terms, as they are of the form $\frac{g^2}{\Delta^2}\epsilon(t)$ which are small compared to the second order terms in (2) provided $\epsilon(t) \ll \Delta$.

f) To simplify the analysis, it is useful to apply a time dependent unitary transformation $T(t)$ to the system so that the Hamiltonian in the transformed representation is time-independent. This is achieved by going to a rotating reference frame both in the qubit and cavity mode. Determine the proper form of $T(t)$ and show that the resulting Hamiltonian (including both the qubit and cavity mode part (2) and the microwave driving (3)) takes the form

$$H_{1q} = \frac{\hbar}{2}\left[\Omega + 2\frac{g^2}{\Delta}\left(a^\dagger a + \frac{1}{2}\right) - \omega_{\mu w}\right]\sigma^z + \hbar\frac{g\epsilon}{\Delta}\sigma^x + \hbar(\omega_r - \omega_{\mu w})a^\dagger a + \hbar\epsilon(a^\dagger + a)$$

g) Show that if we choose the microwave frequency $\omega_{\mu w} = \Omega + (2n+1)\frac{g^2}{\Delta} - 2\frac{g\epsilon}{\Delta}$ (where n is the number of photons in the cavity) and $t = \frac{\pi\Delta}{2\sqrt{2}g\epsilon}$ the action of the microwave pulse is to perform a Hadamard gate (up to a phase) on the qubit.

h) Determine $\omega_{\mu w}$ and t so that a rotation around the x-axis with angle θ is performed.

i) We can also put two qubits inside the same cavity and the cavity can then be used to make a two-qubit gate which will entangle the two qubits. If the two qubits have the same frequency Ω, which is not resonant with the cavity frequency ω_r the Hamiltonian will be of the form

$$H = \hbar\omega_r(a^\dagger a + \frac{1}{2}) + \frac{\hbar\Omega}{2}(\sigma_1^z + \sigma_2^z) + \hbar g[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)]$$

Generalize the transformation (1) to the case of two qubits and show that it generates a two qubit interaction (again dropping constant terms):

$$UHU^\dagger \approx \hbar\left[\omega_r + \frac{g^2}{\Delta}(\sigma_1^z + \sigma_2^z)\right]a^\dagger a + \frac{\hbar}{2}\left(\Omega + \frac{g^2}{\Delta}\right)(\sigma_1^z + \sigma_2^z) + \frac{\hbar g^2}{\Delta}(\sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+)$$

j) Show that in a reference frame rotating at the qubit frequency Ω in the qubit space and ω_r in the cavity mode space, the Hamiltonian takes the form

$$H_{2q} = \frac{\hbar g^2}{\Delta}(\sigma_1^z + \sigma_2^z)(a^\dagger a + \frac{1}{2}) + \frac{\hbar g^2}{\Delta}(\sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+)$$

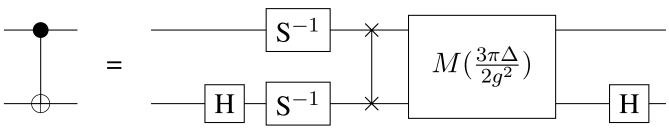
k) Show that this gives the time evolution

$$U_{2q}(t) = e^{-\frac{i}{\hbar}H_{2q}t} = e^{-i\frac{g^2}{\Delta}(\sigma_1^z+\sigma_2^z)(a^\dagger a+\frac{1}{2})}M(t) \otimes 1_r$$

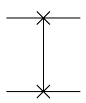
where 1_r is the unit operator in the cavity mode space and

$$M(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\frac{g^2t}{\Delta} & -i\sin\frac{g^2t}{\Delta} & 0 \\ 0 & -i\sin\frac{g^2t}{\Delta} & \cos\frac{g^2t}{\Delta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

l) Since the matrix $M(t)$ is the only place where interactions between the two qubits enter, this is the only place where entanglement is created. To show that our system can perform universal quantum computation we need to show that it can perform the CNOT gate (which we know is universal together with one-qubit operations) using the entangling operation $M(t)$ and operations on individual qubits. Confirm that the following quantum circuit will generate the CNOT gate.



Where the SWAP gate



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

exchanges the two qubit states. It can be implemented by physically exchanging the two qubits, or by known operations with the operation $M(t)$. Here $S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$ is the inverse of the phase gate and $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard gate.

FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2018

Return of solutions:
The problem set is available from Monday morning, 15 October.
Written/printed solutions should be returned to Ekspedisjonskontoret in the Physics Building before Monday, 22 October, at 12:00.
Use candidate numbers rather than full names.

Language:

Solutions may be written in Norwegian or English depending on your preference.

Questions concerning the problems:

Please ask Joakim Bergli (room V405, or on the Piazza page).

The problem set consists of 1 problem written on 6 pages.

Problem 1: Sqeezed states for enhancing the sensitivity of gravitaional wave detectors

We have in the lectures studied coherent states of the harmonic oscillator as examples of minimal uncertainty states. Here we will consider a related class of minimal uncertainty states called squeezed states. We will first study their general properties, and then see how they can be used to enhance the sensitivity of interferometers used in gravitational wave detectors.

We define the squeeze operator

$$S(\zeta) = e^{\frac{1}{2}(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2})}$$

where ζ is a complex number and $\hat{a}$ and $\hat{a}^\dagger$ are the usual annihilation and creation operators of the harmonic oscillator. The squeezed vacuum state is defined as

$$|sq_\zeta\rangle = S(\zeta)|0\rangle$$

a) Show that the action of the squeeze operator on $\hat{a}$ and $\hat{a}^\dagger$ is given by

$$S^\dagger(\zeta)\hat{a}S(\zeta) = \hat{a}\cosh r - e^{i\theta}\hat{a}^\dagger \sinh r$$
$$S^\dagger(\zeta)\hat{a}^\dagger S(\zeta) = \hat{a}^\dagger \cosh r - e^{-i\theta}\hat{a}\sinh r$$

where $\zeta = re^{i\theta}$.

b) In the state $|sq_\zeta\rangle$, find the variance of the position and momentum operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \text{and} \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right).$$

That is, calculate

$$\Delta x^2 = \langle sq_\zeta|\hat{x}^2|sq_\zeta\rangle - \langle sq_\zeta|\hat{x}|sq_\zeta\rangle^2$$
$$\Delta p^2 = \langle sq_\zeta|\hat{p}^2|sq_\zeta\rangle - \langle sq_\zeta|\hat{p}|sq_\zeta\rangle^2$$

c) The Heisenberg uncertainty relation tells us that $\Delta x\Delta p \geq \frac{\hbar}{2}$ with equality only for minimal uncertainty states. Calculate the product $\Delta x\Delta p$ for the states $|sq_\zeta\rangle$ and show that for certain θ they are minimal uncertainty states. For those θ which gives minimal uncertainty, compare Δx and Δp with the corresponding values in vacuum and describe what happens to the uncertainties.

d) Find the expectation value of the number operator $\hat{a}^\dagger\hat{a}$ in the state $|sq_\zeta\rangle$. Later we will apply the theory of squeezed states to a mode of the electromagnetic field, which we know is equivalent to a harmonic oscillator. This expectation value is then interpreted as the mean number of photons in the mode.

The squeezed vacuum state can be displaced to create the squeezed coherent states

$$|\alpha, sq_\zeta\rangle = D(\alpha)S(\zeta)|0\rangle.$$

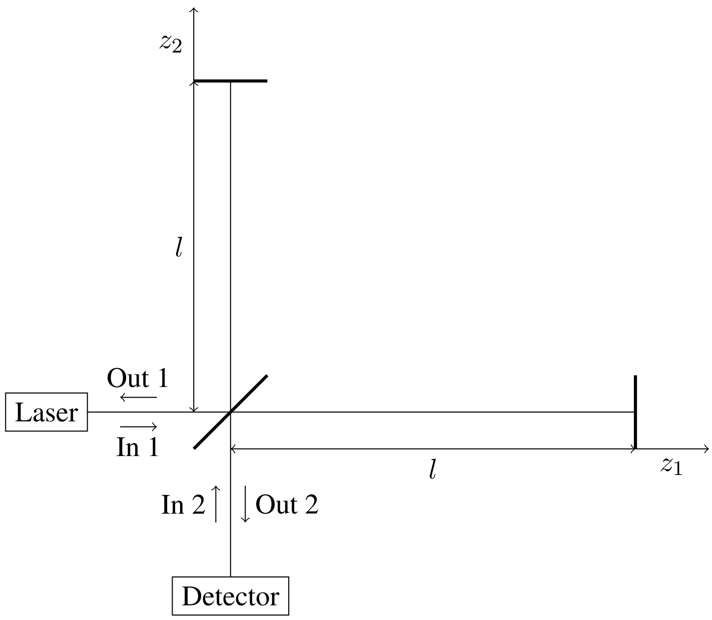
We will now study some properties of these states.

e) Show that these states are still minimal uncertainty states, and that their uncertainties are the same as for the squeezed state $|sq_\zeta\rangle$. Find the expectation values of position and momentum in terms of α and ζ.

f) We have defined the sqeezed coherent states as $|\alpha, sq_\zeta\rangle = D(\alpha)S(\zeta)|0\rangle$. That is, we first sqeeze the vacuum, and then displace. The operators $= D(\alpha)$ and $S(\zeta)$ do not commute. Investigate the states $|sq_\zeta, \alpha\rangle = S(\zeta)D(\alpha)|0\rangle$. That is, we first displace and then squeeze. You may find information on this in the literature, and you should include references to all sources that you use.

