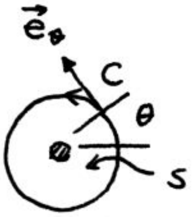
FYS 4110

Midterm exam Oct 2004, Solutions

Problem 1 Particle encircling a magnetic flux

a) Stoke's theorem



$$\phi = \int_S \vec{B}\cdot d\vec{S} = \int_S (\nabla\times\vec{A})\cdot d\vec{S}$$

$$= \oint_C \vec{A}\cdot d\vec{s} = R\int_0^{2\pi} A_\theta \, d\theta$$

Rotational invariance $A_\theta \equiv A$ indep of θ

$$\phi = 2\pi R A \qquad A = \frac{\phi}{2\pi R} \qquad \underline{\vec{A} = \frac{\phi}{2\pi R}\vec{e}_\theta}$$

Outside the solenoid:

$$\vec{B} = \nabla\times\vec{A} = 0$$

$\vec{E} = \vec{B} = 0 \Rightarrow$ no force on the particle

class. eq. of motion not affected by ϕ

b) Momentum operator for particle on the circle:

$$\vec{p} = -i\hbar\nabla \to \cancel{-i\frac{\hbar}{R}\frac{\partial}{\partial\theta}} \; -i\frac{\hbar}{R}\vec{e}_\theta\frac{\partial}{\partial\theta}$$

when acting on wave functions $\psi(\theta)$

$$H = \frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2 \to \frac{1}{2m}\left(-i\frac{\hbar}{R}\frac{\partial}{\partial\theta} - \frac{e}{c}A\right)^2$$

$$= -\frac{\hbar^2}{2mR^2}\left(\frac{\partial}{\partial\theta} - i\frac{e\phi}{2\pi\hbar c}\right)^2$$

$$= \underline{-\frac{\hbar^2}{2mR^2}\left(\frac{\partial}{\partial\theta} - i\varkappa\right)^2} \qquad \varkappa \equiv \frac{e\phi}{2\pi\hbar c}$$

↗

angular mom. eigenstates

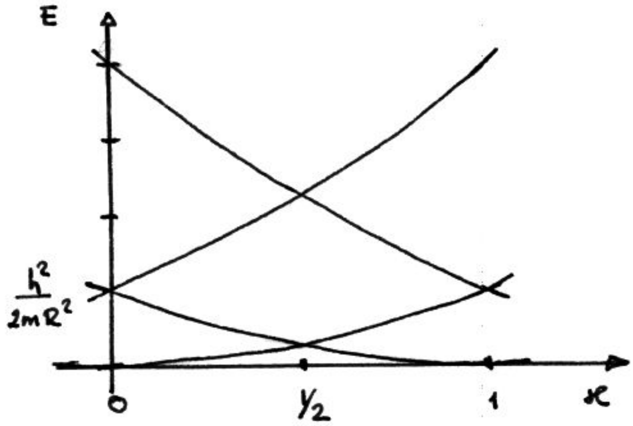
= energy eigenstates

Angular momentum eigenstates

$$-i\hbar\frac{\partial}{\partial\theta}\psi = l\hbar\psi \quad\Rightarrow\quad \psi_l(\theta) = \frac{1}{\sqrt{2\pi}}e^{il\theta}$$

$$H\psi_l = -\frac{\hbar^2}{2mR^2}(il - i\varkappa)^2\psi_l$$

$$\Rightarrow \quad \underline{E_l = \frac{\hbar^2}{2mR^2}(l-\varkappa)^2}$$



Periodic variation :

$\varkappa \to \varkappa + 1$ and $l \to l+1$ leaves the energy unchanged

$\Rightarrow$ the set of energies $\{E_l,\ l = 0, \pm 1, \dots\}$ is invariant

when $\varkappa \to \varkappa + 1$

Expressed in terms of the flux :

$$\frac{e\phi}{2\pi\hbar c} \to \frac{e\phi}{2\pi\hbar c} + 1 \Rightarrow \quad \phi \to \phi + \frac{2\pi\hbar c}{e} \quad , \quad \underline{\phi_0 = \frac{2\pi\hbar c}{e}} \text{ flux quantum}$$

Angular momentum of ground state :

$0 \leq \phi < \frac{\phi_0}{2}$: $l = 0$

$\frac{\phi_0}{2} < \phi \leq \phi_0$: $l = 1$

$\phi = \frac{\phi_0}{2}$: Spectrum is doubly degenerate

Ground state : $l = 0$ and $l = 1$ same energy.

c) Probability current

$$J = -\frac{i\hbar}{2mR}\left(\psi^*\partial_\theta\psi - \psi\partial_\theta\psi^*\right) - \frac{e\phi}{2\pi R mc}\psi^*\psi \qquad \partial_\theta = \frac{\partial}{\partial\theta}$$

in ang. mom state l ; $\psi = \psi_l \quad \partial_\theta\psi_l = il\,\psi_l$

$$\Rightarrow \quad J_l = \left(\frac{\hbar}{mR}\,l - \frac{e\phi}{2\pi R mc}\right)\frac{1}{2\pi}$$

$$= \frac{\hbar}{2\pi m R}\left(l - \varkappa\right)$$

Ground state

$$0 \leq \phi < \frac{\phi_0}{2} \quad (l = 0) \qquad \underline{J_0 = -\frac{\varkappa\hbar}{2\pi m R}}$$

$$\frac{\phi_0}{2} < \phi \leq \phi_0 \quad (l = 1) \qquad \underline{J_1 = \frac{(1-\varkappa)\hbar}{2\pi m R}}$$

Maximum value $(\phi = \frac{\phi_0}{2})$:

$$\underline{J_0 = -\frac{\hbar}{4\pi m R}} \qquad \underline{J_1 = \frac{\hbar}{4\pi m R}}$$

two possible values due to degeneracy

Velocity : $J = \rho v \qquad \rho = \psi^*\psi = \frac{1}{2\pi}$

$$\Rightarrow \quad v_0 = 2\pi J_0 = \underline{-\frac{\hbar}{2mR}} \qquad (l = 0)$$

$$v_1 = 2\pi J_1 = \underline{\frac{\hbar}{2mR}} \qquad (l = 1)$$

d) Propagator

$$G(\theta, t; 0, 0) = \langle\theta|\, e^{-\frac{i}{\hbar}Ht}\,|0\rangle$$

$$= \sum_l \langle\theta|\, e^{-\frac{i}{\hbar}Ht}\,|l\rangle\langle l|0\rangle$$

$$= \sum_l e^{-\frac{i}{\hbar}E_l t}\,\langle\theta|l\rangle\langle l|0\rangle$$

$$\langle \theta | \ell \rangle = \psi_\ell(\theta) = \frac{1}{\sqrt{2\pi}} e^{i\ell\theta}$$

$$\langle \ell | 0 \rangle = \psi_\ell(0)^* = \frac{1}{\sqrt{2\pi}}$$

$$G(\theta t; 0,0) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{+\infty} \exp\Big\{ \underbrace{-\frac{i}{\hbar} \frac{\hbar^2}{2mR^2} (\ell - \varkappa)^2 t + i\ell\theta}_{} \Big\}$$

$$= -i \frac{\hbar}{2mR^2} (\ell^2 - 2\ell\varkappa + \varkappa^2) t + i\ell\theta$$

$$= i \Big\{ -\frac{\hbar t}{2mR^2} \ell^2 + \Big(\theta + \frac{\varkappa \hbar t}{mR^2}\Big) \ell \Big\} - i \frac{\hbar t \varkappa^2}{2mR^2}$$

$$\equiv i \{ \pi w \ell^2 + 2z\ell \} - i \frac{\hbar t \varkappa^2}{2mR^2}$$

$$G(\theta t, 0, 0) = \frac{1}{2\pi} \exp\Big\{ -i \frac{\varkappa^2 \hbar}{2mR^2} t \Big\} \sum_{\ell=-\infty}^{+\infty} \exp\{ i [\pi w \ell^2 + 2z\ell] \}$$

$$= \frac{1}{2\pi} \exp\Big\{ -i \frac{\varkappa^2 \hbar}{2mR^2} t \Big\} \vartheta_3(z, w)$$

$$= \underline{\frac{1}{2\pi} \exp\Big\{ -i \frac{\varkappa^2 \hbar}{2mR^2} t \Big\} \vartheta_3\Big(\frac{1}{2}\Big(\theta + \frac{\varkappa \hbar t}{mR^2}\Big), -\frac{\hbar t}{2\pi m R^2} \Big)}$$

e) Classical paths

$$\theta(t') = \frac{\theta + 2\pi n}{t} t' \qquad n = 0, \pm 1, \ldots$$

$$\dot\theta(t') = \frac{\theta + 2\pi n}{t} \qquad (\text{const})$$

Action

$$S = \frac{1}{2} m v^2 t + \frac{e}{c} A v t \qquad \frac{e}{c} A v = \frac{e}{c} \frac{\Phi}{2\pi R} R \dot\theta$$

$$= \underbrace{\frac{e\Phi}{2\pi c}}_{= \hbar\varkappa} \dot\theta$$

$$S_n = \frac{1}{2} m R^2 \frac{(\theta + 2\pi n)^2}{t} + \frac{e\Phi}{2\pi c} (\theta + 2\pi n)$$

$$= \frac{1}{2} m R^2 \frac{4\pi^2}{t} n^2 + \frac{1}{2} m R^2 \frac{4\pi\theta}{t} n + 2\pi\hbar\varkappa n$$

$$+ \frac{1}{2} m R^2 \frac{\theta^2}{t} + \frac{e\Phi}{2\pi c} \theta$$

$$S_n = 2\pi^2 \frac{mR^2}{t} n^2 + \frac{2\pi\, mR^2}{t}\left(\theta + \frac{\varkappa\hbar t}{mR^2}\right) n + \frac{1}{2}\frac{mR^2}{t}\left(\theta^2 + \cancel{2}\frac{\varkappa\hbar t}{\cancel{2}mR^2}\theta\right)$$

define $\bar\theta = \theta + \frac{\varkappa\hbar t}{mR^2}$

$$S_n = 2\pi^2 \frac{mR^2}{t} n^2 + \frac{2\pi mR^2}{t}\bar\theta n + \frac{1}{2}\frac{mR^2}{t}\bar\theta^2 - \frac{1}{2}\frac{\varkappa^2\hbar^2}{mR^2} t$$

Path integral representation

$$G(\theta t, 00) = N \sum_{n=-\infty}^{+\infty} \exp\left\{\frac{i}{\hbar} S_n\right\} \qquad N = \sqrt{\frac{mR^2}{2\pi i\hbar t}} \quad \text{(prob. 2.41}$$

$$= N \exp\left\{\frac{i}{2}\left(\frac{mR^2}{\hbar t}\bar\theta^2 - \frac{\varkappa^2\hbar}{mR^2}t\right)\right\}$$

$$\times \sum_{n=-\infty}^{+\infty} \exp\Big\{ i \underbrace{\left(2\pi^2 \frac{mR^2}{\hbar t} n^2 + \frac{2\pi mR^2}{\hbar t}\bar\theta n\right)}_{\equiv\ \pi\omega' n^2 + 2z' n} \Big\}$$

$$= N \exp\left\{\frac{i}{2}\left(\frac{mR^2}{\hbar t}\bar\theta^2 - \frac{\varkappa^2\hbar}{mR^2}t\right)\right\} \vartheta_3(z', \omega')$$

$$= \text{——} \quad '' \quad \text{——} \quad \vartheta_3\left(\frac{\pi mR^2}{\hbar t}\bar\theta,\ 2\pi\frac{mR^2}{\hbar t}\right)$$

$$= \sqrt{\frac{mR^2}{2\pi i\hbar t}} \exp\left\{\frac{i}{2}\left(\frac{mR^2}{\hbar t}\bar\theta^2 - \frac{\varkappa^2\hbar}{mR^2}t\right)\right\} \vartheta_3\left(\frac{\pi mR^2}{\hbar t}\bar\theta,\ 2\pi\frac{mR^2}{\hbar t}\right)$$

Apply relation

$$\vartheta_3(z', \varpi') = (-i\varpi')^{-1/2} e^{z'^2/i\pi\varpi'} \vartheta_3\left(\frac{z'}{\varpi'}, -\frac{1}{\varpi'}\right)$$

$$\frac{z'}{\varpi'} = \frac{\pi mR^2}{\hbar t}\bar\theta \,\frac{\hbar t}{2\pi mR^2} = \frac{1}{2}\bar\theta$$

$$-\frac{1}{\varpi'} = -\frac{\hbar t}{2\pi mR^2}$$

Inserted :

$$G(\theta t, 0 0) = \sqrt{\frac{mR^2}{2\pi i\hbar t}} \sqrt{\frac{i\hbar t}{2\pi mR^2}} \exp\left\{ \frac{i}{2} \left(\frac{mR^2}{\hbar t}\bar{\theta}^2 - \frac{\varkappa^2\hbar}{mR^2} \right) t \right\}$$

$$\times \exp\left\{ -\frac{i}{2} \frac{mR^2}{\hbar t} \bar{\theta}^2 \right\} \vartheta_3\left(\tfrac{1}{2}\bar{\theta}, -\frac{\hbar t}{2\pi mR^2} \right)$$

$$= \underline{\frac{1}{2\pi} \exp\left\{ -i \frac{\varkappa\hbar}{2mR^2} t \right\} \vartheta_3\left(\tfrac{1}{2}\left(\theta + \frac{\varkappa\hbar t}{mR^2}\right), -\frac{\hbar t}{2\pi mR^2} \right)}$$

same as obtained by direct calculation.

Problem 2 Entangled photons

a) $\frac{n_1}{N}$ approach P_1 for larg N

$\frac{n_2}{N}$ — " — P_2 — " —

$\frac{n_{12}}{N}$ — " — P_{12} — " —

b) Density operator

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2} \Big(\underbrace{|HV\rangle\langle HV| + |VH\rangle\langle VH|}_{} \overset{|H\rangle_1 \otimes |V\rangle_2}{}$$

$$+ \underline{e^{i\chi} |VH\rangle\langle HV| + e^{-i\chi} |HV\rangle\langle VH|} \Big\}$$

$$\rho_1 = \mathrm{Tr}_2\, \rho = {}_2\langle H| \rho |H\rangle_2 + {}_2\langle V| \rho |V\rangle_2$$

$$= \frac{1}{2} \left(|H\rangle\langle H| + |V\rangle\langle V| \right)_1$$

$$= \underline{\frac{1}{2} \mathbb{1}_1}$$

$$\rho_2 = \mathrm{Tr}_1\, \rho = \underline{\frac{1}{2} \mathbb{1}_2}$$

c) $\chi = \pi$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|HV\rangle - |VH\rangle \right)$$

$$\rho = \frac{1}{2}\left(|HV\rangle\langle HV| + |VH\rangle\langle VH| - |VH\rangle\langle HV| - |HV\rangle\langle VH| \right)$$

$$P_1 = Tr(\rho P_1) = Tr_1(\rho_1 P_1) = \frac{1}{2} Tr P_1 = \frac{1}{2}\langle \theta_1 | \theta_1 \rangle = \underline{\frac{1}{2}}$$

$$P_2 = Tr_2(\rho_2 P_2) = \underline{\frac{1}{2}}$$

$$P_{12} = Tr(\rho P_{12}) = Tr(\rho\, |\theta_1 \theta_2\rangle\langle \theta_1 \theta_2|)$$

$$= \langle \theta_1 \theta_2 | \rho | \theta_1 \theta_2 \rangle$$

$$= \frac{1}{2}\left\{ \cos^2\theta_1 \sin^2\theta_2 + \sin^2\theta_1 \cos^2\theta_2 - 2\cos\theta_1 \sin\theta_1 \cos\theta_2 \sin\theta_2 \right\}$$

$$= \frac{1}{2}\left(\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \right)^2$$

$$= \underline{\frac{1}{2} \sin^2(\theta_1 - \theta_2)}$$

$$P_{12} = \langle P_1 P_2 \rangle = \frac{1}{2}\sin^2(\theta_1 - \theta_2)$$

$$\langle P_1 \rangle \langle P_2 \rangle = \frac{1}{4}$$

$P_{12} \neq \langle P_1 \rangle \langle P_2 \rangle$ unless $\sin(\theta_1 - \theta_2) = \frac{1}{\sqrt{2}}$

shows correlations

d) $\chi = 0$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|HV\rangle + |VH\rangle \right)$$

$$\rho = \frac{1}{2}\left(|HV\rangle\langle HV| + |VH\rangle\langle VH| + |HV\rangle\langle VH| + |VH\rangle\langle HV| \right)$$

$P_1 = Tr_1(\rho_1 P_1) = \underline{\frac{1}{2}}$ $\quad P_2 = Tr_2(\rho_2 P_2) = \underline{\frac{1}{2}}$ as before

$$P_{12} = \langle \theta_1\theta_2 | \rho | \theta_1\theta_2 \rangle$$

$$= \tfrac{1}{2} \left(\cos^2\theta_1 \sin^2\theta_2 + \sin^2\theta_1 \cos^2\theta_2 + 2\cos\theta_1 \sin\theta_1 \cos\theta_2 \sin\theta_2 \right)$$

$$= \tfrac{1}{2} \left(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2 \right)^2$$

$$= \underline{\tfrac{1}{2} \sin^2(\theta_1+\theta_2)}$$

c) is rotationally invariant:

when $\theta_1 \to \theta_1 + \alpha$, $\theta_2 \to \theta_2 + \alpha$

d) is not, but invariant when $\theta_1 \to \theta_1 + \alpha$, $\theta_2 \to \theta_2 - \alpha$

e) $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|HV\rangle + i\, |VH\rangle \right)$

$$\rho = \tfrac{1}{2} \left(|HV\rangle\langle HV| + |VH\rangle\langle VH| + i \left(|VH\rangle\langle HV| - |HV\rangle\langle VH| \right) \right)$$