The use of squeezed states to reduce the noise in gravitational wave interferometers was first proposed by C. Caves, Phys. Rev. D **23**, 1693 (1981). A recent overview is provided by R. Schnabel *et al.*, Nat. Commun. **1**, 121 (2010) and demonstration of the practical use is shown in J. Asai *et al.*, Nat. Photonics **7**, 613 (2013).

To detect gravitational waves, one can use a Michelson interferometer as shown:



Light is aimed at a semitransparent mirror (beam splitter), which splits it into two perpendicular beams. These are reflected back from distant mirrors, and recombined at the beam splitter. Interference beween the two beams will give rise to interference fringes with alternating constructive and destructive interference depending on the exact path length difference. The interferometer is normally operating with the detector at a dark point in the interference pattern, so that in the absence of a signal, there are (ideally) no photons reaching the detector. The end mirrors, where the light is reflected back to the beamspliter, are mounted on large suspended masses (with mass m), which ideally do not move. When a gravitational wave passes through the interferometer, the lengths of the arms change, the fringes move, and the light intensity (photon counting rate) oscillates.

g) In the LIGO-detector (which was the first to detect a real gravitational wave), the distance from the beam splitter to the mirrors is $l = 4$ km. The strain amplitude (ratio of length change to initial length) of a realistic gravitational wave of cosmic origin (inspiraling of two black holes) is 10^{-21}. How small displacement differences $z = z_2 - z_1$ of the interferometer mirrors do we have to detect to see the gravitational wave signal? Compare your answer to some relevant physical dimension.

There are several sources of noise that will reduce the sensitivity of the interferometer. In this problem we will focus on two fundamental quantum mechanical noise limits, and ignore any practical problems (which are not trivial in practice). The first effect is called photon-counting error (or shot noise) and is a consequence of the fact that the laser light used is not in a number eigenstate, but rather close to a coherent state. This means that the photon number is not a sharply defined quantity, and it will fluctuate in time as a result of quantum uncertainty. The second effect is called radiation-pressure error, and is a result of the fluctuating motion of the mirrors because of the fluctuating radiation pressure in the laser beams. This is again because the photon number is not well-defined, and is therefore also a fundamental quantum restriction.

Normally, one would input coherent light (from a powerful laser) in the input port 1, and arrange the interferometer so that in the absense of any gravitational wave signal all the light would exit back in the same direction, while there will be complete destructive interference in the output port 2. Port 2 would be used only for output, with no (that is, the vacuum state) input. Surprisingly, the noise can be modified by the input of a squeezed vacuum state in port 2, instead of the normal vacuum. To investigate this effect, we need to understand how to find the combined state of the field from the two sources. One has to add the electric fields from each source, and it can be shown that this leads to relations between the creation and anihilation operators for the modes.

h) For the radiation pressure noise we need to consider the relation between the field before and after passing the beamsplitter. Let $\hat{a}_1^\dagger$ and $\hat{a}_1$ be the creation and annihilation operators for photons in input mode 1 (moving horizontally in the figure), while $\hat{a}_2^\dagger$ and $\hat{a}_2$ are the corresponding operators for mode 2 (moving vertically). The operators for the horizontal mode after the beamsplitter is $\hat{b}_1^\dagger$ and $\hat{b}_1$, and those for the vertical mode are $\hat{b}_2^\dagger$ and $\hat{b}_2$. The relation between the operators are similar to those we have used to relate states passing beamsplitters:

$$\hat{b}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1 + i\hat{a}_2)$$
$$\hat{b}_2 = \frac{1}{\sqrt{2}}(\hat{a}_2 + i\hat{a}_1)$$

The momentum of a photon is $p = E/c = \hbar\omega/c$. The momentum transfer to the mirror is twice the momentum of a single photon times the number of photons. The change in the interferometer output depends only on the difference in the change in path length, and therefore only on the difference in the transferred momenta to the two end mirrors. The difference in the transferred momentum is then

$$P = \frac{2\hbar\omega}{c}(\hat{b}_2^\dagger\hat{b}_2 - \hat{b}_1^\dagger\hat{b}_1)$$

Find the expectation values of P and P^2 if the input state is

$$|\psi\rangle = S_2(\zeta)D_1(\alpha)|0\rangle$$

where $S_2(\zeta) = e^{\frac{1}{2}(\zeta^*\hat{a}_2^2 - \zeta\hat{a}_2^{\dagger 2})}$ is the squeezing operator in incoming mode 2 and $D_1(\alpha) = e^{\alpha\hat{a}_1^\dagger - \alpha^*\hat{a}_1}$ is the displacement operator in incoming mode 1. That is, we have a coherent state (with typically large intensity) in mode 1 and a squeezed vacuum state in mode 2. You need only consider the case where both α and $\zeta = r$ are real.

i) The effect of the radiation pressure fluctutaions builds up over time as the momentum transferred to the end mirrors leads to displacement. If we define the variance of P as $(\Delta P)^2 = \langle\psi|P^2|\psi\rangle - \langle\psi|P|\psi\rangle^2$, argue that the variance in path difference after a time τ is $\Delta z_{rp} = \frac{\tau}{2m}\Delta P$ and show that it is given by

$$\Delta z_{rp} = \frac{\hbar\omega\tau}{mc}\sqrt{\alpha^2 e^{2r} + \sinh^2 r}.$$

In what way does Δz_{rp} depend on the power of the laser beam in input 1? On the mass of the end mirrors? How can we reduce Δz_{rp}?

j) For the photon counting error we need to consider the output modes, after the light has passed through the beamsplitter, reflected from the mirrors and passed the beamsplitter the second time. We let $\hat{c}_1^\dagger$,$\hat{c}_1$ and $\hat{c}_2^\dagger$,$\hat{c}_2$ denote the creation and annihilation operators of the two output modes. Show that

$$\hat{c}_1 = ie^{i\Phi}\left[-\hat{a}_1 \sin\phi + \hat{a}_2 \cos\phi\right]$$
$$\hat{c}_2 = ie^{i\Phi}\left[\hat{a}_1 \cos\phi + \hat{a}_2 \sin\phi\right]$$

Find the expressions for Φ and ϕ and explain their physical meaning.

k) Show that the expectation value of the number operator $\hat{N}_2 = \hat{c}_2^\dagger \hat{c}_2$ in output port 2 is (for real α and ζ)

$$\langle\psi|\hat{N}_2|\psi\rangle = \alpha^2 \cos^2\phi + \sinh^2 r \sin^2\phi$$

and that the variance is

$$(\Delta N_2)^2 = \langle\psi|\hat{N}_2^2|\psi\rangle - \langle\psi|\hat{N}_2|\psi\rangle^2 = \alpha^2\cos^4\phi + 2\sinh^2 r\cosh^2 r\sin^4\phi + (\alpha^2 e^{-2r} + \sinh^2 r)\cos^2\phi\sin^2\phi.$$

l) As for the radiation pressure noise we can convert this into an uncertainty in the difference in the displacements $z = z_2 - z_1$ of the two mirrors. Show that a change of z by Δz gives a change in ϕ by $\Delta\phi = \frac{\omega}{c}\Delta z$. Show that when $|\alpha\cos\phi| \gg |\sinh r \sin\phi|$ this gives the noise in position difference due to photon counting noise

$$\Delta z_{pc} = \frac{c}{2\omega}\sqrt{\frac{\cot^2\phi}{\alpha^2} + \frac{2\tan^2\phi\sinh^2 r\cosh^2 r}{\alpha^4} + \frac{e^{-2r}}{\alpha^2} + \frac{\sinh^2 r}{\alpha^4}}.$$

We can reduce the photon counting noise by choosing the proper phase difference in the absence of a signal. Working near a dark point in the interference pattern we have $\cos\phi \approx 0$ the first term in the above expression is small. If α is sufficiently large, the second term can also be small provided we are not exactly on the dark point so that $\tan\phi$ is not too large. The last term can also be neglected compared to the third, and we are left with the approximate expression

$$\Delta z_{pc} = \frac{c}{2\omega}\frac{e^{-r}}{\alpha}.$$

Similarly we have for the radiation pressure noise approximately

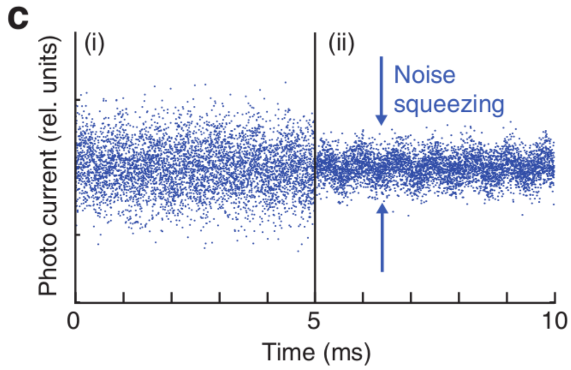
$$\Delta z_{rp} = \frac{\hbar\omega\tau}{mc}\alpha e^r.$$

If we assume that the noise sources are independent (which probably is not true), we get that the total noise is

$$\Delta z = \sqrt{\Delta z_{pc}^2 + \Delta z_{rp}^2}.$$

m) Discuss the dependence of the two noise sources on the laser power and the squezzing parameter r. The power is proportional to the number of photons, which has an average value of α^2. Minimize the total noise as a function of α^2 and determine how the optimal power and the minimal noise depend on r.

Here is a figure [Figure 6c of R. Schnabel *et al.*, Nat. Commun. **1**, 121 (2010)] showing the simulated change in the signal from the detector without (left) and with (right) input of sqeezed light.



As we see, the signal is virtually invisible without squeezing, and is clearly seen with squeezing.

n) Create a plot similar to the one shown above. You are at this point allowed to use any simplifying assumptions you need and any method that you find useful. But you should carefully describe your procedure and any assuptions made, discussing how realistic they are.

FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2019

Return of solutions:
The problem set is available from Monday morning, 14 October.
You may submit handwritten solutions, but they have to be scanned and included in one single file, which is submitted in Inspera before Monday, 21 October, at 12:00.

Language:
Solutions may be written in Norwegian or English depending on your preference.
Questions concerning the problems:
Please ask Joakim Bergli (room V405).

The problem set consists of 3 problems written on 4 pages.

All the problems in this exam are related to the same model of two coupled harmonic oscillators.

Problem 1: Entanglement in the evolution starting from a number state

Two harmonic oscillators, A and B, are coupled with a Hamiltonian

$$H = \hbar\omega_a \hat{a}^\dagger \hat{a} + \hbar\omega_b \hat{b}^\dagger \hat{b} + \hbar\frac{\lambda}{2}(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}). \tag{1}$$

Here $\hat{a}^\dagger$ and $\hat{a}$ are creation and annihilation operators for oscillator A and $\hat{b}^\dagger$ and $\hat{b}$ corresponding operators for oscillator B.

a) Show that the Hamiltonian can be expressed in diagonal form as

$$H = \hbar\omega_c \hat{c}^\dagger \hat{c} + \hbar\omega_d \hat{d}^\dagger \hat{d} \tag{2}$$

where $\hat{c}$ and $\hat{d}$ are linear combinations of $\hat{a}$ and $\hat{b}$

$$\hat{c} = \mu\hat{a} + \nu\hat{b}, \qquad \hat{d} = \nu\hat{a} - \mu\hat{b} \tag{3}$$

where μ and ν are positive real constants satisfying $\mu^2 + \nu^2 = 1$. Determine the constants μ, ν, ω_c and ω_d in terms of ω_a, ω_b and λ. Check that the operators $\hat{c}$ and $\hat{d}$ satisfy the usual harmonic oscillator commutation relations, and that the oscillators C and D are independent of each other (all operators for different oscillators commute).

b) Assume that the initial state of the system is the first excited state of oscillator A. That is, the state $|1_A 0_B\rangle = \hat{a}^\dagger|0\rangle$ where $|0\rangle$ is the ground state. Find the state of the coupled oscillators as a function of time.

c) We define the number operators for the original oscillators as $N_A = \hat{a}^\dagger \hat{a}$ and $N_B = \hat{b}^\dagger \hat{b}$. With the initial state of the system still $\hat{a}^\dagger|0\rangle$, find the expectation values $\langle N_A \rangle$ and $\langle N_B \rangle$ as functions of time. Describe the result and discuss the cases were the oscillators are identical ($\omega_a = \omega_b$) or very different.

d) Calculate the entanglement entropy between oscillators A and B as a function of time. What is the maximal value of the entanglement entropy for different $\Delta = \omega_a - \omega_b$?

e) Assume now that oscillator A initially is in the state $|n\rangle$ while B is in its ground state. Find the state of the oscillators as function of time. Find also $\langle N_A \rangle$ and $\langle N_B \rangle$ and compare to the result of question c). This question is a bit technical, and no further question depends on solving this first, so you may skip it and return to it later if you have time.

Problem 2: Evolution starting from a coherent state

a) Let the initial state $|\psi(0)\rangle$ of the composite system be a coherent state when expressed in terms of the new variables,

$$\hat{c}|\psi(0)\rangle = z_{c0}|\psi(0)\rangle, \qquad \hat{d}|\psi(0)\rangle = z_{d0}|\psi(0)\rangle,$$

Also at a later time the state $|\psi(t)\rangle$ will be a coherent state for both $\hat{c}$ and $\hat{d}$ with eigenvalues

$$z_c(t) = e^{-i\omega_c t} z_{c0}, \qquad z_d(t) = e^{-i\omega_d t} z_{d0}.$$

Show this for $z_c(t)$ (The expression for $z_d(t)$ then follows from symmetry).

b) Show that the state $|\psi(t)\rangle$ from the previous question is also coherent with respect to the operators $\hat{a}$ and $\hat{b}$ and find the eigenvalues $z_a(t)$ and $z_b(t)$ expressed in terms of z_{a0} and z_{b0}.

c) The state $|z_a(t)\rangle \otimes |z_b(t)\rangle$ is clearly coherent with respect to the operators $\hat{a}$ and $\hat{b}$ and with the eigenvalues $z_a(t)$ and $z_b(t)$. But is is not so obvious that this is the only state with this property. Show that this is indeed the case, so that we have $|\psi(t)\rangle = |z_a(t)\rangle \otimes |z_b(t)\rangle$. What does this imply for the entanglement of the two oscillators if we start from a coherent state $|z_{a0}\rangle \otimes |z_{b0}\rangle$?

d) To show that the result of question c) is not trivial, we consider the following situation. We have a system that is composed of two parts, A and B, and a state $|\psi\rangle$ for the full system. We are also given two operators A and B acting on the corresponding subsystems, such that

$$A \otimes \mathbb{1}|\psi\rangle = z_a|\psi\rangle, \qquad \mathbb{1} \otimes B|\psi\rangle = z_b|\psi\rangle. \tag{4}$$

If $A|\psi_A\rangle = z_a|\psi_A\rangle$ and $B|\psi_B\rangle = z_b|\psi_B\rangle$ it is clear that the product state $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ satisfies Eq. (4). Give an example were this is not the unique solution, and show that there can be entangled states $|\psi\rangle$ satisfying Eq. (4).

e) Find $\langle N_A \rangle$ and $\langle N_B \rangle$ for the state $|\psi(t)\rangle$ if the initial state is $|\psi(0)\rangle = |z_{a0}\rangle_A \otimes |0\rangle_B$.