$\underline{P_1 = \tfrac{1}{2}}$, $\underline{P_2 = \tfrac{1}{2}}$ as before

$$P_{12} = \langle \theta_1\theta_2 | \rho | \theta_1\theta_2 \rangle$$

$$= \tfrac{1}{2} \left(\cos^2\theta_1 \sin^2\theta_2 + \sin^2\theta_1 \cos^2\theta_2 \right)$$

$$= \underline{\tfrac{1}{4} \left(\sin^2(\theta_1-\theta_2) + \sin^2(\theta_1+\theta_2) \right)}$$

No contributions from mixed terms $|VH\rangle\langle HV|$, $|HV\rangle\langle VH|$,

Same result as with

$$\rho = \underline{\tfrac{1}{2} \left(|HV\rangle\langle HV| + |VH\rangle\langle VH| \right)}$$

incoherent mixture (mixed state) of $|HV\rangle$ and $|VH\rangle$

f) Bell inequality

$$F(0,\theta,2\theta) = P_{12}(\theta,2\theta) - |P_{12}(0,\theta) - P_{12}(0,2\theta)|$$

case I :

$$P_{12}(\theta_1,\theta_2) = \tfrac{1}{2}\sin^2(\theta_1-\theta_2)$$

$$F_I(0,\theta,2\theta) = \tfrac{1}{2}\{\sin^2\theta - |\sin^2\theta - \sin^2 2\theta|\}$$

case II

$$P_{12}(\theta_1,\theta_2) = \tfrac{1}{2}\sin^2(\theta_1+\theta_2)$$

$$F_{II}(0,\theta,2\theta) = \tfrac{1}{2}\{\sin^2 3\theta - |\sin^2\theta - \sin^2 2\theta|\}$$

case III

$$P_{12}(\theta_1,\theta_2) = \tfrac{1}{4}(\sin^2(\theta_1+\theta_2) + \sin^2(\theta_1-\theta_2))$$

$$F_{III}(0,\theta,3\theta) = \tfrac{1}{4}\{\sin^2 3\theta + \sin^2\theta - 2|\sin^2\theta - \sin^2 2\theta|$$

Plot shows

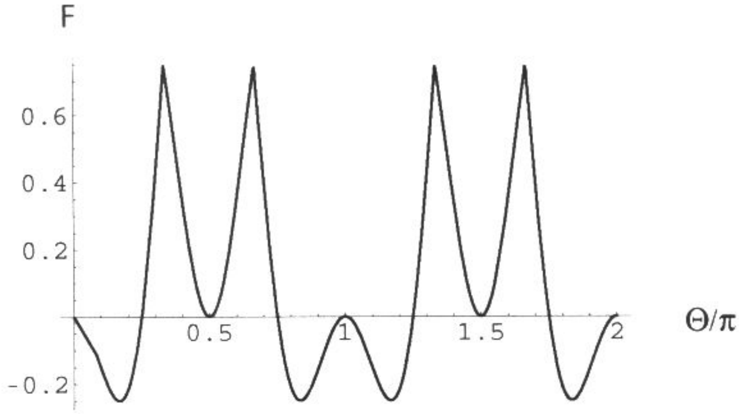
Condition $F(0,\theta,2\theta) \geq 0$

is not satisfied for I and II,

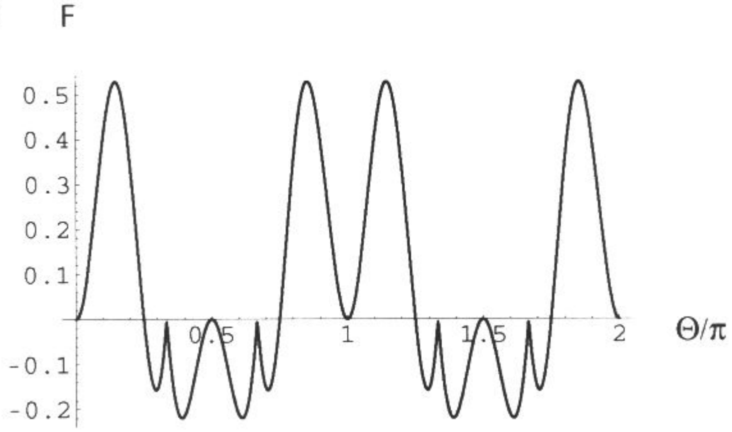
but is satisfied for III

F_{III} is the same as for the non-entangled mixed state $\frac{1}{2}(|HV\rangle\langle HV| + |VH\rangle\langle VH|)$, should <u>not</u> show breaking of the Bell inequality.

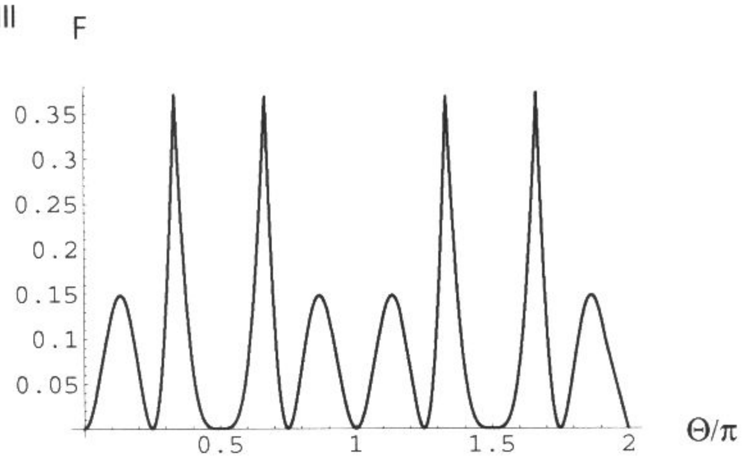
Case I



Case II



Case III



Midterm Exam 2005 Solutions

Problem 1 Spin motion in an oscillating field

a) Time evolution of density matrix

$$\rho(t) = U(t)\,\rho_0\,U^{\dagger}(t)$$

In magnetic field

$$H = -\vec{\mu}\cdot\vec{S} = -\frac{eB}{2mc}\hbar\,\sigma_z$$

$$= \frac{1}{2}\hbar\omega_0\sigma_z \qquad \omega_0 = -\frac{eB}{mc}$$

$$\Rightarrow U(t) = e^{-\frac{i}{\hbar}Ht} = e^{-\frac{i}{2}\omega_0\sigma_z t}$$

$$\vec{r}(t)\cdot\vec{\sigma} = \vec{r}_0\cdot U(t)\,\vec{\sigma}\,U^{\dagger}(t)$$

$$U\sigma_z U^{\dagger} = \sigma_z$$

$$U\sigma_x U^{\dagger} = e^{-\frac{i}{2}\omega_0\sigma_z t}\,\sigma_x\,e^{\frac{i}{2}\omega_0\sigma_z t}$$

$$= \sigma_x - \frac{i}{2}\omega_0 t\,[\sigma_z,\sigma_x] + \frac{1}{2!}\left(-\frac{i}{2}\omega_0 t\right)^2[\sigma_z,[\sigma_z,\sigma_x]] + \cdots$$

$$[\sigma_z,\sigma_x] = 2i\,\sigma_y$$

$$[\sigma_z,\sigma_y] = -2i\,\sigma_x$$

$$\Rightarrow U\sigma_x U^{\dagger} = \sigma_x + \omega_0 t\,\sigma_y - \frac{1}{2}(\omega_0 t)^2\sigma_x - \frac{1}{3!}(\omega_0 t)^3\sigma_y + \cdots$$

$$= \sigma_x\cos\omega_0 t + \sigma_y\sin\omega_0 t$$

$$U\sigma_y U^{\dagger} = \sigma_y - \omega_0 t\,\sigma_x - \frac{1}{2}(\omega_0 t)^2\sigma_y + \frac{1}{3!}(\omega_0 t)^3\sigma_y \cdots$$

$$= -\sigma_x\sin\omega_0 t + \sigma_y\cos\omega_0 t$$

$$\Rightarrow \vec{r}(t)\cdot\vec{\sigma} = (x_0\cos\omega_0 t - y_0\sin\omega_0 t)\,\sigma_x + (x_0\sin\omega_0 t + y_0\cos\omega_0 t)\,\sigma_y + \sigma_z z_0$$

$$\Rightarrow \underline{\vec{r}(t) = (x_0\cos\omega_0 t - y_0\sin\omega_0 t)\,\vec{i} + (x_0\sin\omega_0 t + y_0\cos\omega_0 t)\,\vec{j} + z_0\vec{k}}$$

$\vec{r}$ rotates with angular velocity ω_0 around the z-axis

b) Initial condition $\vec{r}_0 = a\vec{k}$

$$\Rightarrow \quad \rho_0 = \frac{1}{2}(\mathbb{1} + a\sigma_z)$$

$$= \frac{1}{2}\begin{pmatrix} 1+a & 0 \\ 0 & 1-a \end{pmatrix}$$

positivity : $\left.\begin{array}{l} 1+a \geq 0 \Rightarrow a \geq -1 \\ 1-a \geq 0 \Rightarrow a \leq 1 \end{array}\right\} -1 \leq a \leq 1$

Time evolution operator with oscillating field (sect. 1.3.2)

$$U(t) = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}$$

$$A = \left(\cos\frac{\Omega t}{2} - i\cos\theta \sin\frac{\Omega t}{2}\right) e^{-\frac{i}{2}\omega t}$$

$$B = -i\sin\theta \sin\frac{\Omega t}{2}\, e^{-\frac{i}{2}\omega t}$$

$$\cos\theta = \frac{\omega_0 - \omega}{\sqrt{(\omega_0-\omega)^2 + \omega_1^2}} \qquad \sin\theta = \frac{\omega_1}{\sqrt{(\omega_0-\omega)^2+\omega_1^2}}$$

$$\omega_1 = -\frac{eB_1}{mc} \qquad \Omega = \sqrt{(\omega_0-\omega)^2 + \omega_1^2}$$

Time evolution

$$\vec{r}(t)\cdot\vec{\sigma} = \vec{r}_0 \cdot U(t)\vec{\sigma}\, U^+(t)$$

$$= a\, U(t)\sigma_z U^+(t)$$

$$= a \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} A^* & -B \\ B^* & A \end{pmatrix}$$

$$= a \begin{pmatrix} |A|^2 - |B|^2 & -2AB \\ -2A^*B^* & -(|A|^2-|B|^2) \end{pmatrix}$$

$$z(t) = a(|A|^2 - |B|^2) = a\left(\cos^2\frac{\Omega t}{2} + \cos^2\theta \sin^2\frac{\Omega t}{2} - \sin^2\theta\sin^2\frac{\Omega t}{2}\right)$$

$$= \frac{a}{2}\left((1+\cos\Omega t) + (\cos^2\theta - \sin^2\theta)(1-\cos\Omega t)\right)$$

$$\underline{= a\left(\cos^2\theta + \sin^2\theta \cos\Omega t\right)}$$

$$x(t)-iy(t) = -2a\,AB$$

$$= 2a\left[\cos\theta\sin\theta\sin^2\frac{\Omega t}{2} + i\sin\theta\cos\frac{\Omega t}{2}\sin\frac{\Omega t}{2}\right]e^{-i\omega t}$$

$$= 2a\sin\theta\sin\frac{\Omega t}{2}\Big[\left(\cos\theta\sin\frac{\Omega t}{2}\cos\omega t + \cos\frac{\Omega t}{2}\sin\omega t\right)$$

$$+\, i\left(\cos\frac{\Omega t}{2}\cos\omega t - \cos\theta\sin\frac{\Omega t}{2}\sin\omega t\right)\Big]$$

$$\Rightarrow\quad x(t) = \underline{2a\sin\theta\sin\frac{\Omega t}{2}\left(\cos\frac{\Omega t}{2}\sin\omega t + \cos\theta\sin\frac{\Omega t}{2}\cos\omega t\right)}$$

$$y(t) = \underline{-2a\sin\theta\sin\frac{\Omega t}{2}\left(\cos\frac{\Omega t}{2}\cos\omega t - \cos\theta\sin\frac{\Omega t}{2}\sin\omega t\right)}$$

c) Resonance : $\omega = \omega_0$

$$\Rightarrow\quad \Omega = \omega_1\,,\quad \cos\theta = 0\,,\quad \sin\theta = 1$$

$$\Rightarrow\quad z(t) = a\cos\omega_1 t$$

$$x(t) = a\sin\omega_1 t\sin\omega t$$

$$y(t) = -a\sin\omega_1 t\cos\omega t$$

Oscillations in the z coordinate combined with rotation about the z-axis

Problem 2 Charged particle in a strong magnetic field

a) $$m\vec{a} = \frac{e}{c}\vec{v}\times\vec{B}$$

$$\Rightarrow \dot{\vec{v}} = \frac{eB}{mc}\vec{v}\times\vec{k} = \vec{\omega}\times\vec{v} \qquad \vec{\omega} = -\frac{eB}{mc}\vec{k}$$

$$\Rightarrow \dot{\vec{r}} = \vec{\omega}\times\vec{r} + \vec{C} \text{ (const.)}$$

$$\underline{\equiv \vec{\omega}\times(\vec{r}-\vec{r}_0)} \qquad \vec{C} = -\vec{\omega}\times\vec{r}_0$$

Circular motion with angular velocity about a point $\vec{r}_0$.

$$\frac{d}{dt}\left[m\vec{r}\times\vec{v}\right] = m\vec{r}\times\vec{a}$$

$$= \vec{r}\times\left(\frac{e}{c}\vec{v}\times\vec{B}\right)$$

$$= -\frac{e}{c}\,\vec{r}\cdot\vec{v}\,\vec{B} \qquad (\vec{r}\cdot\vec{B} = 0)$$

$$= \frac{d}{dt}\left(-\frac{e\vec{B}}{2c}r^2\right)$$

$$\Rightarrow \frac{d}{dt}L_{mek} = -\frac{d}{dt}\left(\frac{eB}{2c}r^2\right)$$ generally different from 0

conserved only when $r = \text{const}$ $(r_0 = 0)$

$$\frac{d}{dt}L = \underline{\frac{d}{dt}\left(L_m + \frac{eB}{2c}r^2\right) = 0}$$ always conserved

b) $$\vec{R} = \vec{r} + \frac{1}{\omega}\vec{k}\times\vec{v}$$

$$\Rightarrow \dot{\vec{R}} = \vec{v} + \frac{1}{\omega}\vec{k}\times\vec{a}$$

$$= \vec{v} + \frac{1}{m\omega}\frac{e}{c}\vec{k}\times(\vec{v}\times\vec{B})$$

$$= \vec{v} + \frac{1}{\omega}\frac{eB}{mc}\vec{v} \qquad \left(\vec{k}\cdot\vec{v} = 0,\ \omega = -\frac{eB}{mc}\right)$$

$$= \underline{0}$$

Circular orbits

$$\vec{v} = \vec{\omega} \times (\vec{r} - \vec{r}_0)$$

$$\vec{k} \times \vec{v} = \vec{k} \times (\vec{k} \times (\vec{r} - \vec{r}_0))\, \omega$$

$$= -\omega(\vec{r} - \vec{r}_0)$$

$$\Rightarrow \vec{R} = \vec{r} + \frac{1}{\omega}(\vec{k} \times \vec{v}) = \vec{r} - (\vec{r} - \vec{r}_0)) = \underline{\vec{r}_0}$$

$$\vec{\rho} = \frac{1}{\omega} \vec{k} \times \vec{v} = \underline{\vec{r}_0 - \vec{r}}$$

$\vec{R}$ = center of orbit

$\vec{\rho}$ = vector from particle to center of orbit.

c) $$m\vec{v} = \vec{p} - \frac{e}{c}\vec{A} = \vec{p} + \frac{e}{2c}\, \vec{r} \times \vec{B} = \vec{p} + \frac{eB}{2c}\, \vec{r} \times \vec{k}$$

$$\vec{R} = \vec{r} + \frac{1}{\omega} \vec{k} \times \vec{v}$$

$$= \vec{r} + \frac{1}{m\omega} \vec{k} \times (\vec{p} - \frac{e}{c}\vec{A})$$

$$= \vec{r} + \underbrace{\frac{1}{m\omega}\, \frac{eB}{2c}}_{-\frac{1}{2}}\, \underbrace{\vec{k} \times (\vec{r} \times \vec{k})}_{\vec{r}} + \frac{1}{m\omega} \vec{k} \times \vec{p}$$

$$= \underline{\frac{1}{2}\vec{r} + \frac{1}{m\omega} \vec{k} \times \vec{p}}$$

$$\hat{X} = \frac{1}{2}\hat{x} - \frac{1}{m\omega}\hat{p}_y \quad , \quad \hat{Y} = \frac{1}{2}\hat{y} + \frac{1}{m\omega}\hat{p}_x$$

$$[\hat{X}, \hat{Y}] = \frac{1}{2m\omega}\left([\hat{x}, \hat{p}_x] - [\hat{p}_y, \hat{y}]\right)$$

$$= i\frac{\hbar}{m\omega} = i\frac{\hbar c}{|eB|} = \underline{i\, \ell_B^2} \qquad \ell_B = \sqrt{\frac{\hbar c}{|eB|}}$$

$$\vec{\rho} = \vec{R} - \vec{r}$$

$$\Rightarrow \hat{\rho}_x = -\left(\frac{1}{2}\hat{x} + \frac{1}{m\omega}\hat{p}_y\right), \quad \hat{\rho}_y = -\left(\frac{1}{2}\hat{y} - \frac{1}{m\omega}\hat{p}_x\right)$$

$$\Rightarrow [\hat{\rho}_x, \hat{\rho}_y] = \frac{1}{2m\omega}\left(-[\hat{x}, \hat{p}_x] + [\hat{p}_y, \hat{y}]\right) = \underline{-i\ell_B^2}$$