Problem 3: Evolution when the system is coupled to an environment

If the two oscillators are not isolated from the surroundings, the state will evolve from a pure to a mixed state. To describe this, we will use the Lindblad equation

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] - \frac{\gamma}{2}\left(\hat{a}^\dagger\hat{a}\rho + \rho\hat{a}^\dagger\hat{a} - 2\hat{a}\rho\hat{a}^\dagger\right) - \frac{\gamma}{2}\left(\hat{b}^\dagger\hat{b}\rho + \rho\hat{b}^\dagger\hat{b} - 2\hat{b}\rho\hat{b}^\dagger\right)$$

where H is given by (1).

a) What have we assumed about the temperature of the environment when we have writen the Lindblad equation in this form?

b) We denote the matrix elements of ρ in the basis of the eigenstates for the A and B oscillators by

$$\rho_{mn,m'n'} = \langle m_A n_B|\rho|m'_A n'_B\rangle$$

Show that if we start from the state $|1_A 0_B\rangle$, the equations for all nonzero elements of the density matrix are:

$$\begin{aligned}
\dot{\rho}_{10,10} &= -\gamma\rho_{10,10} - i\frac{\lambda}{2}(\rho_{01,10} - \rho_{10,01}) \\
\dot{\rho}_{01,10} &= (i\Delta - \gamma)\rho_{01,10} - i\frac{\lambda}{2}(\rho_{10,10} - \rho_{01,01}) \\
\dot{\rho}_{10,01} &= (-i\Delta - \gamma)\rho_{10,01} + i\frac{\lambda}{2}(\rho_{10,10} - \rho_{01,01}) \\
\dot{\rho}_{01,01} &= -\gamma\rho_{01,01} + i\frac{\lambda}{2}(\rho_{01,10} - \rho_{10,01}) \\
\dot{\rho}_{00,00} &= \gamma(\rho_{10,10} + \rho_{01,01})
\end{aligned}$$

c) Show that the solution to these equations is

$$\begin{aligned}
\rho_{10,10} &= \left[\cos^2\frac{\bar{\lambda}t}{2} + \epsilon^2\sin^2\frac{\bar{\lambda}t}{2}\right]e^{-\gamma t} \\
\rho_{01,10} &= \left[-i\delta\cos\frac{\bar{\lambda}t}{2}\sin\frac{\bar{\lambda}t}{2} + \epsilon\delta\sin^2\frac{\bar{\lambda}t}{2}\right]e^{-\gamma t} \\
\rho_{10,01} &= \left[i\delta\cos\frac{\bar{\lambda}t}{2}\sin\frac{\bar{\lambda}t}{2} + \epsilon\delta\sin^2\frac{\bar{\lambda}t}{2}\right]e^{-\gamma t} \\
\rho_{01,01} &= \delta^2\sin^2\frac{\bar{\lambda}t}{2}e^{-\gamma t} \\
\rho_{00,00} &= 1 - e^{-\gamma t}
\end{aligned} \tag{5}$$

where

$$\bar{\lambda} = \sqrt{\Delta^2 + \lambda^2}, \qquad \delta = \frac{\lambda}{\bar{\lambda}}, \qquad \epsilon = \frac{\Delta}{\bar{\lambda}}.$$

d) The case $\gamma = 0$ is identical to that which we studied in Problem 1b). Check that the solution that you obtained there gives the same density matrix as Eq. (5) in this case.

e) We would like to quantify the entanglement beween the two oscillators as function of time, and see how it depends on the damping rate γ. For a pure state of the combined system, we use the entanglement entropy as a measure of entanglement. Check that our system is in a pure state if and only if $\gamma = 0$.

f) For $\gamma > 0$ we have a mixed state for the combined system, since it is coupled to the environment. The entanglement entropy will then not be a sensible measure of entanglement. Give an example of a mixed state that is not entangled but still has a large entanglement entropy.

g) For mixed states, no simple test that determines if a state is entangled or not is known in general. For two coupled two-level systems it has been shown that the positive partial transpose criterion is sufficient. Search for information on this and explain how it is computed. Describe what you learn for a general system if the partial transpose is positive, and what is special for the case of two coupled two-level systems. Cite your sources.

h) Explain why we can apply the results for coupled two-level systems to the state (5). Calculate the positive partial transpose of this state and show that it is entangled.

i) One possible measure of the entanglement for mixed states is the concurrence. Search for information on this and explain how it is computed for the case of two coupled two-level systems. Cite your sources.

j) Calculate the concurrence for the state (5).

k) Another measure of entanglement is the entropy of formation, E_F. In the case of two coupled two-level systems it is related to the concurrence C by the formula

$$E_F = -x \ln x - (1-x)\ln(1-x)$$

where

$$x = \frac{1}{2}(1 + \sqrt{1 - C^2}).$$

One nice feature of this as opposed to the concurrence is that it agrees with the entropy of entanglement for pure states. Calculate the entropy of formation for the state (5). Show that when $\gamma = 0$ it is equal to the entropy of entanglement as found in Problem 1d).

l) Plot E_F as a function of time for some parameter values that you find give interesting results and discuss what you learn.

FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2020

Return of solutions:
The problem set is available from Friday morning, 16 October.
You may submit handwritten solutions, but they have to be scanned and included in one single file, which is submitted in Inspera before Friday, 23 October, at 12:00.

Language:
Solutions may be written in Norwegian or English depending on your preference.
Questions concerning the problems:
Please ask Joakim Bergli (room V405).

The problem set consists of 2 problems written on 5 pages.

Problem 1: Bloch sphere for three-level system

For a two-level system with density matrix ρ we have defined the Bloch vector $\mathbf{m}$ by the equation

$$\rho = \frac{1}{2}(\mathbb{1} + m_i\sigma_i)$$

where σ_i are the Pauli matrices, and we are summing over the repeated index i. We want to generalize this to arbitrary n-level systems, and in particular study the three-level case. The density matrix is in general a Hermitian matrix with $\mathrm{Tr}(\rho) = 1$, which means that we can write

$$\rho = \frac{1}{n}(\mathbb{1} + \alpha m_i\lambda_i)$$

where α is a numerical constant that will depend on n, and λ_i are traceless Hermitian matrices.

a) How many matrices λ_i do we need for an n-level system? This will also be the number of components of the Bloch vector. That is, the number of dimensions of the space where the Bloch vector is.

One can always choose the matrices λ_i to satisfy the relation

$$Tr(\lambda_i\lambda_j) = 2\delta_{ij}$$

(see e. g. G. Kimura, Physics Letters A **314**, 339 (2003) for an explicit form).

b) Find the value of α so that pure states have $|\mathbf{m}| = 1$.

c) What is the dimension of the space of pure states for n-level systems?

d) Explain why the pure states are on the surface of the Bloch sphere, but do not cover it.

We will now specialize to the case of a 3-level system. In this case the matrices λ_i are known as the Gell-Mann matrices (for the form of all these and more on the present problem, see S. Goyal *et al.* J. Phys. A: Math. Theor. **49**, 165203 (2016)). In question d) you showed that the pure states do not cover the entire surface of the Bloch sphere. We will now see what happens for mixed states. In addition to being Hermitian and having trace 1, the density matrix should not have negative eigenvalues. We will restrict the Bloch vector to lie in the plane spanned by the two Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

so that the density matrix is of the form

$$\rho = \frac{1}{3}\left[\mathbb{1} + \sqrt{3}(m_1\lambda_1 + m_8\lambda_8)\right].$$

e) Find the eigenvalues of density matrices ρ of this form.

f) Plot the cross section of the Bloch sphere spanned by the Gell-Mann matrices λ_1 and λ_8 and mark the area where the density matrix has only positive eigenvalues. These are the only density matrices allowed as physical states.

g) Plot the von Neumann entropy for states in this plane, and determine if the entropy depends only on the length of the Bloch vector, as for a two-level system, or is also a function of the direction of the Bloch vector.

Problem 2: Entanglement transformations using local operations and classical communication

We consider a bipartite system, with subsystems A and B. If we have two pure states, generally both entangled, we can wonder if the entanglement is in some sense equivalent in the two states. By "equivalent" we do not mean quantitatively equal (that is, with the same entanglement entropy), but rather qualitatively equal (but maybe to different degree, so that the entanglement entropy could be different). One way to approach this is to study if one state can be converted to the other if we only apply local operations to each subsystem A and B. Local operations means a unitary operator that acts only on one of the subsystems, or a measurement that measures an observable on one of the subsystems. Most often, one also allows the observers at A and B to exchange classical information in addition to local operations. The combination is referred to as Local Operations and Classical Communication (LOCC).

A pure state for the system can then be Schmidt decomposed as $|\psi\rangle = \sum_i \sqrt{\alpha_i}|i_A\rangle \otimes |i_B\rangle$, which we will write for short $|\psi\rangle = \sum_i \sqrt{\alpha_i}|ii\rangle$. We use the convention that the Schmidt coefficients α_i are ordered, so that $\alpha_1 \geq \alpha_2 \geq \cdots$. Similarly, we write the second state as $|\phi\rangle = \sum_i \sqrt{\beta_i}|i'i'\rangle$. A vector $\beta = (\beta_1, \ldots, \beta_n)$ is said to majorize another vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ if

$$\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i \tag{1}$$

for all k. This is written as $\alpha \prec \beta$.

If we write the fact that $|\psi\rangle$ can be transformed to $|\phi\rangle$ using LOCC as $|\psi\rangle \to |\phi\rangle$, the following theorem (M. Nielsen, Phys. Rev. Lett., **83**, 436 (1999)) gives the necessary and sufficient conditions for one state to be converted to another using LOCC:

$$|\psi\rangle \to |\phi\rangle \quad \text{if and only if} \quad \alpha \prec \beta. \tag{2}$$

a) Show that if both A and B are 2-level systems, then either $|\psi\rangle \to |\phi\rangle$ or $|\phi\rangle \to |\psi\rangle$, or both. That is, one of the states can always be converted to the other. This means that the entanglement is in some sense of the same type in all the states.

b) Show that if both A and B are 2-level systems there exist states that can be converted to any other state using LOCC, and find one example of such a state.

c) What local operations should you apply to transform

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \to \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)?$$

   How much classical information do you need to transfer?

In general, the local operations that are needed include measurement on one side, with the result of the measurement transmitted as classical information to the other side. We know that if we make a standard projective measurement on one of the particles in an entangled pair, we will end in an eigenstate of the corresponding operator, and entanglement disappears. So if we want to reduce entanglement without eliminating it entirely, we need to make a type of measurement that is affecting the state less (and necessarily giving us less precise information at the same time). One way to achieve this is to let the particle interact and get entangled with another particle, and then measuring on this particle. Consider a 2-level system (we call it system 1) in the state

$$|\psi\rangle_1 = \cos\phi|0\rangle + \sin\phi|1\rangle.$$

We want to make a non-projective measurement of the state by entangling it with a second 2-level system (system 2), which initially is in the state

$$|\chi\rangle_2 = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

The entangling operation is given by the unitary transformation which in the tensor product basis $|i\rangle_1 \otimes |j\rangle_2$ is given by the matrix

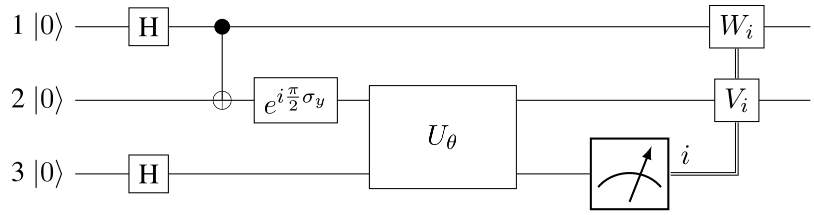
$$U_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{3}$$

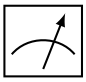
That is, we evolve the system to the final state $U_\theta|\psi\rangle_1 \otimes |\chi\rangle_2$ and measure the second 2-level system.

d) What is the final state of the first particle for each measurement outcome on the second?

e) Give an interpretation of your answer to the previous question. Explain in words what happens physically (Hint: What interaction between the two particles would generate the given unitary transformation U_θ?).

f) We have the following quantum circuit.



Here U_θ is the unitary transformation given in Eq. (3) with the lower line corresponding to the first qubit.  indicates that the qubit is measured in the $\{|0\rangle, |1\rangle\}$-basis, and the rightmost part of the circuit means that the operations V_i and W_i are dependent on the outcome i of the measurement. We choose the operations $V_0 = W_0 = \mathbb{1}$. What operations must V_1 and W_1 be, so that the final state of the system consisting of qubits 1 and 2 is the same, independent of the measurement outcome? What will the final state be?

g) What are the probabilities for each of the measurement outcomes?

h) Describe with words what the different gates in the circuit do and how we can claim that it is realizing the transformation of one state to another using LOCC. Which state is the initial state of the LOCC transformation?

i) Prove that the entropy of entanglement can never be increased using LOCC. If you find a proof, or helpful fact, in the literature, cite your sources.

The following are states of two 3-level systems

$$|\psi\rangle = \sqrt{\frac{1}{2}}|11\rangle + \sqrt{\frac{2}{5}}|22\rangle + \sqrt{\frac{1}{10}}|33\rangle$$
$$|\phi\rangle = \sqrt{\frac{3}{5}}|11\rangle + \sqrt{\frac{1}{5}}|22\rangle + \sqrt{\frac{1}{5}}|33\rangle$$

j) Show that neither $|\psi\rangle \to |\phi\rangle$ nor $|\phi\rangle \to |\psi\rangle$. This means that the entanglement in the two states is qualitatively different (in the sense of non-conversion using LOCC).

k) The suggested classification of entanglement based om LOCC is not ideal, as it suffers from at least one serious drawback. Search in the literature and find criticism of this classification. Cite you sources.

One consequence of the theorem (2) is the so-called entanglement catalysis. Local transformations on a composite quantum system can be enhanced in the presence of certain entangled states. These extra states act much like catalysts in a chemical reaction: they allow otherwise impossible local transformations to be realized, without being consumed in any way.

The following are states of two 4-level systems

$$|\psi_1\rangle = \sqrt{0.4}|11\rangle + \sqrt{0.4}|22\rangle + \sqrt{0.1}|33\rangle + \sqrt{0.1}|44\rangle$$
$$|\psi_2\rangle = \sqrt{0.5}|11\rangle + \sqrt{0.25}|22\rangle + \sqrt{0.25}|33\rangle$$

l) Show that neither $|\psi_1\rangle \to |\psi_2\rangle$ nor $|\psi_2\rangle \to |\psi_1\rangle$.

m) We assume now that the two parties, in addition to the above 4-level systems also share a pair of entangled 2-level systems in the state

$$|\phi\rangle = \sqrt{0,6}|55\rangle + \sqrt{0.4}|66\rangle$$

Show that

$$|\psi_1\rangle|\phi\rangle \to |\psi_2\rangle|\phi\rangle$$

This means that the presence of the state $|\phi\rangle$ enables the transformation from $|\psi_1\rangle$ to $|\psi_2\rangle$ without being changed in the process.

FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2021

Return of solutions:
The problem set is available from Friday morning, 22 October.
You may submit handwritten solutions, but they have to be scanned and included in one single file, which is submitted in Inspera before Friday, 29 October, at 12:00.

Language:
Solutions may be written in Norwegian or English depending on your preference.
Questions concerning the problems:
Please ask Joakim Bergli (room V405).

The problem set consists of 1 problem written on 4 pages.

Problem 1: Superradiance

In this problem we will study the phenomenon of superradiance. This is the modifiaction of the emission form an atom in the presence of other identical atoms. To aid you in solving the problems, you may consult any material that you can find on the topic, but as always you should cite the sources you use.

As introduction, we will first consider the emission from a single atom. The interaction beteen the atom and the electromagnetic field is given by the Hamiltonian

$$H_{int} = -\frac{e}{m}\mathbf{A}(\mathbf{r}) \cdot \mathbf{p}.$$

In the following, we will only consider transitions between two atomic states, the ground state $|0\rangle$ and one excited state $|1\rangle$.

a) Show that in the subspace spanned by the atomic states $\{|0\rangle, |1\rangle\}$ and for the purpose of calculating transition rates of spontaneous emission, we can replace the interaction Hamiltonian by

$$H_{int} = \sum_{\mathbf{k}a} g_{\mathbf{k}a}(\hat{a}_{\mathbf{k}a}\sigma^+ + \hat{a}^\dagger_{\mathbf{k}a}\sigma^-)$$

and determine the coupling constants $g_{\mathbf{k}a}$. Here, $\sigma^\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

b) Find the rate of spontaneous emission w_1 from the state $|1\rangle$ to the state $|0\rangle$, summing over all possible final photon states.