$(\hat{X}, \hat{Y})$ commutes like phase space variables $(\hat{X}, \hat{P})$

with $\hat{Y} = \frac{1}{m\omega}\hat{P}$

d) $\hat{a} = \frac{1}{\sqrt{2}\ell_B}(\hat{X} + i\hat{Y})$, $\hat{b} = \frac{1}{\sqrt{2}\ell_B}(\hat{\rho}_x - i\hat{\rho}_y)$

$$\Rightarrow \quad [\hat{a}, \hat{a}^+] = \frac{1}{2\ell_B^2}(-i[\hat{X},\hat{Y}] + i[\hat{Y},\hat{X}]) = 1$$

$$[\hat{b}, \hat{b}^+] = \frac{1}{2\ell_B^2}(i[\hat{\rho}_x,\hat{\rho}_y] - i[\hat{\rho}_y,\hat{\rho}_x]) = 1$$

$$[\hat{a}, \hat{b}] = \frac{1}{2\ell_B^2}([\hat{X},\hat{\rho}_x] + [\hat{Y},\hat{\rho}_y] + i([\hat{Y},\hat{\rho}_x] - [\hat{X},\hat{\rho}_y]))$$

$$[\hat{X},\hat{\rho}_x] = [\hat{Y},\hat{\rho}_y] = 0$$

$$[\hat{Y},\hat{\rho}_x] = [\tfrac{1}{2}\hat{y} + \tfrac{1}{m\omega}\hat{p}_x , -(\tfrac{1}{2}\hat{x} + \tfrac{1}{m\omega}\hat{p}_y)]$$

$$= -\frac{1}{2m\omega}([\hat{y},\hat{p}_y] + [\hat{p}_x,\hat{x}]) = 0$$

$[\hat{X},\hat{\rho}_y] = 0$ similary

$$\Rightarrow \quad [\hat{a},\hat{b}] = 0$$

$[\hat{a},\hat{b}^+] = 0$ similar calculations

$(\hat{a},\hat{a}^+)$, $(b,\hat{b}^+)$ commut. relations as for two indep. harm. osc.

e) $H = \frac{1}{2}mv^2$, $\vec{v} = \vec{\omega} \times (\vec{r} - \vec{R})$ $= v^2 = \omega^2(\vec{r}-\vec{R})^2 = \omega^2\vec{\rho}^{\,2}$

$$\hat{H} = \frac{1}{2}m\omega^2(\hat{\rho}_x^2 + \hat{\rho}_y^2) \qquad \hat{\rho}_x = \frac{\ell_B}{\sqrt{2}}(b + b^+)$$

$$= \frac{1}{2}m\omega^2\ell_B^2(bb^+ + b^+b) \qquad \hat{\rho}_y = i\frac{\ell_B}{\sqrt{2}}(b - b^+)$$

$= \underline{\hbar\omega(b^+b + \frac{1}{2})}$ harm osc. spectrum $\underline{E_n = \hbar\omega(n + \frac{1}{2})}$

Note energy spectrum independent of m

$$L = (m\vec{r}\times\vec{v})_z + \frac{eB}{2c}r^2 \qquad \vec{r} = \vec{R}-\vec{\rho}\ ,\ \vec{v} = -\vec{\omega}\times\vec{\rho}$$

$$\vec{r}\times\vec{v} = -(\vec{R}-\vec{\rho})\times(\vec{\omega}\times\vec{\rho})$$

$$= \omega(\rho^2 - \vec{R}\cdot\vec{\rho})$$

$$r^2 = R^2 + \rho^2 - 2\vec{R}\cdot\vec{\rho}$$

$$\hat{L} = m\omega\left((\hat{\rho}^2 - \hat{\vec{R}}\cdot\hat{\vec{\rho}}) - \tfrac{1}{2}(\hat{R}^2 + \hat{\rho}^2 - 2\hat{\vec{R}}\cdot\hat{\vec{\rho}})\right)$$

$$= \tfrac{1}{2}m\omega(\hat{\rho}^2 - \hat{R}^2)$$

$$\underline{= \hbar(b^+b - a^+a)}$$

Ground state = lowest Landau level:

$n=0 \Rightarrow E_0 = \frac{1}{2}\hbar\omega$, no restriction on m

$|m\rangle = |m,0\rangle$ $\quad m = 0,1,2\ldots$ orthonorm. basis in the lowest Landau level

Coherent state

$$\hat{a}|z\rangle = z|z\rangle\ ,\quad \hat{b}|z\rangle = 0$$

$$\hat{x} = \hat{X} - \hat{\rho}_x = \frac{l_B}{\sqrt{2}}(\hat{a} + \hat{a}^+ + \hat{b} + \hat{b}^+)$$

$$\hat{y} = \hat{Y} - \hat{\rho}_y = i\frac{l_B}{\sqrt{2}}(-\hat{a} + \hat{a}^+ + \hat{b} - \hat{b}^+)$$

$$\langle z|\hat{x}|z\rangle = \langle z|\hat{X}|z\rangle = \frac{l_B}{\sqrt{2}}(z+z^*) = \underline{\sqrt{2}\,l_B\,\mathrm{Re}\,z}$$

$$\langle z|\hat{y}|z\rangle = \langle z|\hat{Y}|z\rangle = -i\frac{l_B}{\sqrt{2}}(z-z^*) = \underline{\sqrt{2}\,l_B\,\mathrm{Im}\,z}$$

Expanded in $|m\rangle$-states

$$|z\rangle = \sum_m c_m|m\rangle \qquad a|z\rangle = \sum_m c_m\sqrt{m}\,|m-1\rangle$$

$$z|z\rangle = \sum_m z\,c_{m-1}\,|m-1\rangle$$

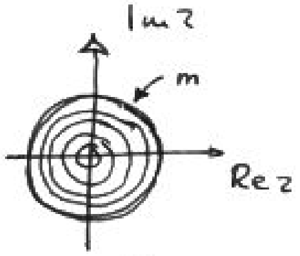
$$\Rightarrow c_m = \frac{c_{m-1}}{\sqrt{m}}z = \frac{c_{m-2}}{\sqrt{m(m-1)}}z^2 = \ldots = \frac{c_0}{\sqrt{m!}}z^m\,|m\rangle$$

Normalization $\quad 1 = \langle z|z\rangle = |c_0|^2\sum_m\frac{|z|^{2m}}{m!} = |c_0|^2 e^{-|z|^2}$

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_m \frac{z^m}{\sqrt{m!}} |m\rangle$$

$$|\langle m|z\rangle|^2 = \frac{|z|^{2m}}{m!} e^{-|z|^2}$$

In the z-plane: maximum around a circle of radius $|z|^2 = m$



In the x, y plane: maximum at $r^2 = 2m\ell_B^2$

Area within state m:

$$A_m = \pi r_m^2 = 2\pi m \ell_B^2$$ increases linearly with m

Number of states = m

$\Rightarrow$ density of states $\sigma = \frac{m}{A_m} = \underline{\frac{1}{2\pi\ell_B^2}}$

g) $\hat{H} = \hat{H}_0 - eE\hat{x}$ $\qquad H_0 = \cancel{\tfrac{1}{2}}\hbar\omega(b^\dagger b + \tfrac{1}{2})$

in the lowest Landau level $H_0 \to \frac{1}{2}\hbar\omega$, $\hat{x} \to \hat{X}$

$$\Rightarrow \underline{\hat{H} = \tfrac{1}{2}\hbar\omega - \tfrac{1}{\sqrt{2}}\ell_B eE(a+a^\dagger)}$$

Time evolution

$$U(t) = \exp\{-\tfrac{i}{\hbar}\hat{H}t\} = e^{-\frac{i}{2}\omega t} e^{i\frac{\ell_B}{\sqrt{2}\hbar}eE(a+a^\dagger)t}$$

$$a(t) = U^\dagger(t)\, a\, U(t) = e^{-i\frac{\ell_B}{\sqrt{2}\hbar}eE(a+a^\dagger)t}\, a\, e^{i\frac{\ell_B}{\sqrt{2}\hbar}eE(a+a^\dagger)t}$$

$$= a - i\frac{\ell_B}{\sqrt{2}\hbar}eE[a+a^\dagger, a]\,t$$

$$= a + i\frac{\ell_B}{\sqrt{2}\hbar}eEt$$

$$a^\dagger(t) = a^\dagger - i\frac{\ell_B}{\sqrt{2}\hbar}eEt$$

Heisenberg picture

$$\hat{X}(t) = \frac{l_B}{\sqrt{2}}\left(\hat{a}(t) + \hat{a}^{+}(t)\right) = \frac{l_B}{\sqrt{2}}\left(a + a^{+}\right) = \underline{\hat{X}(0)}$$

$$\hat{Y}(t) = -i\frac{l_B}{\sqrt{2}}\left(\hat{a}(t) - a^{+}(t)\right) = \hat{Y}(0) + \frac{l_B^2}{\hbar} eEt$$

$$= \underline{\hat{Y}(0) + \frac{E}{B}ct}$$

Drift in the y-direction with constant velocity $\underline{v_{drift} = \frac{E}{B}c}$

FYS 4110, 2006

Midterm exam, solutions

Problem 1, Spin coherent states

a) Eigenvalue equation $\hat{J}_- |\psi\rangle = \lambda |\psi\rangle$

$$|\psi\rangle = \sum_m c_m |j, m\rangle \qquad \text{expansion in } |j,m\rangle \text{ basis}$$

$m \leq j \Rightarrow$ there is a maximum value m_{max} in the expansion.

When $\hat{J}_-$ is applied to $|\psi\rangle$ that will reduce the max. value, $m_{max} \rightarrow m_{max} - 1$, since $\hat{J}_-$ lowers the m value

This creates a conflict between the RHS and LHS of the eigenvalue equation. Only solution: $\lambda = 0$

$$\Rightarrow |\psi\rangle = |j, -j\rangle$$

b) $$(\Delta \vec{J})^2 = j(j+1)\hbar^2 - \langle \hat{\vec{J}} \rangle^2$$

min. value of $(\Delta \vec{J})^2 \Rightarrow$ max value of $\langle \hat{\vec{J}} \rangle^2$

Assume $\langle \hat{J}_x \rangle = \langle \hat{J}_y \rangle = 0$, $\langle \hat{J}_z \rangle = J$

We have $-j\hbar \leq \langle \hat{J}_z \rangle \leq j\hbar$

Equality: $\hat{J}_z |j,j\rangle = j\hbar |j,j\rangle$; $\hat{J}_z |j,-j\rangle = -j\hbar |j,-j\rangle$

Max. value of $\langle \hat{J}_z \rangle^2$; $j^2\hbar^2$ for $|j,j\rangle$ and $|j,-j\rangle$

Min value for $(\Delta \vec{J})^2$: $j(j+1)\hbar^2 - j^2\hbar^2 = j\hbar^2$

for $|j,j\rangle$ and $|j,-j\rangle$

c) The solution in b) is valid for any choice of the z-direction (rotational invariance).

Assume $\vec{n}$ is a unit vector in an arbitrary direction. We choose this to be the (new) z-axis: $\vec{n} = \vec{k}$

The results of b) applies to this situation and we translate to $\vec{n}$-variable:

$$\hat{J}_z = \vec{k} \cdot \hat{\vec{J}} = \vec{n} \cdot \hat{\vec{J}}$$

minimum uncertainty state:

$$\hat{J}_z \, |j,j\rangle = j\hbar \, |j,j\rangle$$

$$\Leftrightarrow \quad \vec{n} \cdot \hat{\vec{J}} \, |\vec{n},j\rangle = j\hbar \, |\vec{n},j\rangle$$

with $|\vec{n},j\rangle$ as max spin state in the $\vec{n}$-direction

$$\langle j,j \, | \, \hat{\vec{J}} \, | \, j,j\rangle = J\vec{k} \quad , \; J = j\hbar$$

$$\Leftrightarrow \quad \langle \vec{n} j \, | \, \hat{\vec{J}} \, | \, \vec{n} j\rangle = J\vec{n}$$

d) Choose an arbitrary spin ½ state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

($|0\rangle$ = spin down, $|1\rangle$ = spin up in the z-direction

$$\langle \hat{\vec{J}} \rangle = \frac{\hbar}{2} (\beta^* \; \alpha^*) \, \vec{\sigma} \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$= \frac{\hbar}{2} \Big\{ (\beta^* \alpha^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \vec{i} + (\beta^* \alpha^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \vec{j}$$

$$+ (\beta^* \alpha^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \vec{k} \Big\}$$

$$= \frac{\hbar}{2} \Big\{ (\alpha^*\beta + \alpha\beta^*) \, \vec{i} + i(\alpha^*\beta - \alpha\beta^*) \, \vec{j} + (|\beta|^2 - |\alpha|^2) \, \vec{k} \Big\}$$

$$\langle \vec{J} \rangle^2 = \frac{\hbar^2}{4}\left((\alpha^*\beta + \alpha\beta^*)^2 - (\alpha^*\beta - \alpha\beta^*)^2 + (|\beta|^2 - |\alpha|^2)^2 \right)$$

$$= \frac{\hbar^2}{4}\left(4|\alpha|^2|\beta|^2 + (|\beta|^2 - |\alpha|^2)^2 \right)$$

$$= \frac{\hbar^2}{4}\left(|\alpha|^2 + |\beta|^2 \right)^2$$

$$= \frac{\hbar^2}{4}$$

$$\Rightarrow (\Delta \vec{J})^2 = \frac{1}{2}\frac{3}{2}\hbar^2 - \frac{1}{4}\hbar^2 = \underline{\frac{1}{2}\hbar^2}$$

Result the same for all states (independent of α and β)

$\Rightarrow$ all states have min value (and max value)
for $(\Delta \vec{J})^2$

e) Coherent state defined by

$$\vec{\sigma}\cdot\vec{n}\,|z\rangle = |z\rangle$$

with $|z\rangle = \alpha|0\rangle + \beta|1\rangle$ write this as a two-component eq. in the k-basis

$$\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1$$

$$\Rightarrow \cos\theta\,\beta + e^{-i\varphi}\sin\theta\,\alpha = \beta$$

$$\Rightarrow \frac{\beta}{\alpha}(1-\cos\theta) = e^{-i\varphi}\sin\theta \qquad 1-\cos\theta = 2\sin^2\frac{\theta}{2}\ ,\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\Rightarrow \frac{\beta}{\alpha}\sin\frac{\theta}{2} = e^{-i\varphi}\cos\theta/2$$

$$\frac{\beta}{\alpha} = e^{-i\varphi}\cot\theta/2 = z$$

$$\Rightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = N\begin{pmatrix} z \\ 1 \end{pmatrix} \qquad N^2(|z|^2+1) = 1$$

$$= \frac{1}{\sqrt{|z|^2+1}}$$

$$\underline{\langle 0|z\rangle = \alpha = \frac{1}{\sqrt{1+|z|^2}}}\ ,\ \underline{\langle 1|z\rangle = \beta = \frac{z}{\sqrt{1+|z|^2}}}$$

f) $$\langle z|z_0\rangle = \sum_{k=0}^{1} \langle z|k\rangle\langle k|z_0\rangle$$

$$= \frac{1+z^* z_0}{\sqrt{(1+|z|^2)(1+|z'|^2)}}$$

$$\Rightarrow \underline{|\langle z|z_0\rangle|^2 = \frac{1+z^*z_0+zz_0^*+|z|^2|z_0|^2}{(1+|z|^2)(1+|z_0|^2)}}$$

g) $$\int d^2z \, \frac{1}{(1+|z|^2)^2} \, |z\rangle\langle z|$$

$$= \sum_{kk'} \int d^2z \, \frac{\langle k|z\rangle\langle z|k'\rangle}{(1+|z|^2)^3} \, |k\rangle\langle k'|$$

$$= \sum_{kk'} |k\rangle\langle k'| \int d^2z \, \frac{z^k z^{*k'}}{(1+|z|^2)^3}$$

change to polar coordinates: $z = e^{i\varphi} r$, $d^2z = d\varphi \, dr \, r$

$$= \sum_{kk'} |k\rangle\langle k'| \underbrace{\int_0^{2\pi} d\varphi \, e^{i\varphi(k-k')}}_{2\pi\,\delta_{kk'}} \int_0^{\infty} dr \, \frac{r^{k+k'+1}}{(1+r^2)^3}$$

change of variable $t = r^2 \Rightarrow r\,dr = \frac{1}{2}\,dt$

$$= \sum_k \pi \int_0^{\infty} dt \, \frac{t^k}{(1+t)^3} \, |k\rangle\langle k|$$

$k = 0$ $$\int_0^{\infty} dt \, \frac{1}{(1+t)^3} = \left[-\frac{1}{2}\frac{1}{(1+t)^2}\right]_0^{\infty} = \frac{1}{2}$$

$k = 1$ $$\int_0^{\infty} dt \, \frac{t}{(1+t)^3} = \int_0^{\infty} dt \left(\frac{1}{(1+t)^2} - \frac{1}{(1+t)^3}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \int d^2z \, \frac{1}{(1+|z|^2)^2} \, |z\rangle\langle z| = \sum_k \frac{\pi}{2} \, |k\rangle\langle k| = \frac{\pi}{2} \, \mathbb{1}$$

$$\Rightarrow \underline{\int \frac{d^2z}{2\pi} \, \frac{4}{(1+|z|^2)^2} \, |z\rangle\langle z| = \mathbb{1}}$$

Problem 2 , Entanglement in a three-particle system

a) Correlated state is not a product state:

$\rho \neq \rho_A \otimes \rho_B \otimes \rho_C$ mixed state

$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$ pure state

$\Rightarrow \langle \hat{A}\hat{B}\hat{C} \rangle \neq \langle \hat{A} \rangle \langle \hat{B} \rangle \langle \hat{C} \rangle$

with $\hat{A}$ operating on subsystem A etc

Entangled state: not a statistical mixture of product states

$$\rho \neq \sum_k p_k \, \rho_k^A \otimes \rho_k^B \otimes \rho_k^C \qquad p_k \geq 0 \quad \sum_k p_k = 1$$

b)

$$\rho_A = \mathrm{Tr}_{BC}\, \rho = \tfrac{1}{2}\left(|u\rangle_A \langle u|_A + |d\rangle_A \langle d|_A \right) = \tfrac{1}{2} \mathbb{1}_A$$

$$\rho_{AB} = \mathrm{Tr}_A\, \rho = \tfrac{1}{2}\left(|uu\rangle_{BC} \langle uu|_{BC} + |dd\rangle_{BC} \langle dd|_{BC} \right)$$

Von Neuman entropy $S_A = S_{BC} = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = \underline{\log 2}$

Degree of entanglement measured by entropy of subsystem (when the total state is pure).