We will next study the curious way in which the emission of a photon from an excited atom is modified by the presence of other identical atoms at nearby points in space. If the distance between the atoms is much less than the wavelenght of the emitted light, there is no way to determine which atom the light is emitted from, and we can consider the coupling of each atom to the radiation field to be the same. The interaction Hamiltonian is then

$$H_{int} = \sum_{\mathbf{k}a} g_{\mathbf{k}a}(\hat{a}_{\mathbf{k}a} D^{+} + \hat{a}^{\dagger}_{\mathbf{k}a} D^{-})$$

with $D^{\pm} = \sum_i \sigma_i^{\pm}$ where the sum is over all the atoms and $\sigma_i^{\pm}$ is the $\sigma^{\pm}$ acting on atom i.

c) Consider first two atoms in the initial state $|10\rangle$ where one atom is excited and the other in the ground state. Initially, one would expect the presence of the second atom not to affect the first, but this is not correct. Show that the state $|10\rangle$ will not always decay to the ground state $|00\rangle$, but sometimes only to the state $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$.

   Hint: write the inital state as

$$|10\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) + \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)\right]$$

   and show that only one of the components decay, while the other is not changed by the interaction.

d) On average, how many photons are emitted every time the experiment is repeated?

We consider now a large number N of atoms (we omly consider even N), all so close in space that the interaction with the field is the same. We study the evolution from the initial state $|11\cdots 1\rangle$ where all the atoms are excited. The permutation symmetry of the inial state as well as the interaction Hamiltonian means that only states symmetric under permutation of the atoms will be populated. These take the form

$$|JM\rangle = \sqrt{\frac{(J+M)!}{N!(J-M)!}}(D^{-})^{J-M}|11\cdots 1\rangle$$

with $J = N/2$ and $-J \leq M \leq J$ (M is an integer for even N and half integer for odd N).

e) Show that these states are orthogonal and normalized.

f) Show that the decay rate from the state $|JM\rangle$ is

$$w_{JM} = (J+M)(J-M+1)w_1$$

   where w_1 is the decay rate of a single atom that we studied in question b).

g) For a given N, which M will give the largest decay rate (and consequently the largest photon emission rate)? Show that in some cases the rate of photon emission is much larger than what you would expect from N independent atoms, hence the term superradiance.

The fact that the collection of N atoms radiates at a rate much larger than N independent atoms indicates that there are correlations between the atoms. We will study these correlations using two different approaches.

First, we consider the correlation of operators on pairs of atoms. The decay rate w_{JM} that you calculated in question f) is proportional to the matrix element squared,

$$|\langle J, M-1|D^{-}|JM\rangle|^2$$

h) Show that this is equal to the expectation value

$$\langle JM|D^+D^-|JM\rangle$$

i) Show that

$$\langle JM|\sum_i \sigma_i^+\sigma_i^-|JM\rangle = J + M.$$

j) The permutation symmetry of the state implies that the expectation value $\langle JM|\sigma_i^+\sigma_j^-|JM\rangle$ (for $i \neq j$) is independent of i and j. Show that

$$\langle JM|\sigma_i^+\sigma_j^-|JM\rangle = \frac{J^2 - M^2}{N(N-1)}.$$

k) The physical interpretation of this correlation is not so easy to see. To make it more concrete, we can ask what is the probability of measuring some property of atom j given that we know the result of some measurement on atom i. Imagine that we measure σ_x on atom i (if it was a real spin, we know how to do that. Never mind how to do it on an atom, just assume that it can be done). What is the probability that we will get the same result if we then measure σ_x on atom j? Give the answer as a function of J and M. What is the maximal value for having the same result on measuring both atoms and for what J and M does it occur?

A second approach to study the correlations between atoms in the state $|JM\rangle$ is to study the entanglement beween different subsystems. In this part we will limit ourselves to the case $N = 4$.

l) Find the reduced density matrix for the first atom for all possible values of M. Calculate the entanglement entropy between the first atom and the rest of the system.

m) For $M = 0$, find the reduced density matrix of the first two atoms, and the entanglement entropy with the rest of the system.

The above argument follows closely the original ideas of Dicke (Phys. Rev. **93**, 99 (1954)) and is dependent on the fact that the coupling to the field is perfectly identical for all the atoms, and that there is no interaction between them. In reality, this may be difficult to achieve. The presence of interactions between the atoms means that the atomic eigenstates are no longer eigenstates of the Hamiltonian with interactions included. This implies that we have to take into account the internal dynamis of the atoms during the emission process. One approach to this is to use the master equation in the Lindblad form. To get any real benefit from this method requires more work than we have time for during the exam, so we will limit ourselves to a simple situation without interactions.

We study two identical atoms coupled identically to the electromagnetic field as in question c). The Hamiltonian is then

$$H = -\frac{1}{2}\omega_0(\sigma_1^z + \sigma_2^z).$$

The interaction with the field gives rise to a Lindblad equation of the usual form

$$\frac{d\rho}{dt} = -i[H,\rho] - \frac{\gamma}{2}(D^+D^-\rho + \rho D^+D^- - 2D^-\rho D^+)$$

n) Determine all stationary states (states with $\frac{d\rho}{dt} = 0$) of the system.

o) Use the result of the preceeding question to confirm the result in question c) that the initial state $|10\rangle$ will not decay to the ground state and find the final state at $t \to \infty$.

p) The conclusion that not all states will decay to the ground state is a consequence of the identity of the atomic couplings to the electromagnetic field. This means that it is impossible to determine which atom has emitted a given photon. A different situation arises if there is a way to discriminate the photons coming from the two atoms. One way to achieve this is to have independent environments for each atom. Write the Lindblad equation for two two-level atoms coupled to independent environments. A full derivation is not needed, only a reasonable justification. Use this equation to show that any initial state will decay to the ground state.

FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2022

Return of solutions:
The problem set is available from Friday morning, 30 September.
You may submit handwritten solutions, but they have to be scanned and included in one single file, which is submitted in Inspera before Friday, 7 October, at 14:00.

Language:
Solutions may be written in Norwegian or English depending on your preference.
Questions concerning the problems:
Please ask Joakim Bergli or Maria Markova.

Problem 1: Supersymmetric quantum mechanics

In this problem we will study a construction that is known as supersymmetric quantum mechanics (SUSYQM). For the harmonic oscillator, we can construct the operators $\hat{a}$ and $\hat{a}^\dagger$ that allows us to find the energy spectrum algabraically without explicitly solving the Schödinger equation. SUSYQM will allow us a similar construction for certain other potentials.

We study the motion of a particle in a potential $V_-(x)$ with the Hamiltonian in the position basis (we use units where $\hbar = 1$)

$$H_- = \frac{p^2}{2m} + V_-(x) = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V_-(x).$$

Assume that there exists a function $W(x)$ such that the operators

$$A = \frac{ip}{\sqrt{2m}} + W(x)$$
$$A^\dagger = -\frac{ip}{\sqrt{2m}} + W(x)$$

factorize the Hamiltonian so that $H_- = A^\dagger A$. The function $W(x)$ is known as the superpotential.

a) Show that $W(x)$ must satisfy the equation

$$W^2 - \frac{1}{\sqrt{2m}}\frac{dW}{dx} = V_-.$$

We define the partner Hamiltonian to H_- as

$$H_+ = AA^\dagger = \frac{p^2}{2m} + V_+,$$

where V_+ is called the partner potential to V_-. We now define an extended system with the Hamiltonian

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \tag{1}$$

and the two operators

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \qquad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}.$$

b) Show that we have the following supersymmetry algebra (which is identical to Eq. (1.233) in the lecture notes)

$$\begin{aligned} \left\{Q, Q^\dagger\right\} &= H \\ [Q, H] = \left[Q^\dagger, H\right] &= 0 \\ \{Q, Q\} = \left\{Q^\dagger, Q^\dagger\right\} &= 0. \end{aligned}$$

For this reason we say that the extended system is supersymmetric.

c) Explain what is the difference between the extended system described by the Hamiltonian (1) and a system of two particles, one in the potential V_- and one on the potential V_+.

d) Show that the ground state energy of H is nonnegative. That is, if $H|\Psi_0\rangle = E_0|\Psi_0\rangle$ then $E_0 \geq 0$.

If $E_0 = 0$ we say that SUSY is unbroken. If $E_0 > 0$ we have broken SUSY. The spectra of the partner Hamiltonians are related.

e) Show that we have the intertwinig relations

$$AH_- = H_+A, \qquad A^\dagger H_+ = H_-A^\dagger.$$

f) Show that this implies that if we have an eigenstate $|\psi_n^-\rangle$ of H_- with eigenvalue E_n^- then $A|\psi_n^-\rangle$ is an eigenstate of H_+ with the same eigenvalue.