Subsystem A is maximally mixed ⇒ S_A is maximal

⇒ A+BC is maximally entangled

Subsystem BC is a statistical mixture of product state

⇒ No separate entanglement between B and C

c)

$$|u\rangle = \tfrac{1}{\sqrt{2}}(|f\rangle + |b\rangle) = \tfrac{1}{\sqrt{2}}(|r\rangle + |l\rangle)$$

$$|d\rangle = \tfrac{1}{\sqrt{2}}(|f\rangle - |b\rangle) = -\tfrac{i}{\sqrt{2}}(|r\rangle - |l\rangle)$$

GHZ state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|uuu\rangle - |ddd\rangle\right)$$

$$= \frac{1}{2}\left(|ffb\rangle + |fbf\rangle + |bff\rangle + |bbb\rangle\right)$$

$$= \frac{1}{2}\left(|rrf\rangle + |llf\rangle + |rlb\rangle + |lrb\rangle\right)$$

d) In all three cases, the expressions for $|\psi\rangle$ show that if the spin component of B and C are determined (by measurement) then the spin component of A is also uniquely determined.

1) Measurement of the spin in the z-direction of <u>either</u> B or C will determine the spin in the z-direction for the two other particles. (Strict correlation in the z-component of the spin.)

2) Measurement of the spin in the x-direction for B <u>and</u> C will determine the spin in the x-direction for A (For example fb measured for BC implies f for A)

3) Measurement of the spin in the y-direction for B and the x-component for C will determine the y component for A. (For example lf measured for BC implies l for A)

e) $\sigma_x |u\rangle = |d\rangle$, $\sigma_x |d\rangle = |u\rangle$

$\sigma_y |u\rangle = i|d\rangle$, $\sigma_y |d\rangle = -i|u\rangle$

$$\Rightarrow \hat{O}_1 |\psi\rangle = \hat{O}_2 |\psi\rangle = \hat{O}_3 |\psi\rangle = |\psi\rangle$$

all three have eigenvalue 1

$$\hat{O}_1 \hat{O}_2 \hat{O}_3 = \sigma_x \sigma_y^2 \otimes \sigma_y \sigma_x \sigma_y \otimes \sigma_y^2 \sigma_x$$

$$\sigma_y^2 = \mathbb{1} \ , \ \sigma_x \sigma_y = - \sigma_y \sigma_x$$

$$\Rightarrow \quad \hat{O}_1 \hat{O}_2 \hat{O}_3 = - \sigma_x \otimes \sigma_x \otimes \sigma_x = - \hat{O}_4 \qquad \text{eigenvalue of } \underline{\hat{O}_4 = -1}$$

f) Eigenvalue equations for $\hat{O}_1, \hat{O}_2, \hat{O}_3$

1 : $m_x^A m_y^B m_y^C = 1$

2 : $m_y^A m_x^B m_y^C = 1$

3 : $m_y^A m_y^B m_x^C = 1$

product of equations

$$m_x^A m_y^{A2} m_x^B m_y^{B2} m_x^C m_y^{C2} = 1$$

$$m_y^{A2} = m_y^{B2} = m_y^{C2} = 1 \Rightarrow$$

$$\underline{m_x^A m_x^B m_x^C = 1}$$

Eigenvalue equation for $\hat{O}_4$

$$\underline{m_x^A m_x^B m_x^C = -1}$$

contradicts equations for $\hat{O}_1, \hat{O}_2, \hat{O}_3$

Cannot assume spin components to have sharp, but undetermined values before the measurements.

FYS 4110 Midterm Exam 2007

Solutions

Problem 1, Density operators

a) Density operator, matrix form

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$\rho_{11} = \langle + | \rho | + \rangle = \underline{\tfrac{1}{2}(1+z)}$$

$$\rho_{12} = \langle + | \rho | - \rangle = \underline{\tfrac{1}{2}(x-iy)}$$

$$\rho_{21} = \langle - | \rho | + \rangle = \underline{\tfrac{1}{2}(x+iy)}$$

$$\rho_{22} = \langle - | \rho | - \rangle = \underline{\tfrac{1}{2}(1-z)}$$

b) Reduced density matrices

$$\rho^A = \mathrm{Tr}_B\, \rho = \frac{1}{4}\Big(\mathbb{1}\cdot \mathrm{Tr}_B \mathbb{1} + \sum_i a_i \sigma_i \,\mathrm{Tr}_B \mathbb{1} + \sum_j b_j \mathbb{1}\, \mathrm{Tr}_B \sigma_j + \sum_{ij} c_{ij}\, \sigma_i\, \mathrm{Tr}_B \sigma_j\Big)$$

use: $\mathrm{Tr}\, \mathbb{1} = 2$ (2x2 matrix)

$\mathrm{Tr}\, \sigma_i = 0 \quad i = 1, 2, 3$

$$\Rightarrow \quad \rho^A = \underline{\tfrac{1}{2}(\mathbb{1} + \vec{a}\cdot\vec{\sigma})}$$

In the same way

$$\rho^B = \underline{\tfrac{1}{2}(\mathbb{1} + \vec{b}\cdot\vec{\sigma})}$$

Completely uncorrelated means ρ is a product,

$$\rho = \rho^A \otimes \rho^B$$

$$= \frac{1}{4}\Big(\mathbb{1}\otimes\mathbb{1} + \sum_i a_i \sigma_i \otimes \mathbb{1} + \sum_j b_j \mathbb{1}\otimes\sigma_j + \sum_{ij} a_i b_j\, \sigma_i \otimes \sigma_j\Big)$$

The two subsystems are uncorrelated if $\underline{c_{ij} = a_i b_j}$

c) Density operators for the Bell states

$$\rho_{c\pm} = |c\pm\rangle\langle c\pm| = \tfrac{1}{2}\left(|++\rangle\langle ++| \pm |++\rangle\langle --| \pm |--\rangle\langle ++| + |--\rangle\langle --|\right)$$

$$\rho_{a\pm} = |a\pm\rangle\langle a\pm| = \tfrac{1}{2}\left(|+-\rangle\langle +-| \pm |+-\rangle\langle -+| \pm |-+\rangle\langle +-| + |-+\rangle\langle -+|\right)$$

Expressed in terms of Pauli matrices

$$|+\rangle\langle +| = \tfrac{1}{2}(\mathbb{1}+\sigma_z) \quad |-\rangle\langle -| = \tfrac{1}{2}(\mathbb{1}-\sigma_z)$$

$$|+\rangle\langle -| = \tfrac{1}{2}(\sigma_x+i\sigma_y) \quad |-\rangle\langle +| = \tfrac{1}{2}(\sigma_x-i\sigma_y)$$

for composite system

$$|++\rangle\langle ++| = |+\rangle\langle +| \otimes |+\rangle\langle +| = \tfrac{1}{4}(\mathbb{1}+\sigma_z)\otimes(\mathbb{1}+\sigma_z)$$

$$= \tfrac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z + \sigma_z\otimes\sigma_z)$$

$$|--\rangle\langle --| = \tfrac{1}{4}(\mathbb{1}\otimes\mathbb{1} - \sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z + \sigma_z\otimes\sigma_z)$$

$$\Rightarrow \quad |++\rangle\langle ++| + |--\rangle\langle --| = \tfrac{1}{2}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\sigma_z)$$

$$|++\rangle\langle --| = |+\rangle\langle -| \otimes |+\rangle\langle -| = \tfrac{1}{4}(\sigma_x+i\sigma_y)\otimes(\sigma_x+i\sigma_y)$$

$$= \tfrac{1}{4}(\sigma_x\otimes\sigma_x + i\,\sigma_x\otimes\sigma_y + i\,\sigma_y\otimes\sigma_x - \sigma_y\otimes\sigma_y)$$

$$|--\rangle\langle ++| = \tfrac{1}{4}(\sigma_x\otimes\sigma_x - i\,\sigma_x\otimes\sigma_y - i\,\sigma_y\otimes\sigma_x - \sigma_y\otimes\sigma_y)$$

$$\Rightarrow \quad |++\rangle\langle --| + |--\rangle\langle ++| = \tfrac{1}{2}(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y)$$

$$\Rightarrow \quad \underline{\rho_{c\pm} = \tfrac{1}{4}(\mathbb{1}\otimes\mathbb{1} \pm \sigma_x\otimes\sigma_x \mp \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z)}$$

$$|+-\rangle\langle +-| = |+\rangle\langle +| \otimes |-\rangle\langle -| = \tfrac{1}{4}(\mathbb{1}+\sigma_z)\otimes(\mathbb{1}-\sigma_z)$$

$$= \tfrac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z)$$

$$|-+\rangle\langle -+| = \tfrac{1}{4}(\mathbb{1}\otimes\mathbb{1} - \sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z)$$

$$\Rightarrow \quad |+-\rangle\langle +-| + |-+\rangle\langle -+| = \tfrac{1}{2}(\mathbb{1}\otimes\mathbb{1} - \sigma_z\otimes\sigma_z)$$

$$|+-\rangle\langle-+| = |+-\rangle \otimes \langle-+| = \frac{1}{4}(\sigma_x + i\sigma_y) \otimes (\sigma_x - i\sigma_y)$$

$$= \frac{1}{4}(\sigma_x \otimes \sigma_x - i\sigma_x \otimes \sigma_y + i\sigma_y \otimes \sigma_x + \sigma_y \otimes \sigma_y)$$

$$|-+\rangle\langle+-| = \frac{1}{4}(\sigma_x \otimes \sigma_x + i\sigma_x \otimes \sigma_y - i\sigma_y \otimes \sigma_y + \sigma_y \otimes \sigma_y)$$

$$\Rightarrow |+-\rangle\langle-+| + |-+\rangle\langle+-| = \frac{1}{2}(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)$$

$$\Rightarrow \underline{\rho_{a\pm} = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} \pm \sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y \mp \sigma_z \otimes \sigma_z)}$$

Note: no terms of the form $\sigma_i \otimes \mathbb{1}$ or $\mathbb{1} \otimes \sigma_i$ $(\vec{a} = \vec{b} = 0)$

$\Rightarrow$ $\underline{\rho^A = \frac{1}{2}\mathbb{1}}$, $\underline{\rho^B = \frac{1}{2}\mathbb{1}}$ $\Rightarrow$ entropy $\underline{S^A = S^B = \ln 2}$

for all the four Bell states

The Bell states are pure states: Correlations are due to entanglement; $S = 0$ (entropy of full system)

$S^A = S^B$ are maximal for subsystems $\Rightarrow$ maximal entanglement.

d) Check of conditions:

1) $\rho = \rho^+$, 2) $\rho \geq 0$ (non-neg. eigenv.), 3) $\mathrm{Tr}\,\rho = 1$

Satisfied for ρ_1 and ρ_2

1) $\rho^+ = x^* \rho_1^+ + (1-x^*)\rho_2^+ = x\rho_1 + (1-x)\rho_2 = \rho$ (x real)

2) $0 < x < 1 \Rightarrow x > 0$ & $1 - x > 0$

positive combination of positive operators

$\Rightarrow$ general state $\langle\psi|\rho|\psi\rangle = x\langle\psi|\rho_1|\psi\rangle + (1-x)\langle\psi|\rho_2|\psi\rangle \geq 0$

$\Rightarrow \rho \geq 0$

3) $\mathrm{Tr}\,\rho = x\,\mathrm{Tr}\,\rho_1 + (1-x)\,\mathrm{Tr}\,\rho_2 = x + (1-x) = \underline{1}$

If $x < 0$ or $1 - x > 0$: 1) and 3) still ok, but positivity not satisfied.

e) Choose f. ex. ρ_{c+} and ρ_{c-} :

$$\rho = \frac{1}{2}(\rho_{c+} + \rho_{c-})$$

$$= \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\sigma_z)$$

$$= \frac{1}{8}\left((\mathbb{1}+\sigma_z)\otimes(\mathbb{1}+\sigma_z) + (\mathbb{1}-\sigma_z)\otimes(1-\sigma_z)\right)$$

This is of the form

similar results for other choices.

$$\rho = \sum_k p_k \rho_k^A \otimes \rho_k^B$$

seperable, per definition non-entangled.

f) The Bell states have density operators that are all combinations of $\sigma_x\otimes\sigma_x$, $\sigma_y\otimes\sigma_y$, $\sigma_z\otimes\sigma_z$ (and identity $\mathbb{1}$). These all commute:

$$\sigma_x\sigma_y = i\sigma_z = -\sigma_y\sigma_x \Rightarrow$$

$$(\sigma_x\otimes\sigma_x)(\sigma_y\otimes\sigma_y) = \sigma_x\sigma_y\otimes\sigma_x\sigma_y = -\sigma_z\otimes\sigma_z$$

$$(\sigma_y\otimes\sigma_y)(\sigma_x\otimes\sigma_x) = \sigma_y\sigma_x\otimes\sigma_y\sigma_x = -\sigma_z\otimes\sigma_z$$

$$\Rightarrow [\sigma_x\otimes\sigma_x, \sigma_y\otimes\sigma_y] = 0$$ similar argument for other operators

More general argument:

Orthogonal pure states $\rho_1 = |\psi_1\rangle\langle\psi_1|$, $\rho_2 = |\psi_2\rangle\langle\psi_2|$

$$\Rightarrow \rho_1\rho_2 = |\psi_1\rangle\underbrace{\langle\psi_1|\psi_2\rangle}_{=0}\langle\psi_2| = 0 = \rho_2\rho_1$$

All four Bell states are orthogonal $\Rightarrow$ density op. commute

$\Rightarrow$ all linear combinations of these commute.

Problem 2, Jaynes-Cummings model

a) Eigenstates of H_0

$$H_0|m,n\rangle = \hbar\left(\tfrac{1}{2}m\omega_0 + n\omega\right)|m,n\rangle \qquad m=\pm 1,\ n=0,1,2,\dots$$

$$|1\rangle = |1,n-1\rangle, \qquad |2\rangle = |-1,n\rangle \Rightarrow$$

$$H_0|1\rangle = \hbar\left(\tfrac{1}{2}\omega_0 + (n-1)\omega\right)|1\rangle \equiv \left(\tfrac{1}{2}\hbar\Delta + \varepsilon\right)|1\rangle$$

$$H_0|2\rangle = \hbar\left(-\tfrac{1}{2}\omega_0 + n\omega\right)|2\rangle \equiv \left(-\tfrac{1}{2}\hbar\Delta + \varepsilon\right)|2\rangle$$

$$\Rightarrow \underline{\Delta = \omega_0 - \omega} \qquad \underline{\varepsilon = (n-\tfrac{1}{2})\hbar\omega}$$

$$H_1|1\rangle = i\hbar\lambda\,\sigma_-|1\rangle \otimes a^{+}|n-1\rangle$$

$$= i\hbar\lambda\sqrt{n}\,|2\rangle \longleftarrow = \tfrac{1}{2}i\hbar g\,|2\rangle$$

$$H_1|2\rangle = -i\hbar\lambda\sqrt{n}\,|1\rangle$$

$$\Big\} \Rightarrow \underline{g = 2\lambda\sqrt{n}}$$

2x2 matrix form

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \underline{\tfrac{1}{2}\hbar \begin{pmatrix} \Delta & -ig \\ ig & -\Delta \end{pmatrix} + \varepsilon\,\mathbb{1}}$$

b) Eigenvalue problem

new parameters $\Delta = \Omega\cos\theta$, $g = \Omega\sin\theta \Rightarrow \Omega = \sqrt{\Delta^2+g^2}$

$$\Rightarrow H = \tfrac{1}{2}\hbar\Omega \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix} + \varepsilon\,\mathbb{1}$$

eigenvalue problem to solve:

$$\begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \Lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} \cos\theta - \Lambda & -i\sin\theta \\ i\sin\theta & -\cos\theta - \Lambda \end{vmatrix} = 0$$

$$(\cos\theta-\Lambda)(-\cos\theta-\Lambda)-\sin^2\theta = 0$$

$$\Lambda^2-\cos^2\theta-\sin^2\theta = 0$$

$$\Lambda^2 = 1 \Rightarrow \Lambda_\pm = \pm 1$$ two eigenvalues

Energies

$$E_\pm = \tfrac{1}{2}\hbar\Omega\Lambda_\pm + \varepsilon$$

$$= (n-\tfrac{1}{2})\hbar\omega \pm \tfrac{1}{2}\hbar\sqrt{(\omega_0-\omega)^2+4n\lambda^2}$$

Eigenvectors

$$\cos\theta\, a_\pm - i\sin\theta\, b_\pm = \pm a_\pm$$

$$\mp(1\mp\cos\theta)a_\pm = i\sin\theta\, b_\pm$$

Use :

$$\sin\theta = 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$

$$1-\cos\theta = 2\sin^2\tfrac{\theta}{2}$$

$$1+\cos\theta = 2\cos^2\tfrac{\theta}{2}$$

$$\Rightarrow 2\sin^2\tfrac{\theta}{2}\, a_+ = -i\, 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\, b_+$$

$$\sin\tfrac{\theta}{2}\, a_+ = -i\cos\tfrac{\theta}{2}\, b_+$$

matrix $$\underline{\psi_+ = \begin{pmatrix} a_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} i\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{pmatrix}}$$

$$2\cos^2\tfrac{\theta}{2}\, a_- = i\, 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\, b_-$$

$$\cos\tfrac{\theta}{2}\, a_- = i\sin\tfrac{\theta}{2}\, b_-$$

matrix $$\underline{\psi_- = \begin{pmatrix} a_- \\ b_- \end{pmatrix} = \begin{pmatrix} i\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}}$$

General state

$$\psi = d_+ \psi_+ + d_- \psi_-$$

$$= \begin{pmatrix} i(d_+ \cos\frac{\theta}{2} + d_- \sin\frac{\theta}{2}) \\ -d_+ \sin\frac{\theta}{2} + d_- \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Initial condition

$$c_1(0) = 0 \qquad c_2(0) = 1$$

$$\Rightarrow \quad d_+(0) \cos\frac{\theta}{2} + d_-(0) \sin\frac{\theta}{2} = 0$$

$$-d_+(0) \sin\frac{\theta}{2} + d_-(0) \cos\frac{\theta}{2} = 1$$

$$\Rightarrow \quad d_+(0) = -\sin\frac{\theta}{2}, \quad d_-(0) = \cos\frac{\theta}{2}$$

Time evolution

$$d_+(t) = e^{-\frac{i}{\hbar}E_+ t} d_+(0) = -\sin\frac{\theta}{2}\, e^{-\frac{i}{2}\Omega t}\, e^{-\frac{i}{\hbar}\varepsilon t}$$

$$d_-(t) = e^{-\frac{i}{\hbar}E_- t} d_-(0) = \cos\frac{\theta}{2}\, e^{\frac{i}{2}\Omega t}\, e^{-\frac{i}{\hbar}\varepsilon t}$$

$$\Rightarrow \quad c_1(t) = i\,(d_+(t) \cos\frac{\theta}{2} + d_-(t) \sin\frac{\theta}{2})$$

$$= i\, e^{-\frac{i}{\hbar}\varepsilon t} \left(-\sin\frac{\theta}{2} \cos\frac{\theta}{2}\, e^{-\frac{i}{2}\Omega t} + \cos\frac{\theta}{2} \sin\frac{\theta}{2}\, e^{\frac{i}{2}\Omega t}\right)$$

$$\underline{= -\, e^{-\frac{i}{\hbar}\varepsilon t} \sin\theta \, \sin\frac{\Omega}{2} t}$$

$$c_2(t) = -d_+(t) \sin\frac{\theta}{2} + d_-(t) \cos\frac{\theta}{2}$$

$$= e^{-\frac{i}{\hbar}\varepsilon t} \left(\sin^2\frac{\theta}{2}\, e^{-\frac{i}{2}\Omega t} + \cos^2\frac{\theta}{2}\, e^{\frac{i}{2}\Omega t}\right)$$

$$= \underline{e^{-\frac{i}{2}\varepsilon t} \left(\cos\frac{\Omega}{2} t + i \cos\theta \sin\frac{\Omega}{2} t\right)}$$

$$\underline{|c_1(t)|^2 = \sin^2\theta \, \sin^2\frac{\Omega}{2} t}$$

d) The atom is initially in the lowest energy state. The interaction with the electromagnetic field introduces oscillations between this state and the excited atomic state, with oscillation frequency Ω.
The situation is similar to that of Sect. 1.3.2 of the lecture notes, where the oscillations are induced by a time-dependent magnetic field.
Connection between the two expressions

$$g = \omega_1 \Rightarrow$$

$$2\lambda\sqrt{n} = -\frac{eB_1}{m_e c} \Rightarrow \underline{B_1 = \text{const}\cdot\sqrt{n}}$$

the amplitude of the magnetic field is proportional to the square root of the photon number.

e) $$|\psi(t)\rangle = c_1(t)\,|+1\rangle_A \otimes |n-1\rangle_B + c_2(t)\,|-1\rangle_A \otimes |n\rangle_B$$

$$\Rightarrow \rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

$$= |c_1(t)|^2\, |+1\rangle_A\langle+1|_A \otimes |n-1\rangle_B\langle n-1|_B$$

$$+ |c_2(t)|^2\, |-1\rangle_A\langle-1|_A \otimes |n\rangle_B\langle n_B|$$

$$+ c_1(t)\,c_2(t)^*\, |+1\rangle_A\langle-1|_A \otimes |n-1\rangle_B\langle n|_B$$

$$+ c_1^*(t)\,c_2(t)\, |-1\rangle_A\langle+1|_A \otimes |n\rangle_B\langle n-1|_B$$

f) Reduced density matrix

$$\rho_A = \mathrm{Tr}_B\,\rho = {}_B\langle n-1|\,\rho\,|n-1\rangle_B + {}_B\langle n|\,\rho\,|n\rangle_B$$

$$= |c_1|^2\,|+1\rangle_A\langle +1|_A + |c_2|^2\,|-1\rangle_A\langle -1|_A$$

matrix form

$$\rho_A(t) = \begin{pmatrix} |c_1(t)|^2 & 0 \\ 0 & |c_2(t)|^2 \end{pmatrix}$$

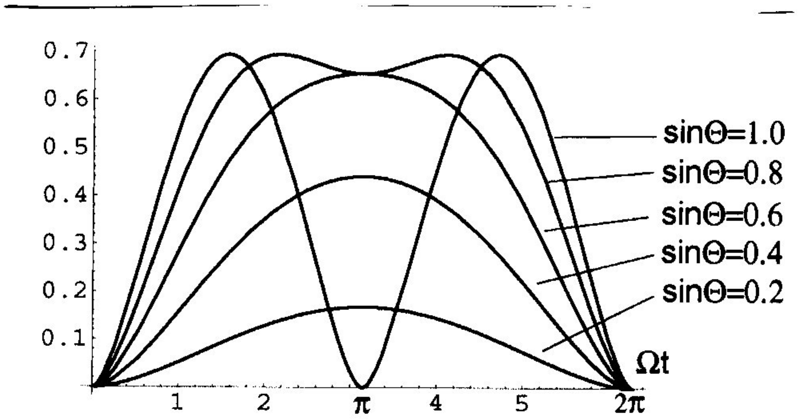
Entropy

$$S_A(t) = -\left(|c_1(t)|^2 \log|c_1(t)|^2 + |c_2(t)|^2 \log|c_2(t)|^2\right)$$

$$= \underline{-\left(|c_1(t)|^2 \log|c_1(t)|^2 + (1-|c_1(t)|^2)\log(1-|c_1(t)|^2)\right)}$$

$$|c_1(t)|^2 = \sin^2\theta\,\sin^2\tfrac{\Omega}{2}t$$

Plot of $S_A(t)$ for different values of $\sin\theta$:



For a pure state (of the composite system), the entropy of the subsystem gives a measure of the degree of entanglement. The figure shows: Entanglement increases from a minimum at $\Omega t = 0$ ($2\pi, 4\pi, \ldots$). A maximum is reached at $\Omega t = \pi$ for small θ ($\sin\theta < \frac{1}{\sqrt{2}}$). For larger θ $\Omega t = \pi$ is instead a minimum and the maxima moves towards $\Omega t = \frac{\pi}{2}, 3\frac{\pi}{2}$.

FYS4110 Midterm Exam 2008

Solutions

Problem 1 Spin splitting in positronium

a) $\langle ij | \vec{\Sigma}_e \cdot \vec{\Sigma}_p | kl \rangle$

$$= \sum_{mn} \langle ij | \vec{\sigma}_e \otimes \mathbb{1}_p | mn \rangle \langle mn | \mathbb{1}_e \otimes \vec{\sigma}_p | kl \rangle$$

$$= \sum_{mn} \left(\langle i | \vec{\sigma}_e | m \rangle \delta_{jn} \right) \cdot \left(\delta_{mk} \langle n | \vec{\sigma}_p | l \rangle \right.$$

$$= \underline{\langle i | \vec{\sigma}_e | k \rangle \cdot \langle j | \vec{\sigma}_p | l \rangle}$$

b) matrix elements

$$\vec{\sigma} = \sigma_x \vec{i} + \sigma_y \vec{j} + \sigma_z \vec{k} \quad \Rightarrow$$

$$\langle + | \vec{\sigma} | + \rangle = \vec{k} \, , \quad \langle - | \vec{\sigma} | - \rangle = -\vec{k}$$

$$\langle + | \vec{\sigma} | - \rangle = \vec{i} - i\vec{j} \, , \quad \langle - | \vec{\sigma} | + \rangle = \vec{i} + i\vec{j}$$

$$\Rightarrow \quad \langle ++ | \vec{\Sigma}_e \cdot \vec{\Sigma}_p | ++ \rangle = \vec{k} \cdot \vec{k} = 1$$

$$\langle ++ | -''- | +- \rangle = \vec{k} \cdot (\vec{i} - i\vec{j}) = 0$$

$$\langle ++ | -''- | -+ \rangle = -''- = 0$$

$$\langle ++ | -''- | -- \rangle = (\vec{i} - i\vec{j}) \cdot (\vec{i} - i\vec{j}) = 0$$

$$\langle +- | -''- | +- \rangle = \vec{k} \cdot (-\vec{k}) = -1$$

$$\langle +- | -''- | -+ \rangle = (\vec{i} - i\vec{j}) \cdot (\vec{i} + i\vec{j}) = 2$$

$$\langle +- | -''- | -- \rangle = (\vec{i} - i\vec{j}) \cdot (-\vec{k}) = 0$$

$$\langle -+ | -''- | -+ \rangle = (-\vec{k}) \cdot \vec{k} = -1$$

$$\langle -+ | -''- | -- \rangle = (-\vec{k}) \cdot (\vec{i} - i\vec{j}) = 0$$

$$\langle -- | -''- | -- \rangle = (-\vec{k}) \cdot (-\vec{k}) = 1$$

other matrix elements determined by hermiticity of $\vec{\Sigma}_e \cdot \vec{\Sigma}_p$

Matrix representation

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

---

c) From b) follows

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |0,0\rangle = \cancel{4}\frac{1}{\sqrt{2}}\left(\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |+-\rangle - \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |-+\rangle\right)$$

$$= \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}\Big(\left(-|+-\rangle + 2|-+\rangle\right) - \left(-|-+\rangle + 2|+-\rangle\right)\Big)$$

$$= -\frac{3}{4}\hbar^2 |0,0\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,1\rangle = \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |++\rangle = \frac{\hbar^2}{4}|1,1\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}\Big(\left(-|+-\rangle + 2|-+\rangle\right) + \left(-|-+\rangle + 2|+-\rangle\right)\Big)$$

$$= \frac{1}{4}\hbar^2 |1,0\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,-1\rangle = \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |--\rangle = \frac{\hbar^2}{4}|1,-1\rangle$$

matrix form in the spin basis

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p = \frac{\hbar^2}{4}\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Total spin $\hat{\vec{S}} = \hat{\vec{S}}_e + \hat{\vec{S}}_p \Rightarrow \hat{\vec{S}}^2 = \hat{\vec{S}}_e^{\,2} + \hat{\vec{S}}_p^{\,2} + 2\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p$

$$\hat{\vec{S}}_e^{\,2} = \frac{\hbar^2}{4}\vec{\sigma}_e^{\,2} \otimes \mathbb{1}_p = 3\frac{\hbar^2}{4}\mathbb{1}_e \otimes \mathbb{1}_p = \frac{3}{4}\hbar^2 \mathbb{1} = \hat{\vec{S}}_p^{\,2}$$

$$\Rightarrow \hat{\vec{S}}^2 = \frac{3}{2}\hbar^2 \mathbb{1} + 2\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p$$

$$\Rightarrow \hat{\vec{S}}^2 |0,0\rangle = 0 \,, \quad \hat{\vec{S}}^2 |1,m\rangle = 2\hbar^2 |1,m\rangle \qquad m = 0, \pm 1$$

$$\hat{S}_z |0,0\rangle = 0 \,, \quad \hat{S}_z |1,m\rangle = m\hbar |1,m\rangle$$

$\hat{\vec{S}}^2 = S(S+1)\hbar^2 \Rightarrow S = 0$ for $|0,0\rangle$, $S = 1$ for $|1,m\rangle$

d) Need to find the matrix elements of $(S_e)_z - (S_p)_z \equiv D$

$$D|1,1\rangle = D|1,-1\rangle = 0$$

$$D|0,0\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(2|+-\rangle - (-2)|-+\rangle\right) = \hbar|1,0\rangle$$

$$D|1,0\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(2|+-\rangle + (-2)|-+\rangle\right) = \hbar|0,0\rangle$$

mixes only $|0,0\rangle$ and $|1,0\rangle$

Hamiltonian in the spin basis

$$H = \begin{pmatrix} E_0 - \frac{3}{4}\hbar^2\kappa & 0 & \lambda\hbar^2 & 0 \\ 0 & E_0 + \frac{1}{4}\hbar^2\kappa & 0 & 0 \\ \lambda\hbar^2 & 0 & E_0 + \frac{1}{4}\hbar^2\kappa & 0 \\ 0 & 0 & 0 & E_0 + \frac{1}{4}\hbar^2\kappa \end{pmatrix}$$

e) · $|1,1\rangle$ and $|1,-1\rangle$ are eigenvectors with eigenvalues $\underline{E = E_0 + \frac{1}{4}\hbar^2\kappa}$ (indep. of λ)

Eigenvalue problem for the remaining two states

$$\begin{pmatrix} E_0 - \frac{3}{4}\hbar^2\kappa & \lambda\hbar^2 \\ \lambda\hbar^2 & E_0 + \frac{1}{4}\hbar^2\kappa \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

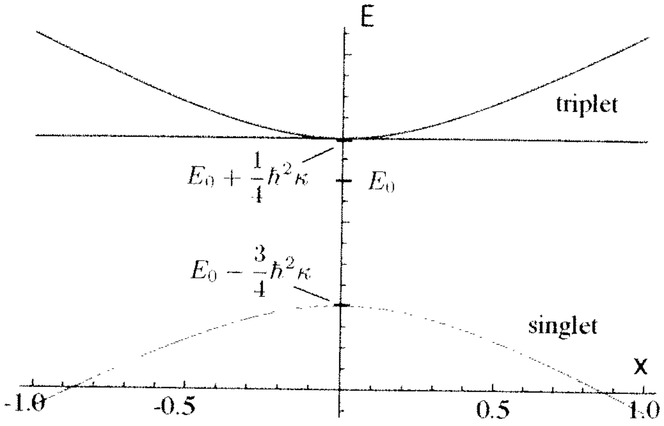
write this as $\left(E_0 - \frac{1}{4}\hbar^2\kappa\right)\mathbb{1} + \frac{1}{2}\hbar^2\kappa\begin{pmatrix} -1 & 2x \\ 2x & 1 \end{pmatrix}$ $\quad x = \lambda/\kappa$

$$\Rightarrow \begin{pmatrix} -1 & 2x \\ 2x & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \mu\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{with} \quad E = E_0 - \frac{1}{4}\hbar^2\kappa + \frac{1}{2}\hbar^2\kappa\mu$$

eigenvalues $\begin{vmatrix} -1-\mu & 2x \\ 2x & 1-\mu \end{vmatrix} = 0 \Rightarrow \mu^2 = 4x^2 + 1$

$$E_\pm = E_0 - \frac{1}{4}\hbar^2\kappa \pm \frac{1}{2}\hbar^2\kappa\sqrt{4x^2+1}$$

$$= \underline{E_0 - \frac{1}{4}\hbar^2\kappa \pm \frac{1}{2}\hbar^2\sqrt{\kappa^2 + 4\lambda^2}}$$



f) $\hat{\rho}_A = |A\rangle\langle A| = |a|^2 |+-\rangle\langle +-| + |b|^2 |-+\rangle\langle -+|$

$+ ab^* |+-\rangle\langle -+| + a^* b |-+\rangle\langle +-|$

$\hat{\rho}_B = |B\rangle\langle B| = |b|^2 |+-\rangle\langle +-| + |a|^2 |-+\rangle\langle -+|$

$- ab^* |+-\rangle\langle -+| - a^* b |-+\rangle\langle +-|$

Reduced density operators

$\hat{\rho}_{Ae} = \mathrm{Tr}_p\, \hat{\rho}_A = |a|^2 |+\rangle\langle +| + |b|^2 |-\rangle\langle -|$

$\hat{\rho}_{Ap} = \mathrm{Tr}_e\, \hat{\rho}_A = |a|^2 |-\rangle\langle -| + |b|^2 |+\rangle\langle +|$

$\hat{\rho}_{Be} = \mathrm{Tr}_p\, \hat{\rho}_B = |b|^2 |+\rangle\langle +| + |a|^2 |-\rangle\langle -|$

$\hat{\rho}_{Bp} = \mathrm{Tr}_e\, \hat{\rho}_B = |b|^2 |-\rangle\langle -| + |a|^2 |+\rangle\langle +|$

g. Entropy

$S_{Ae} = S_{Ap} = S_{Be} = S_{Bp} = -\left(|a|^2 \log |a|^2 + |b|^2 \log |b|^2\right)$

$$= \underline{-\left(|a|^2 \log |a|^2 + (1-|a|^2) \log (1-|a|^2)\right)}$$

g) Eigenstates

$$|A\rangle = \alpha|0,0\rangle + \beta|1,0\rangle = a|+-\rangle + b|-+\rangle$$

$$\Rightarrow a = \frac{\alpha+\beta}{\sqrt{2}}, \quad b = \frac{\alpha-\beta}{\sqrt{2}}$$

α, β determined by eigenvalue eq. in e):

$$-\alpha + 2x\beta = \mu\alpha \quad \Rightarrow \quad \beta = \frac{\mu+1}{2x}\alpha$$

$\mu = \pm\sqrt{4x^2+1}$; choose $\mu = -\sqrt{4x^2+1}$ (+ gives $|B\rangle$)

gives $\beta \to 0$ for $x \to 0$

Note α, β real.