The partner Hamiltonians H_- and H_+ therefore have the same eigenvalues and are called essentially isospectral. The only exception to this is the ground state of one of the Hamiltonians for a system with unbroken SUSY. Let us assume that this is the case for H_-, in which case we can choose the ground state of H on the form

$$|\Psi_0\rangle = \begin{pmatrix} |\psi_0^-\rangle \\ 0 \end{pmatrix}$$

g) Show that if we are to have unbroken SYSY with this ground state we must have $A|\psi_0^-\rangle = 0$ so that there is no corresponding eigenstate of H_+.

h) Show that we have the relation

$$|\psi_n^+\rangle = \frac{A}{\sqrt{E_n^+}} |\psi_{n+1}^-\rangle$$

between the eigenstates of H_+ and H_- with the eigenvalues related by

$$E_{n-1}^+ = E_n^- .$$

i) Show that the ground state wave function is then determined by the equation

$$\psi_0^-(x) = \langle x|\psi_0^-\rangle = N e^{-\sqrt{2m}\int_0^x W(x')dx'}.$$

We will now study the superpotential

$$W(x) = \begin{cases} \infty & x < 0 \\ -\frac{b}{\tan x} & 0 \le x \le \pi \\ -\infty & x > \pi \end{cases} \quad (2)$$

j) Show that for a specific choice of b, the partner potentials take the form

$$V_- = -\frac{1}{2m}$$
$$V_+ = -\frac{1}{2m} + \frac{1}{m \sin^2 x}$$

on the interval $0 \le x \le \pi$ with both potentials being ∞ outside this interval. Specify the value of b that gives these potentials.

k) The potential V_- corresponds to the well known case of a particle in an infinite square well. Either solve this problem or write down the solution (eigenstates and eigenvalues) from your favorite textbook.

l) Use the relations between V_- and V_+ derived above to determine the eigenvalues and eigenstates of H_+.

The above calculation gives an example where we could determine exact analytic expressions for the eigenstates and eigenvalues of the potential V_+ from the fact that we knew the corresponding results for the simpler potential V_-. In some cases we can do even better and determine the full spectrum and eigenstates directly. Assume that we have a family of partner potentials $V_\pm(a_o, x)$ that depends on some real parameter a_0. This family is called shape invariant if there exist real functions $a_1 = f(a_0)$ and $g(a_0)$ such that

$$V_+(a_0, x) + g(a_0) = V_-(a_1, x) + g(a_1)$$

with the corresponding relation

$$H_+(a_0, x) + g(a_0) = H_-(a_1, x) + g(a_1)$$

for the Hamiltonians.

m) Show that for a shape invariant superpotential, the Hamiltonians $H_+(a_0, x)$ and $H_-(a_1, x)$ have the same eigenstates, and that the energy eigenvalues are related by

$$E_n^+(a_0) = E_n^-(a_1) + g(a_1) - g(a_0).$$

n) Assume that SUSY is unbroken for all a_n generated recursively from a_0 by $a_{n+1} = f(a_n)$ and that $E_0^-(a_n) = 0$. Show that the energy spectrum of $H_-(a_0)$ is given by

$$E_n^-(a_0) = g(a_n) - g(a_0). \tag{3}$$

o) Show that the corrsponding energy eigenstates are

$$|\psi_n^-(a_0)\rangle = \frac{A^\dagger(a_0)}{\sqrt{E_{n-1}^+(a_0)}} \cdots \frac{A^\dagger(a_{n-2})}{\sqrt{E_1^+(a_{n-2})}} \frac{A^\dagger(a_{n-1})}{\sqrt{E_0^+(a_{n-1})}} |\psi_0^-(a_n)\rangle \tag{4}$$

We now return to the superpotential (2) that we studied above. To simplify the expressions, we will choose the mass so that $\sqrt{2m} = 1$.

p) Show that we have

$$V_+(b, x) = V_-(b + 1, x) + (b + 1)^2 - b^2.$$

Use this to show that we can generate a family of shape invariant potentials by choosing the initial parameter value a_0 to be any b, and determine the associated functions $f(b)$ and $g(b)$.

q) Choosing the starting value $a_0 = 1$ corresponds to the infinite square well. Using the relation (3), find the energy eigenvalues of this system. Use (4) to find the wave functions for the two lowest energy eigenstates. Admittedly, this is a very complicated way to calculate something that we know in advance how to derive in a simple way.

FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2023

Return of solutions:
The problem set is available from Friday morning, 29 September.
You may submit handwritten solutions, but they have to be scanned and included in one single file, which is submitted in Inspera before Friday, 6 October, at 14:00.

Language:
Solutions may be written in Norwegian or English depending on your preference
Questions concerning the problems:
Please ask Joakim Bergli or Maria Markova.

To aid you in solving the problems, you may consult any material that you can find on the topic, but as always you should cite the sources you use.

Problem 1: Quantum error correction

In this problem we will study some basic properties of quantum error correction. To understand the principles, and also the special features of quantum codes it is useful first to get a basic understanding of classical error correction codes. We therefore first consider a classical 3-bit repetition code. Imagine that we are to transmit a message consisting of a string of classical bits. There is a certain (hopefully small) probability p that a bit is changed in transfer and received as the opposite of what it was originally (for simplicity we consider symmetric communication channels, so the probability of a bit flip is the same for both 0 and 1). A simple way to detect and correct such errors is to repeat the same bit, replacing it by a number of identical copies.

a) The simplest classical error correction code is the 3-bit repetition code where each bit is encoded by three identical copies, so that $0 \to 000$ and $1 \to 111$. Explain that using this code one can correct 1 bit-flip error in each triplet. What happens if there are 2 bit-flip errors? What is the probability for this to happen?

In trying to construct similar codes for quantum systems, we face three challenges:

- The no-cloning theorem prevents us from making copies of the original state
- We can not measure the state of the system without collapsing the wavefunction, which means that information encoded in superpositions will be lost
- There are additional errors types possible for a quantum system as compared to a classical.

To illustrate the last point, consider transmitting a qubit which during transfer is changed by the unitary operation

$$U_x(\theta) = e^{i\theta\sigma_x} = \cos\theta I + i \sin\theta \sigma_x \tag{1}$$

where I is the identity operator. if $\theta = \pi/2$ this corresponds to a classical bit flip. Other values of θ are errors with no classical analog.

Nevertheless, it is possible to construct quantum error correction codes, and we will study a simple example. We consider first a 3-qubit repetition code, where each logical state consists of three physical qubits

$$|0\rangle_b = |000\rangle \qquad |1\rangle_b = |111\rangle.$$

The subscipt b on the states $|\cdot\rangle_b$ expresses the fact that this code will correct bit-flip errors. When encoding a general state

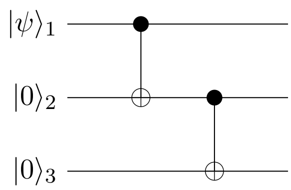
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

we prepare the three qubits in the state

$$|\psi\rangle_b = \alpha|000\rangle + \beta|111\rangle.$$

b) Explain why this encoding process does not violate the no-cloning theorem.

c) Show that the following quantum circuit will implement the encoding if the original state $|\psi\rangle_1$ initially is stored in qubit 1.

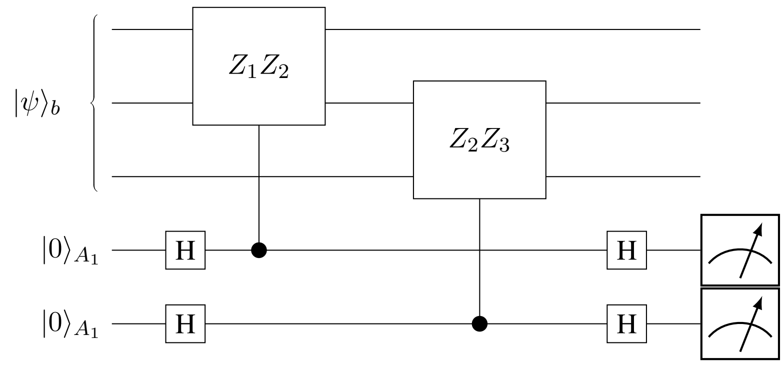


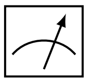
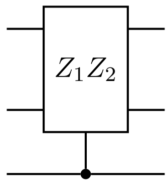
We study first errors that correspond to classical bit-flip errors. These are described by the unitary operators X_i, flipping qubit i. That is, $X_i = \sigma_x^{(i)}$, the Pauli matrix σ_x acting on qubit i. Similarly, we write $Y_i = \sigma_y^{(i)}$ and $Z_i = \sigma_z^{(i)}$. The two operators Z_1Z_2 and Z_2Z_3 are called the stabilizers of the code, and the two logical states $|0\rangle_b$ and $|1\rangle_b$ are eigenstates of both stabilizers with eigenvalue 1.

d) Consider all states that are reached from the two logical states by a single bit flip on one of the qubits. Check that they are all eigenstates of the stabilizers and find the corresponding eigenvalues. Explain how we can use the measurement of the stabilizers to determine an operation that will correct the error. Explain why this will still work if the initial state is a superposition of the two logical states.

e) Explain the difference between measuring the stabilizer Z_1Z_2 and measuring the operators Z_1 and Z_2 separately. What information about the state do we get in the two cases?

f) Measuring the stabilizers directly is not always possible, as it involves a joint measurement on two qubits. Instead, one can transfer the information to ancillas (additional qubits) and measure their state instead. Show that the following circuit is equivalent to measuring the two stabilizers.



 indicates that the qubit is measured in the $\{|0\rangle, |1\rangle\}$-basis and 

means that the operation Z_1Z_2 is executed on the first two qubits if the third qubit is in state $|1\rangle$ while no operation is done if the third qubit is in state $|0\rangle$ (controlled Z_1Z_2 operation).