Normalization : $\alpha^2 + \beta^2 = \left(1 + \left(\frac{\mu+1}{2x}\right)^2\right)\alpha^2 = 1$

$$\Rightarrow \alpha^2 = \frac{4x^2}{4x^2+(\mu+1)^2}$$

$$a^2 = \frac{1}{2}\left(1 + \frac{\mu+1}{2x}\right)^2\alpha^2 = \frac{1}{2}\frac{(2x+\mu+1)^2}{4x^2+(\mu+1)^2}$$

$$(2x+\mu+1)^2 = 4x^2 + 1 + 4x + \mu^2 + 2(2x+1)\mu$$
$$= 2\left(\mu^2 + 2x(\mu+1) + \mu\right) = 2(\mu+1)(\mu+2x)$$

$$4x^2 + (\mu+1)^2 = 4x^2 + 1 + \mu^2 + 2\mu = 2(\mu^2+\mu) = 2\mu(\mu+1)$$

$$\Rightarrow a^2 = \frac{1}{2}\frac{2(\mu+2x)(\mu+1)}{2\mu(\mu+1)} = \underline{\frac{1}{2}\left(1 + \frac{2x}{\sqrt{4x^2+1}}\right)}$$

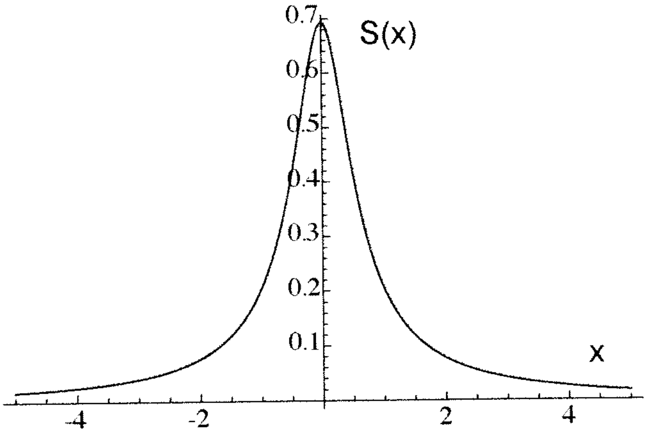
$$b^2 = 1 - a^2 = \frac{1}{2}\left(1 - \frac{2x}{\sqrt{4x^2+1}}\right)$$

Entropy of reduced density matrices

$$\underline{S(x) = -\left[a(x)^2 \log a(x)^2 + b(x)^2 \log b(x)^2\right]}$$

For $x = 0$: $\hat{\rho}_{Ae} = \hat{\rho}_{Be} = \frac{1}{2}\mathbb{1}_e$, $\hat{\rho}_{Ap} = \hat{\rho}_{Bp} = \frac{1}{2}\mathbb{1}_p$

maximal entanglement $S(0) = \log 2$



Entanglement of states |A> and |B> as functions of $x = \lambda/\kappa$

Problem 2 Driven harmonic oscillator

a) $$\hat{U} e^{\hat{A}} \hat{U}^{-1} = \hat{U} \left(\mathbb{1} + \hat{A} + \tfrac{1}{2} \hat{A}^2 + \cdots \right) \hat{U}^{-1}$$

$$= \mathbb{1} + \hat{U}\hat{A}\hat{U}^{-1} + \tfrac{1}{2} (\hat{U}\hat{A}\hat{U}^{-1})^2 + \cdots = e^{\hat{U}\hat{A}\hat{U}^{-1}}$$

special case: $\hat{U}_0(t)\hat{D}(z)\hat{U}_0(t)^\dagger$ $\qquad \hat{U}_0(t)^\dagger = \hat{U}_0(t)^{-1}$

$$= \hat{U}_0(t)\, e^{z\hat{a}^\dagger - z^*\hat{a}}\, \hat{U}_0(t) = e^{\hat{U}_0(t)(z\hat{a}^\dagger - z^*\hat{a})\hat{U}_0(t)^\dagger}$$

$$\hat{U}_0(t)\,\hat{a}\,\hat{U}_0(t)^\dagger = e^{-i\omega_0 t(\hat{a}^\dagger\hat{a}+\frac{1}{2})}\,\hat{a}\,e^{i\omega_0 t(\hat{a}^\dagger\hat{a}+\frac{1}{2})}$$

$$= \hat{a} - i\omega_0 t\,[\hat{a}^\dagger\hat{a}+\tfrac{1}{2}, \hat{a}] + \tfrac{1}{2}(-i\omega_0 t)^2\,[\hat{a}^\dagger\hat{a}+\tfrac{1}{2}, [\hat{a}^\dagger\hat{a}+\tfrac{1}{2}, \hat{a}]] + \cdots$$

$$= \hat{a} + i\omega_0 t\,\hat{a} + \tfrac{1}{2}(i\omega_0 t)^2\,\hat{a} + \cdots$$

$$= e^{i\omega t}\,\hat{a}$$

$$\Rightarrow \hat{U}_0(t)\,\hat{a}^\dagger\,\hat{U}_0(t)^\dagger = e^{-i\omega_0 t}\,\hat{a}^\dagger$$

$$\Rightarrow \hat{U}_0(t)\,\hat{D}(z)\,\hat{U}_0(t)^\dagger = \hat{D}(z e^{-i\omega_0 t})$$

$$|\psi(t)\rangle = \hat{U}_0(t)|z_0\rangle = \hat{U}_0(t)\,\hat{D}(z_0)\,|0\rangle$$

$$= \hat{U}_0(t)\,\hat{D}(z_0)\,\hat{U}_0(t)^\dagger\,\hat{U}(t_0)|0\rangle$$

$$= \hat{D}(z_0 e^{-i\omega_0 t})\, e^{-\frac{i}{2}\omega_0 t}\,|0\rangle$$

$$= \underline{e^{-\frac{i}{2}\omega_0 t}\,|z_0 e^{-i\omega_0 t}\rangle}$$

remains a coherent state during the evolution.

b) $$\dot{\hat{x}} = \frac{i}{\hbar}[\hat{H}, \hat{x}] = \frac{i}{\hbar}\frac{1}{2m}[\hat{p}^2, \hat{x}] = \frac{i}{\hbar}\frac{1}{2m}(\hat{p}[\hat{p}, \hat{x}] + [\hat{p}, \hat{x}]\hat{p})$$

$$= \frac{\hat{p}}{m}$$

$$\dot{\hat{p}} = \frac{i}{\hbar}[\hat{H}, \hat{p}] = -\frac{d}{dx}\left(\tfrac{1}{2}m\omega_0^2\hat{x}^2 + W(\hat{x}, t)\right) = -m\omega_0^2\hat{x} - \frac{\partial W}{\partial x}(\hat{x}, t)$$

$$\Rightarrow \underline{m\ddot{\hat{x}} + m\omega_0^2\hat{x} = -\frac{\partial \hat{W}}{\partial x} = -A \overset{\sin}{\cancel{\cos}}\,\omega t\, \mathbb{1}}$$

Driven harmonic oscillator, force $f(t) = -A \overset{\sin}{\cancel{\cos}}\,\omega t$

c) Time evolution in the Schrödinger picture (S) and interaction picture (I)

$|\psi_I(t)\rangle = \hat{U}_0(t)^+ |\psi_S(t)\rangle$ def. of transf. S → I

$|\psi_S(t)\rangle = \hat{U}(t)|\psi_S(0)\rangle$ & $|\Psi_I(0)\rangle = |\psi_S(0)\rangle$

$\Rightarrow |\psi_I(t)\rangle = \hat{U}_0(t)^+ \hat{U}(t) |\psi_I(0)\rangle$

$$\Rightarrow \underline{\hat{U}_I(t) = \hat{U}_0(t)^+ \hat{U}(t)}$$

Schrödinger eq. ⇒

$$i\hbar \frac{d}{dt} \hat{U}(t) = \hat{H}(t)\hat{U}(t)$$

$$i\hbar \frac{d}{dt} \hat{U}_0(t) = \hat{H}_0 \hat{U}_0(t) \Rightarrow i\hbar \frac{d}{dt} \hat{U}_0(t)^+ = -\hat{U}_0(t)^+ \hat{H}_0$$

$$\Rightarrow i\hbar \frac{d}{dt} \hat{U}_I(t) = i\hbar \frac{d}{dt} \hat{U}_0(t)^+ \hat{U}(t) + \hat{U}_0(t)\, i\hbar \frac{d}{dt} \hat{U}(t)$$

$$= \hat{U}_0(t)^+ \hat{H}(t)\hat{U}(t) - \hat{U}_0(t) H_0 \hat{U}(t)$$

$$= \hat{U}_0(t)^+ \hat{W}(t)\hat{U}(t)$$

$$\equiv \hat{H}_I(t)\hat{U}_I(t)$$

$$\Rightarrow \underline{\hat{H}_I(t) = \hat{U}_0^+(t)\hat{W}(t)\hat{U}_0(t)} \qquad \hat{U}_0(t) = e^{-\frac{i}{\hbar}\hat{H}_0 t}$$

$$\hat{W} = A\hat{x}\sin\omega t = A\sqrt{\frac{\hbar}{2m\omega_0}}\,(\hat{a} + \hat{a}^+)\sin\omega t$$

$$\Rightarrow \hat{H}_I(t) = A\sqrt{\frac{\hbar}{2m\omega_0}}\, e^{i\omega_0 t \hat{a}^+\hat{a}}(\hat{a} + \hat{a}^+)e^{-i\omega_0 t \hat{a}^+\hat{a}}\sin\omega t$$

$$= \underline{A\sqrt{\frac{\hbar}{2m\omega_0}}\,(e^{-i\omega_0 t}\hat{a} + e^{i\omega_0 t}\hat{a}^+)\sin\omega t}$$

$$= \theta(t)^*\hat{a} + \theta(t)\hat{a}^+ \quad \text{with} \quad \underline{\theta(t) = A\sqrt{\frac{\hbar}{2m\omega_0}}\, e^{i\omega_0 t}\sin\omega t}$$

d) Assume

$$\hat{U}_I = e^{\xi a^+ - \xi^* \hat{a}}\, e^{i\varphi} = e^{\xi \hat{a}^+} e^{-\xi^* \hat{a}}\, e^{i\varphi - \frac{1}{2}\xi^*\xi}$$

$$\Rightarrow \frac{d\hat{U}_I}{dt} = \dot{\xi}\, \hat{a}^+ e^{\xi\hat{a}^+} e^{-\xi^*\hat{a}}\, e^{i\varphi - \frac{1}{2}\xi^*\xi}$$

$$- e^{\xi\hat{a}^+} \dot{\xi}^* \hat{a}\, e^{-\xi^*\hat{a}}\, e^{i\varphi - \frac{1}{2}\xi^*\xi}$$

$$+ e^{\xi\hat{a}^+} e^{-\xi^*\hat{a}} \left(i\dot{\varphi} - \tfrac{1}{2}(\dot{\xi}^*\xi + \xi^*\dot{\xi})\right) e^{i\varphi - \frac{1}{2}\xi^*\xi}$$

use $\quad e^{\xi\hat{a}^+}\hat{a}\, e^{-\xi\hat{a}^+} = \hat{a} - \xi$

$$\frac{d\hat{U}_I}{dt} = \left[(\dot{\xi}\hat{a}^+ - \dot{\xi}^*\hat{a}) + (i\dot{\varphi} + \tfrac{1}{2}(\dot{\xi}^*\xi - \xi^*\dot{\xi}))\right] \hat{U}_I(t)$$

Of the form

$$i\hbar \frac{d\hat{U}_I}{dt} = \hat{H}_I(t)\,\hat{U}_I(t)$$

if: 1) $\theta = i\hbar\dot{\xi} \quad \Rightarrow \quad \xi(t) = -\frac{i}{\hbar}\int_0^t \theta(t')\,dt'$

2) $\underline{\dot{\varphi} = \frac{i}{2}(\dot{\xi}^*\xi - \xi^*\dot{\xi})}$

e) Note: $\hat{U}_I(t) = e^{i\varphi(t)}\,\hat{D}(\xi(t))$

Time evolution in the Schrödinger picture

$$|\psi(t)\rangle = \hat{U}_0(t)\,\hat{U}_I(t)\,|\psi(0)\rangle$$

$$= \hat{U}_0(t)\, e^{i\varphi(t)}\hat{D}(\xi(t))\,|z_0\rangle$$

$$= e^{i\varphi}\,\hat{U}_0(t)\,\hat{D}(\xi)\,\hat{D}(z_0)\,|0\rangle$$

Product of displacements operators

$$\hat{D}(\xi)\,\hat{D}(z_0) = e^{\xi\hat{a}^+ - \xi^*\hat{a}}\, e^{z_0\hat{a}^+ - z_0^*\hat{a}}$$

$$= e^{(\xi+z_0)\hat{a}^+ - (\xi+z_0)^*\hat{a} + \frac{1}{2}(\xi z_0^* - \xi^* z)}$$

$$= e^{\frac{1}{2}(\xi z_0^* - \xi^* z_0)}\,\hat{D}(z_0 + \xi)$$

Use results from a):

$$\hat{U}_0(t)\,|z\rangle = e^{-\frac{i}{2}\omega_0 t}\,|e^{-i\omega_0 t} z\rangle$$

$$\Rightarrow \quad |\psi(t)\rangle = \exp\left(i\varphi + \tfrac{1}{2}(\xi z_0^* - \xi^* z_0)\right) \hat{U}_0(t)\,|z_0 + \xi(t)\rangle$$

$$= \exp\left(i(\varphi - \tfrac{1}{2}\omega_0 t) + \tfrac{1}{2}(\xi z_0^* - \xi^* z_0)\right) |e^{-i\omega_0 t}(z_0 + \xi(t))\rangle$$

$$\Rightarrow \quad \gamma(t) = \varphi(t) - \tfrac{1}{2}\omega_0 t + \frac{1}{2i}\left(\xi(t) z_0^* - \xi(t)^* z_0\right)$$

$$z(t) = e^{-i\omega_0 t}(z_0 + \xi(t))$$

f) The function $\xi(t)$

$$\xi(t) = -\frac{i}{\hbar}\int_0^t \theta(t')\,dt'$$

with $$\theta(t) = -\frac{i}{2} A \sqrt{\frac{\hbar}{2m\omega_0}}\left(e^{i(\omega_0+\omega)t} - e^{i(\omega_0-\omega)t}\right)$$

$$\Rightarrow \quad \xi(t) = \frac{i}{2} A \sqrt{\frac{1}{2m\omega_0\hbar}}\left(\frac{e^{i(\omega_0+\omega)t} - 1}{\omega_0 + \omega} - \frac{e^{i(\omega_0-\omega)t} - 1}{\omega_0 - \omega}\right)$$

$$z(t) = z_0 e^{-i\omega_0 t} + \frac{i}{2} A \sqrt{\frac{1}{2m\omega_0\hbar}}\left(\frac{e^{i\omega t} - e^{-i\omega_0 t}}{\omega_0 + \omega} - \frac{e^{-i\omega t} - e^{-i\omega_0 t}}{\omega_0 - \omega}\right)$$

$$= \left(z_0 + i\frac{A}{\sqrt{2m\omega_0\hbar}}\,\frac{\omega}{\omega_0^2 - \omega^2}\right) e^{-i\omega_0 t}$$

$$- i\frac{A}{\sqrt{2m\omega_0\hbar}}\,\frac{1}{\omega_0^2 - \omega^2}\left(\omega \cos\omega t - i\omega_0 \sin\omega t\right)$$

For the coherent state: $x(t) = \langle\psi(t)|\,\hat{x}\,|\psi(t)\rangle$ with $x(t) = \sqrt{2\hbar/m\omega_0}\,\mathrm{Re}\,z(t)$. $x(t)$ satisfies the same eq. of motion as the Heisenberg eq. of motion for $\hat{x}$. This is identical to the class. eq. of motion.

Can also be verified by explicit calculation.

FYS4110 Midttermineksamen, høsten 2009

Løsninger

Oppgave 1

a) Egenverdier til $\hat{H}_0 = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \frac{1}{2}\hbar\omega_0\sigma_z$

Benytter $\hat{a}^\dagger\hat{a}\,|n,m\rangle = n|n,m\rangle$

$\sigma_z\,|n,m\rangle = 2m|n,m\rangle \qquad m = \pm\frac{1}{2}$

$\Rightarrow$ egenverdier: $\underline{E^0_{nm} = \hbar\left[(n+\frac{1}{2})\omega + m\omega_0\right]}$

Operatorene $\hat{a}\sigma_+$ og $\hat{a}^\dagger\sigma_-$ kobler (har matriseelementer) mellom tilstander med energiforskjell $\Delta E = \pm\hbar(\omega-\omega_0)$, mens operatorene $\hat{a}\sigma_-$ og $\hat{a}^\dagger\sigma_+$ kobler tilstander med energiforskjell $\Delta E = \pm\hbar(\omega+\omega_0)$. $\hat{H}_1$ blander sammen egentilstandene til $\hat{H}_0$ mer effektivt når energidifferansen er liten enn når den er stor. (Se f.eks. uttrykk i perturbasjonsteori). Når $|\omega+\omega_0| \gg |\omega-\omega_0|$ er derfor betydningen av leddene som er strøket mye mindre enn betydningen av de som er beholdt.

b) Operatorene $\hat{a}\sigma_+$ og $\hat{a}\sigma_-$ kobler sammen par av tilstander $|n,-\frac{1}{2}\rangle$ og $|n-1,+\frac{1}{2}\rangle$ for $n = 1, 2, \ldots$:

$\hat{a}\sigma_+\,|n,-\frac{1}{2}\rangle = \sqrt{n}\,|n-1,+\frac{1}{2}\rangle$

$\hat{a}^\dagger\sigma_-\,|n,-\frac{1}{2}\rangle = 0$

$\hat{a}\sigma_+\,|n-1,+\frac{1}{2}\rangle = 0$

$\hat{a}^\dagger\sigma_-\,|n-1,+\frac{1}{2}\rangle = \sqrt{n}\,|n,-\frac{1}{2}\rangle$

ingen kobling mellom $|n,-\frac{1}{2}\rangle$, $|n-1,+\frac{1}{2}\rangle$ og andre egentilstander til $\hat{H}_0$. Operatoren $\hat{H}_1$ kobler derfor også bare disse parene av tilstander.

Spesielt ; n = 1 :

$$\langle 0,+\tfrac{1}{2}|\hat{H}_1|1,-\tfrac{1}{2}\rangle = \langle 1,-\tfrac{1}{2}|\hat{H}_1|0,+\tfrac{1}{2}\rangle = \tfrac{1}{2}\hbar\lambda$$

$$\langle 0,+\tfrac{1}{2}|\hat{H}_1|0,+\tfrac{1}{2}\rangle = \langle 1,-\tfrac{1}{2}|\hat{H}_1|1,-\tfrac{1}{2}\rangle = 0$$

Egenverdiligning i to-dimensjonalt underrom

$$\hat{H}|\psi\rangle = E|\psi\rangle \quad \text{med} \quad |\psi\rangle = c_1|0,+\tfrac{1}{2}\rangle + c_2|1,-\tfrac{1}{2}\rangle$$

på matriseform

$$\tfrac{1}{2}\hbar\begin{pmatrix}\omega+\omega_0 & \lambda \\ \lambda & 3\omega-\omega_0\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = E\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}\omega_0-\omega-\varepsilon & \lambda \\ \lambda & \omega-\omega_0-\varepsilon\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = 0$$

med $\varepsilon = 2\left(\frac{E}{\hbar} - \omega\right)$

Determinant-betingelse

$$\begin{vmatrix}\omega_0-\omega-\varepsilon & \lambda \\ \lambda & \omega-\omega_0-\varepsilon\end{vmatrix} = 0$$

$$\Rightarrow \quad \varepsilon^2 - (\omega-\omega_0)^2 - \lambda^2 = 0 \quad \Rightarrow \quad \varepsilon_\pm = \pm\sqrt{(\omega-\omega_0)^2+\lambda^2} \equiv \pm\Omega$$

$$E_\pm = \hbar(\omega + \tfrac{1}{2}\varepsilon_\pm)$$
$$= \underline{\hbar(\omega \pm \tfrac{1}{2}\Omega)} = \underline{\hbar(\omega \pm \tfrac{1}{2}\sqrt{\Delta\omega^2+\lambda^2})}$$

c) Koeffisienter c_1 og c_2

$$E_+ : \quad (\omega_0-\omega-\varepsilon_+)c_{1+} + \lambda c_{2+} = 0$$

$$\Rightarrow \quad (\Delta\omega+\Omega)c_{1+} = \lambda c_{2+}$$

$$\Rightarrow \quad c_{1+} = N\lambda \; ; \quad c_{2+} = N(\Delta\omega+\Omega)$$

normalisering : $|c_{1+}|^2 + |c_{2+}|^2 = 1 \Rightarrow N = [(\Delta\omega+\Omega)^2+\lambda^2]^{-1/2}$

Def: $c_{1+} = \cos\beta \,, \quad c_{2+} = \sin\beta$

$$\Rightarrow \quad \cos\beta = \underline{\frac{\lambda}{\sqrt{(\Delta\omega+\Omega)^2+\lambda^2}}} = \underline{\frac{\lambda}{\sqrt{2(\Delta\omega^2+\lambda^2+\Delta\omega\sqrt{\Delta\omega^2+\lambda^2}}}} = \underline{\frac{\lambda}{\sqrt{2\Omega(\Omega+\Delta\omega)}}}$$

$$\sin\beta = -\underline{\frac{\Delta\omega+\Omega}{\sqrt{(\Delta\omega+\Omega)^2+\lambda^2}}} = -\underline{\frac{\Delta\omega+\sqrt{\Delta\omega^2+\lambda^2}}{\sqrt{2(\Delta\omega^2+\lambda^2+\Delta\omega\sqrt{\Delta\omega^2+\lambda^2}}}} = \underline{-\sqrt{\frac{\Omega+\Delta\omega}{2\Omega}}}$$

Egentilstand med egenverdi E_-

ortogonalitet $\langle\psi_+|\psi_-\rangle = 0 \Rightarrow c_{1+}^* c_{1-} + c_{2+}^* c_{2-} = 0$

$$\Rightarrow \quad c_{1-} = -c_{2+} = +\sin\beta$$

$$\underline{c_{2-} = c_{1+} = \cos\beta}$$

(Entydig opp til multiplikasjon med en felles fasefaktor.)

d) Initialtilstand

$$|\psi(0)\rangle = |0,+\tfrac{1}{2}\rangle = \cos\beta\,|\psi_+\rangle + \sin\beta\,|\psi_-\rangle$$

Tidsutvikling

$$|\psi(t)\rangle = \cos\beta\, e^{-\frac{i}{\hbar}E_+t}|\psi_+\rangle + \sin\beta\, e^{-\frac{i}{\hbar}E_-t}|\psi_-\rangle$$

$$= (\cos^2\beta\, e^{-\frac{i}{\hbar}E_+t} + \sin^2\beta\, e^{-\frac{i}{\hbar}E_-t})|0,+\tfrac{1}{2}\rangle$$

$$- \cos\beta\sin\beta\,(e^{-\frac{i}{\hbar}E_+t} - e^{-\frac{i}{\hbar}E_-t})|1,-\tfrac{1}{2}\rangle$$

$$= c_1(t)|0,+\tfrac{1}{2}\rangle + c_2(t)|1,-\tfrac{1}{2}\rangle$$

koeffisienter

$$c_1(t) = e^{-i\omega t}(\cos^2\beta\, e^{-\frac{i}{2}\Omega t} + \sin^2\beta\, e^{\frac{i}{2}\Omega t})$$

$$= \underline{e^{-i\omega t}(\cos\tfrac{\Omega}{2}t - i\cos2\beta\,\sin\tfrac{\Omega}{2}t)}$$

$$= \underline{e^{-i\omega t}(\cos\tfrac{\Omega}{2}t + i\,\frac{\Delta\omega}{\Omega}\sin\tfrac{\Omega}{2}t}$$

$$c_2(t) = \cos\beta\sin\beta\, e^{-i\omega t}(e^{\frac{i}{2}\Omega t} - e^{-\frac{i}{2}\Omega t}) = \underline{i\, e^{-i\omega t}\sin2\beta\,\sin\tfrac{\Omega}{2}t}$$

$$= \underline{-i\, e^{-i\omega t}\,\frac{\lambda}{\Omega}\sin\tfrac{\Omega}{2}t}$$

e) $$|c_2(t)|^2 = \sin^2 2\beta \, \sin^2 \frac{\Omega}{2} t$$

$$= \frac{1}{2} \sin^2 2\beta \, (1 - \cos\Omega t)$$

$\cos\Omega t$ - periodisk funksjon med periode $T = \frac{2\pi}{\Omega} = \frac{2\pi}{\sqrt{\Delta\omega^2 + \lambda^2}}$

Maksimalverdi for $|c_2|^2$:

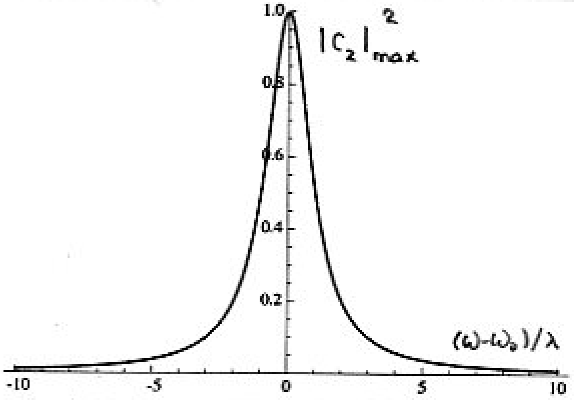
for $\sin^2 \frac{\Omega}{2} t = 1 \Rightarrow t = T(n + \frac{1}{2})$ n-heltall

$$|c_2|^2_{max} = \sin^2 2\beta = (2\sin\beta\cos\beta)^2$$

$$2\sin\beta\cos\beta = -2 \frac{\lambda}{\sqrt{2\Omega(\Omega+\Delta\omega)}} \sqrt{\frac{\Omega+\Delta\omega}{2\Omega}} = -\frac{\lambda}{\Omega}$$

$$\Rightarrow |c_2|^2_{max} = \frac{\lambda^2}{\Omega^2} = \frac{\lambda^2}{\Delta\omega^2 + \lambda^2} = \frac{\lambda^2}{(\omega-\omega_0)^2 + \lambda^2}$$

Som funksjon av ω_0, med ω og λ fast:



Resonans for $\omega_0 = \omega \Rightarrow |c_2|^2_{max} = 1$, størst mulig verdi.

f) Tetthetsoperator

$$\hat{\rho}(t) = |c_1(t)|^2 |0,+\tfrac{1}{2}\rangle\langle 0,+\tfrac{1}{2}| + |c_2(t)|^2 |1,-\tfrac{1}{2}\rangle\langle 1-\tfrac{1}{2}|$$
$$+ c_1(t)c_2(t)^* |0,+\tfrac{1}{2}\rangle\langle 1,-\tfrac{1}{2}| + c_1(t)^* c_2(t) |1,-\tfrac{1}{2}\rangle\langle 0,+\tfrac{1}{2}|$$

Redusert tetthetsoperator for spinn

$$\hat{\rho}_s(t) = \sum_n \langle n|\hat{\rho}(t)|n\rangle$$

$$= |c_1(t)|^2 |+\tfrac{1}{2}\rangle\langle+\tfrac{1}{2}| + |c_2(t)|^2 |-\tfrac{1}{2}\rangle\langle-\tfrac{1}{2}|$$

$$= (1 - \sin^2 2\beta \sin^2 \tfrac{\Omega}{2} t) |+\tfrac{1}{2}\rangle\langle+\tfrac{1}{2}| + \sin^2 2\beta \sin^2 \tfrac{\Omega}{2} t \, |-\tfrac{1}{2}\rangle\langle-\tfrac{1}{2}|$$

Redusert posisjons-tetthetsmatrise

$$\hat{\rho}_P(t) = \sum_{m=-1/2}^{+1/2} \langle m|\hat{\rho}(t)|m\rangle$$

$$= |c_1(t)|^2 |0\rangle\langle 0| + |c_2(t)|^2 |1\rangle\langle 1|$$

$$= \underline{(1-\sin^2 2\beta \sin^2 \tfrac{\Omega}{2}t)|0\rangle\langle 0| + \sin^2 2\beta \sin^2\tfrac{\Omega}{2}t\, |1\rangle\langle 1|}$$

g) Forventningsverdier

$$\langle \vec{\sigma}(t)\rangle = Tr_s(\vec{\sigma}\hat{\rho}_s(t))$$

$$= |c_1(t)|^2 \langle +\tfrac{1}{2}|\vec{\sigma}|+\tfrac{1}{2}\rangle + |c_2(t)|^2\langle -\tfrac{1}{2}|\vec{\sigma}|-\tfrac{1}{2}\rangle$$

$$= (|c_1(t)|^2 - |c_2(t)|^2)\,\vec{k}$$

$$= \underline{(1-2\sin^2 2\beta\sin^2\tfrac{\Omega}{2}t)\,\vec{k}}$$

$$\langle x(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\, Tr_P((\hat{a}+\hat{a}^\dagger)\hat{\rho}_P(t)) \underline{\;= 0\;}$$

$$\langle x\vec{\sigma}(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\, Tr((\hat{a}+\hat{a}^\dagger)\vec{\sigma}\hat{\rho}(t))$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\Big\{\langle 0,\tfrac{1}{2}|\hat{a}\,\sigma_+|1,-\tfrac{1}{2}\rangle\, c_1(t)c_2(t)^*(\vec{i}+i\vec{j})$$

$$+\langle 1,-\tfrac{1}{2}|\hat{a}\,\sigma_-|0,+\tfrac{1}{2}\rangle\, c_1(t)^* c_2(t)(\vec{i}-i\vec{j})$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\Big\{(c_1(t)c_2(t)^* + c_1(t)^*c_2(t))\,\vec{i} + i(c_1(t)c_2(t)^* - c_1(t)^*c_2(t))\,\vec{j}\Big\}$$

$$= \underline{-\sqrt{\frac{\hbar}{2m\omega}}\,(\sin 4\beta\sin^2\tfrac{\Omega}{2}t\,\vec{i} - \sin 2\beta\sin\Omega t\,\vec{j})}$$

Partikkelen oscillerer i potensialet samtidig som spinnet preseserer rundt $\vec{B}$-feltet. Variasjonen i $\langle\vec{\sigma}(t)\rangle$ viser at energien oscillerer mellom spinn-energi og bevegelsesenergi. Tidsmidlet posisjon er $\langle x\rangle = 0$, mens $\langle x\vec{\sigma}(t)\rangle$ viser at oscillasjonene i x-koordinaten er korrelert med spinnbevegelsen.