So far, we considered only spin-flip errors on the individual qubits, but we know that in contrast to classical bits, qubits can have arbitrary rotations of the state, as described by the unitary operation $U_x(\theta)$ given in Eq. (1).

g) Study the what happens when measuring the stabilizers after such an error has occurred. Explain why the same code as before also can correct errors of this type.

Qubits can also experience phase errors which do not have any classical analog. These are the effect of the operators

$$U_z(\phi) = e^{i\phi\sigma_z} = \cos\phi I + i\sin\phi\sigma_z$$

acting on the qubit.

h) Investigate what happens if a phase error affects one of the qubits in the 3-qubit bit-flip repetition code that we have studied. Show that this code will not be able to handle phase errors.

We can construct a code that detects and corrects phase errors in the same way as the 3-qubit bit flip code corrects bit flip errors. This is not difficult once we realize that the operator Z is acting like a bit flip between the eigenstates of X, so we can use the X eigenstates as the basis states in the same way as we used the Z eigenstates for the bit flip code. We denote the X eigenstates as

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

In the 3-qubit phase flip code the logical states are then

$$|0\rangle_p = |+++\rangle \qquad |1\rangle_p = |---\rangle$$

and the stabilizers are X_1X_2 and X_2X_3.

Protecting against both bit flip and phase flip errors at the same time is then achieved by what is called concatenation of the two codes. Three logical qubits in the bit flip code are then used in the phase flip code to get a 9-qubit code with the logical states

$$|0\rangle_L = |+++\rangle_b \qquad |1\rangle_L = |---\rangle_b$$

where

$$|+\rangle_b = \frac{1}{\sqrt{2}}(|0\rangle_b + |1\rangle_b), \qquad |-\rangle_b = \frac{1}{\sqrt{2}}(|0\rangle_b - |1\rangle_b).$$

i) Explain why the stabilizers are the operators Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.

j) Show that all the stabilizers of the 9-qubit code commute so that they can be simultaneously measured.

k) Show that any error (unitary operation) on a single qubit can be written as

$$U = a_0I + a_1X + a_2XZ + a_3Z$$

with complex coefficients a_i. This means that the 9-qubit code that corrects bit-flip and phase-flip errors will correct any 1-qubit error.

Problem 2: Encoding a qbit in an oscillator

In this second problem we will derive some properties of a different type of quantum error correcting code: the Gottesman-Kitaev-Preskill code. This code was one of the first propositions for a so-called bosonic code: a scheme in which the logical qubit - a finite-dimensional quantum system - is encoded into an infinite-dimensional system such as a harmonic oscillator.

a) We first define the displacement operator $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$, where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators respectively such that $[\hat{a}, \hat{a}^\dagger] = 1$, and α is a complex number. Show that $\hat{D}(\beta)\hat{D}(\alpha) = e^{(\beta\alpha^* - \beta^*\alpha)/2}\hat{D}(\alpha + \beta) = e^{\beta\alpha^* - \beta^*\alpha}\hat{D}(\alpha)\hat{D}(\beta)$ (show both equalities).

The GKP code is designed to protect against small displacement errors. For this purpose, it encodes a qubit's states into a lattice of regular displacements from the origin in the phase space of a harmonic oscillator. This means that the if we denote the eigenstates of the position operator $\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$ by $|x\rangle$, the logical states are

$$|0\rangle_L = \sum_{j=-\infty}^{\infty} |2j\alpha\rangle,$$

$$|1\rangle_L = \sum_{j=-\infty}^{\infty} |(2j+1)\alpha\rangle$$

b) Show that for real α we have that

$$\hat{D}(\alpha)|x\rangle = |x+\alpha\rangle.$$

c) Show that for real α we have

$$\hat{D}(\alpha)|0\rangle_L = |1\rangle_L, \qquad \hat{D}(\alpha)|1\rangle_L = |0\rangle_L.$$

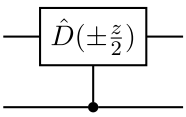
The above result means that the operator $\bar{X} = \hat{D}(\alpha)$ acts on the logical states in the same way as the usual Pauli operator σ_x. We define the logical Pauli operators $\bar{X} = \hat{D}(\alpha)$, $\bar{Z} = \hat{D}(\beta)$, and $\bar{Y} = i\bar{X}\bar{Z}$.

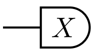
d) Check that if $\beta\alpha^* - \beta^*\alpha = i\pi$, these operators have the same commutation relations as the usual Pauli matrices.

e) If we take the logical states is the position representation, the wavefunctions consist of a set of equally spaced δ-functions, often called a Dirac comb. Show that if we instead use the momentum representation wavefunctions, they also have the same structure, with δ-functions separated by the distance 2β.

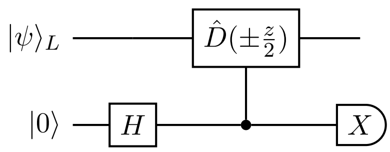
   It may be useful to know that

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{-2\pi nit/T}.$$

f) Phase estimation circuit: In the following circuit, the upper line represents the harmonic oscillator while the lower line is a qubit (two level system). The controlled displacement gate $C\hat{D}(\pm\frac{z}{2})$ represented by 

applies a displacement $\hat{D}(\pm\frac{z}{2})$ on the target oscillator depending on the value of the control qubit: $C\hat{D}(\pm\frac{z}{2}) = \hat{D}(z/2)\otimes|0_a\rangle\langle 0_a| + \hat{D}(-z/2)\otimes|1_a\rangle\langle 1_a|$, and 

represents a measurement in the X basis.



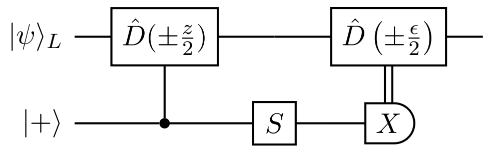
(2)

Show that the measurement outcome probabilities for this circuit are:

$$P(\pm) = \frac{1}{2}\left[1 \pm \frac{1}{2}\left(\langle\psi_L|\hat{D}(z)|\psi_L\rangle + \langle\psi_L|\hat{D}^\dagger(z)|\psi_L\rangle\right)\right] \tag{3}$$

.

The following circuit is the so-called "Sharpen" circuit used for the preparation of GKP states.



(4)

$S = \text{diag}(1, i)$ is the phase-shift gate. ϵ is a small parameter with $\arg(\epsilon) = \arg(z) - \pi/2$ such that $\hat{D}(\epsilon)$ is a small displacement in a direction orthogonal to $\hat{D}(z)$. The double line from the X-measurement symbol means that the operation $\hat{D}\left(\pm\frac{\epsilon}{2}\right)$ is made with the $\pm$-sign depending on the outcome of the measurement.

g) Show that the probabiliy of getting $\pm$ outcome on the ancilla qubit is $P(\pm) = \frac{1}{2}\left(1 \pm \text{Im}\left(\langle\psi_L|\hat{D}(z)|\psi_L\rangle\right)\right)$. Show that if the input state $|\psi\rangle_L$ is an eigenstate of $\hat{D}(z)$ with eigenvalue $e^{i\theta}$, then for a $\pm$ outcome, the output is an eigenstate with eigenvalue $e^{i(\theta \mp z\epsilon)}$. Explain how repeated use of this circuit will result in the state approaching an eigenstate with eigenvalue 1.