Oppgave 2

a) Unitaritet

$$S_\lambda^\dagger = e^{\frac{1}{2}(\lambda\hat{a}^\dagger - \lambda^*\hat{a})} = S_\lambda^{-1} \Rightarrow S_\lambda^\dagger S_\lambda = \mathbb{1}$$

Transformasjon av senkeoperator

$$\hat{b}_\lambda = S_\lambda \hat{a} S_\lambda^\dagger = e^{\chi}\hat{a}e^{-\chi} \qquad \chi = \tfrac{1}{2}(\lambda^*\hat{a}^2 - \lambda\hat{a}^{\dagger 2})$$

$$= \hat{a} + [\chi, \hat{a}] + \frac{1}{2!}[\chi, [\chi, \hat{a}]] + \cdots$$

$$[\chi, \hat{a}] = -\tfrac{1}{2}\lambda[\hat{a}^{\dagger 2}, \hat{a}] = \lambda\hat{a}^\dagger$$

$$[\chi, \hat{a}^\dagger] = \tfrac{1}{2}\lambda^*[\hat{a}^2, \hat{a}^\dagger] = \lambda^*\hat{a}$$

$$\Rightarrow \hat{b}_\lambda = \hat{a} + \lambda\hat{a}^\dagger + \tfrac{1}{2}|\lambda|^2 a + \frac{1}{3!}\lambda|\lambda|^2\hat{a}^\dagger +$$

$$= \hat{a}\left(1 + \frac{1}{2!}|\lambda|^2 + \frac{1}{4!}|\lambda|^4 + \cdots\right)$$

$$+ \frac{\lambda}{|\lambda|}\hat{a}^\dagger\left(|\lambda| + \frac{1}{3!}|\lambda|^3 + \cdots\right)$$

$$= \underline{\cosh|\lambda|\,\hat{a} + \frac{\lambda}{|\lambda|}\sinh|\lambda|\,\hat{a}^\dagger}$$

$$\Rightarrow \underline{\hat{b}_\lambda^\dagger = \cosh|\lambda|\,\hat{a}^\dagger + \frac{\lambda^*}{|\lambda|}\sinh|\lambda|\,\hat{a}}$$

$$[\hat{a}, \hat{a}^\dagger] = \mathbb{1} \Rightarrow$$

$$[\hat{b}_\lambda, \hat{b}_\lambda^\dagger] = [S_\lambda\hat{a}S_\lambda^\dagger, S_\lambda\hat{a}^\dagger S_\lambda^\dagger]$$

$$= S_\lambda[\hat{a}, \hat{a}^\dagger]S_\lambda^\dagger = S_\lambda S_\lambda^\dagger = \underline{\mathbb{1}}$$

Samme kommutator

b) Egenvektor til $\hat{b}_\lambda$,

$$\hat{b}_\lambda |z,\lambda\rangle = \hat{b}_\lambda S_\lambda |z\rangle$$

$$= S_\lambda S_\lambda^+ \hat{b}_\lambda S_\lambda |z\rangle$$

$$\hat{b}_\lambda = S_\lambda \hat{a} S_\lambda^+ \Rightarrow \hat{a} = S_\lambda^+ \hat{b}_\lambda S_\lambda .$$

$$\hat{b}_\lambda |z,\lambda\rangle = S_\lambda \hat{a} |z\rangle$$

koherent tilstand: $\hat{a}|z\rangle = z|z\rangle$

$$\Rightarrow \quad \hat{b}_\lambda |z,\lambda\rangle = z S_\lambda |z\rangle = \underline{z|z,\lambda\rangle}$$

$|z,\lambda\rangle$ egentilstand, med z som egenverdi

c) $\lambda = \lambda^*$ reell:

$$\Rightarrow \quad \hat{b}_\lambda = \cosh\lambda\, \hat{a} + \sinh\lambda\, \hat{a}^+$$

$$\hat{b}_\lambda^+ = \cosh\lambda\, \hat{a}^+ + \sinh\lambda\, \hat{a}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^+) \qquad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - i\hat{a}^+)$$

$$S_\lambda \hat{x} S_\lambda^+ = \sqrt{\frac{\hbar}{2m\omega}}(\hat{b}_\lambda + \hat{b}_\lambda^+)$$

$$= (\cosh\lambda + \sinh\lambda)\sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^+)$$

$$= \underline{e^\lambda \hat{x}}$$

$$S_\lambda \hat{p} S_\lambda^+ = -i\sqrt{\frac{\hbar}{2m\omega}}(\hat{b}_\lambda - \hat{b}_\lambda^+)$$

$$= (\cosh\lambda - \sinh\lambda)\left(-i\sqrt{\frac{\hbar}{2m\omega}}(\hat{a} - \hat{a}^+)\right)$$

$$= \underline{e^{-\lambda}\hat{p}}$$

dvs $S_\lambda \hat{x} S_\lambda^+ = d\,\hat{x}$, $S_\lambda \hat{p} S_\lambda^+ = \frac{1}{d}\hat{p}$, med $\underline{d = e^\lambda}$

$$\Rightarrow \quad \Delta x_{z\lambda}^2 = \langle z|(S_\lambda^+ \hat{x} S_\lambda)^2|z\rangle - \langle z|S_\lambda^+ \hat{x} S_\lambda|z\rangle^2 = \frac{1}{d^2}\Delta x_z^2 \,; \quad \Delta p_{z\lambda}^2 = d^2 \Delta p_z^2$$

$$\Rightarrow \quad \Delta x_{z\lambda}\Delta p_{z\lambda} = \Delta x_z \Delta p_z = \frac{\hbar}{2}$$, samme som for koherent tilstand

d) Presset grunntilstand

$$|0,\lambda\rangle = S_\lambda |0\rangle$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\tfrac{1}{2}(\lambda^* \hat{a}^2 - \lambda \hat{a}^{+2})\right)^n |0\rangle$$

S_λ inneholder bare kvadratiske operatorer i $\hat{a}$ og $\hat{a}^+$.
Kan derfor bare heve n med et <u>like</u> antall trinn fra $n=0$.

Benytter $\hat{b}_\lambda |0,\lambda\rangle = 0$ fra b)
og $\hat{b}_\lambda = \cosh|\lambda|\, \hat{a} + \frac{\lambda}{|\lambda|} \sinh|\lambda|\, \hat{a}^+$ fra a)

$$\hat{b}_\lambda \sum_n c_n |2n\rangle = 0 \Rightarrow$$

$$\cosh|\lambda| \sum_n c_n \sqrt{2n}\, |2n-1\rangle + \frac{\lambda}{|\lambda|} \sinh|\lambda| \sum_n c_n \sqrt{2n+1}\, |2n+1\rangle = 0$$

$$\Rightarrow \sum_n \left(\cosh|\lambda| \sqrt{2n}\, c_n + \frac{\lambda}{|\lambda|} \sinh|\lambda| \sqrt{2n-1}\, c_{n-1}\right) |2n-1\rangle = 0$$

Hver koeffisient i rekken må forsvinne:

$$\cosh|\lambda| \sqrt{2n}\, c_n + \frac{\lambda}{|\lambda|} \sinh|\lambda| \sqrt{2n-1}\, c_{n-1} = 0$$

$$\Rightarrow \quad c_n = -\frac{\lambda}{|\lambda|} \tanh|\lambda| \sqrt{\frac{2n-1}{2n}}\, c_{n-1}$$

$$= \left(-\frac{\lambda}{|\lambda|} \tanh|\lambda|\right)^n \sqrt{\frac{(2n-1)(2n-3)\cdots 1}{2n(2n-2)\cdots 2}}\, c_0$$

$$= \left(-\frac{\lambda}{|\lambda|} \tanh|\lambda|\right)^n \frac{\sqrt{(2n)!}}{2^n\, n!}\, c_0$$

$$= \left(-\frac{\lambda}{2|\lambda|} \tanh|\lambda|\right)^n \frac{\sqrt{(2n)!}}{n!}\, c_0$$

Normalisering $\quad \sum_n |c_n|^2 = 1$

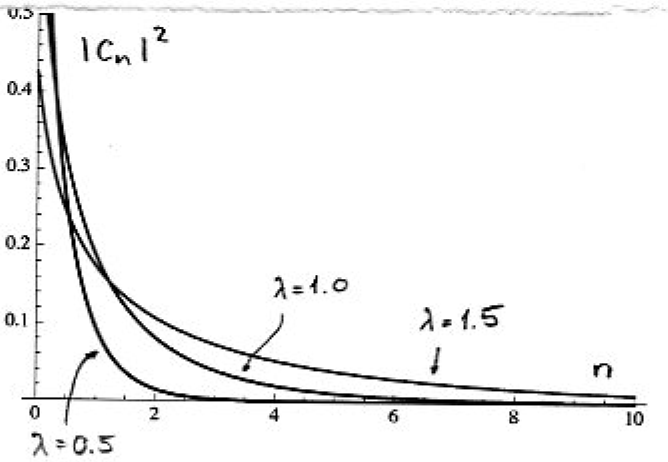
$$\Rightarrow |c_0|^{-2} = \sum_{n=0}^{\infty} |a|^{2n} \frac{2n!}{(n!)^2} \qquad |a| = \tfrac{1}{2}\tanh|\lambda|$$

$$= \frac{1}{\sqrt{1-4|a|^2}} = \frac{1}{\sqrt{1-\tanh^2|\lambda|}} = \cosh|\lambda|$$

koeffisienter

$$c_n = \frac{1}{\sqrt{\cosh|\lambda|}}\left(-\frac{\lambda}{2|\lambda|}\tanh|\lambda|\right)^n \frac{\sqrt{(2n)!}}{n!}$$

e) Plot av $|c_n|^2$



$|c_n|^2$ faller monotont av med økende n

For $\lambda = 0$ er bare $|c_0|^2 \neq 0$ $(=1)$

når $\lambda \neq 0$ er $|c_n|^2 \neq 0$ for alle n,

og jo større λ desto langsommere avtar $|c_n|^2$ med økende n

f) Benytter at $|z,\lambda\rangle$ er egentilstand for $\hat b_\lambda$, for generell λ. Studerer

$$\hat b_\lambda e^{-\frac{i}{\hbar}\hat H t}|z_0,\lambda_0\rangle \qquad \text{for uspesifisert } \lambda$$

$$= e^{-\frac{i}{\hbar}\hat H t} e^{\frac{i}{\hbar}\hat H t}\left(\cosh|\lambda|\,\hat a + \frac{\lambda}{|\lambda|}\sinh|\lambda|\,\hat a^\dagger\right) e^{-\frac{i}{\hbar}\hat H t}|z_0,\lambda\rangle$$

$$e^{\frac{i}{\hbar}\hat H t}\,\hat a\, e^{-\frac{i}{\hbar}\hat H t} = e^{i\hat a^\dagger \hat a t}\,\hat a\, e^{-i\hat a^\dagger \hat a t} = e^{-i\omega t}\,\hat a$$

$$e^{\frac{i}{\hbar}\hat H t}\,\hat a^\dagger e^{-\frac{i}{\hbar}\hat H t} = e^{i\omega t}\,\hat a^\dagger$$

$$\Rightarrow \hat b_\lambda e^{-\frac{i}{\hbar}\hat H t}|z_0,\lambda\rangle = e^{-\frac{i}{\hbar}\hat H t}e^{-i\omega t}\left(\cosh|\lambda|\hat a + \frac{\lambda e^{2i\omega t}}{|\lambda|}\sinh|\lambda|\right)|z_0,\lambda\rangle$$

$$= e^{-i\omega t}e^{-\frac{i}{\hbar}\hat H t}\hat b_{(\lambda e^{2i\omega t})}|z_0,\lambda_0\rangle$$

Uttrykket gjelder for vilkårlig valgt λ.

Velger nå $\lambda e^{2i\omega t} = \lambda_0$, dvs $\lambda = \lambda_0 e^{-2i\omega t}$ $\Rightarrow$

$$\hat b_{(\lambda e^{2i\omega t})}|z_0,\lambda\rangle = \hat b_{\lambda_0}|z_0,\lambda_0\rangle = z_0|z_0,\lambda\rangle$$

$$\Rightarrow \hat b_{(\lambda_0 e^{-2i\omega t})}e^{-\frac{i}{\hbar}\hat H t}|z_0,\lambda\rangle = e^{-i\omega t}z_0 e^{-\frac{i}{\hbar}\hat H t}|z_0,\lambda\rangle$$

dvs $e^{-\frac{i}{\hbar}\hat H t}|z_0,\lambda\rangle$ er egenvektor for $\hat b_{(\lambda_0 e^{-2i\omega t})}$

med egenverdi $z_0 e^{-i\omega t}$

$$\Rightarrow \underline{e^{-\frac{i}{\hbar}\hat H t}|z_0,\lambda\rangle = e^{i\alpha(t)}|z_0 e^{-i\omega t}, \lambda_0 e^{-2i\omega t}\rangle}$$

$\alpha(t)$ ubestemt kompleks fase

Tidsutvikling, presset tilstand på formen $e^{i\alpha(t)}|z(t),\lambda(t)\rangle$

med $z(t) = z_0 e^{-i\omega t}$ og $\lambda(t) = \lambda_0 e^{-2i\omega t}$

g) $z_0 = 0$

$\langle\hat x\rangle = \langle\hat p\rangle = 0$; $\hat x$ og $\hat p$ er lineære i $\hat a$ og $\hat a^+$

$\Rightarrow$ alle matriseelementer mellom tilstandene $|2n\rangle$ forsvinner.

$$\Rightarrow \Delta x^2 = \langle\hat x^2\rangle = \frac{\hbar}{2m\omega}\langle(\hat a+\hat a^+)^2\rangle$$

$$\Delta p^2 = \langle\hat p^2\rangle = -\frac{\hbar m\omega}{2}\langle(\hat a-\hat a^+)^2\rangle$$

benytter:

$\hat a + \hat a^+ = c b_\lambda + c^* b_\lambda^+$; $c = \cosh|\lambda| - \frac{\lambda^*}{|\lambda|}\sinh|\lambda|$

$\hat a - \hat a^+ = d b_\lambda - d^* b_\lambda^+$; $d = \cosh|\lambda| + \frac{\lambda^*}{|\lambda|}\sinh|\lambda|$

$$\langle(\hat{a}+\hat{a}^{+})^{2}\rangle = c^{2}\langle 0,\lambda|\,\hat{b}_{\lambda}^{2}\,|0,\lambda\rangle + c^{*2}\langle 0,\lambda|\,\hat{b}_{\lambda}^{+2}\,|0,\lambda\rangle$$

$$+\, cc^{*}\langle 0,\lambda|\,\hat{b}_{\lambda}^{+}\hat{b}_{\lambda} + \hat{b}_{\lambda}^{+} b_{\lambda}\,|0,\lambda\rangle$$

$$\lambda = \lambda(t) = \lambda_{0}e^{-2i\omega t}$$

benytter $\langle\hat{b}_{\lambda}^{2}\rangle = \langle\hat{b}_{\lambda}^{+2}\rangle = \langle\hat{b}_{\lambda}^{+} b_{\lambda}\rangle = 0$

$$\langle\hat{b}_{\lambda}\hat{b}_{\lambda}^{+}\rangle = 1$$

$$\Rightarrow \quad \langle(\hat{a}+\hat{a}^{+})^{2}\rangle = |c|^{2} = \cosh 2|\lambda| - \sinh 2|\lambda|\,\frac{\mathrm{Re}\,\lambda}{|\lambda|}$$

$$= \cosh 2\lambda_{0} - \sinh 2\lambda_{0}\,\cos 2\omega t$$

Tilsvarende

$$\langle(\hat{a}-\hat{a}^{+})^{2}\rangle = |d|^{2} = \cosh 2\lambda_{0} + \sinh 2\lambda_{0}\cos 2\omega t$$

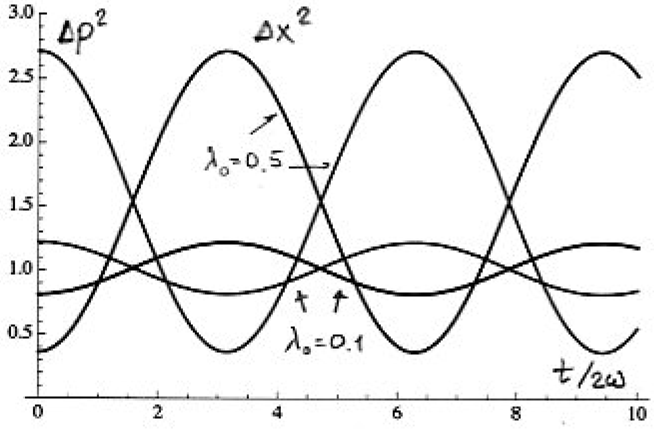
$$\underline{\Delta x^{2} = \frac{\hbar}{2m\omega}\left(\cosh 2\lambda_{0} - \sinh 2\lambda_{0}\cos 2\omega t\right)}$$

$$\underline{\Delta p^{2} = \frac{\hbar m\omega}{2}\left(\cosh 2\lambda_{0} + \sinh 2\lambda_{0}\cos 2\omega t\right)}$$

Plot av Δx^{2} og Δp^{2}, normalisert med faktorene:

$$\Delta x^{2} \rightarrow \frac{2m\omega}{\hbar}\Delta x^{2}$$

$$\Delta p^{2} \rightarrow \frac{2}{\hbar m\omega}\Delta p^{2}$$



Variansene Δx^{2} og Δp^{2} varierer periodisk i t, med periode $\frac{\pi}{\omega}$; de varierer i motfase. Amplituden i oscillasjonene øker med λ_{0}.

Midtermineksamen, FYS 4110, høsten 2010
Løsninger

OPPGAVE 1

a) Hamiltonoperatoren i $\{|\psi_L\rangle, |\psi_R\rangle\}$ basis er

$$H = \begin{pmatrix} E_0 & \lambda \\ \lambda & E_0 \end{pmatrix} \tag{1}$$

Egenverdiene E er bestemt av ligningen,

$$\begin{vmatrix} E_0 - E & \lambda \\ \lambda & E_0 - E \end{vmatrix} = 0 \quad \Rightarrow \quad (E - E_0)^2 - \lambda^2 = 0 \tag{2}$$

Løsninger

$$E_0^{\pm} = E_0 \pm \lambda \tag{3}$$

Egenvektorer på matriseform

$$\psi_0^{\pm} = \begin{pmatrix} \alpha_0^{\pm} \\ \beta_0^{\pm} \end{pmatrix}, \quad |\alpha_0^{\pm}|^2 + |\beta_0^{\pm}|^2 = 1 \tag{4}$$

Koeffisientene er bestemt av egenverdiligningen

$$\begin{aligned} \begin{pmatrix} E_0 & \lambda \\ \lambda & E_0 \end{pmatrix} \begin{pmatrix} \alpha_0^{\pm} \\ \beta_0^{\pm} \end{pmatrix} &= E_0^{\pm} \begin{pmatrix} \alpha_0^{\pm} \\ \beta_0^{\pm} \end{pmatrix} \\ &\Rightarrow \\ (E_0 - E_{=}^{\pm})\alpha_0^{\pm} &= -\lambda\beta_0^{\pm} \\ &\Rightarrow \\ \alpha_0^{\pm} = \pm\beta_0^{\pm} &= \frac{1}{\sqrt{2}} \end{aligned} \tag{5}$$

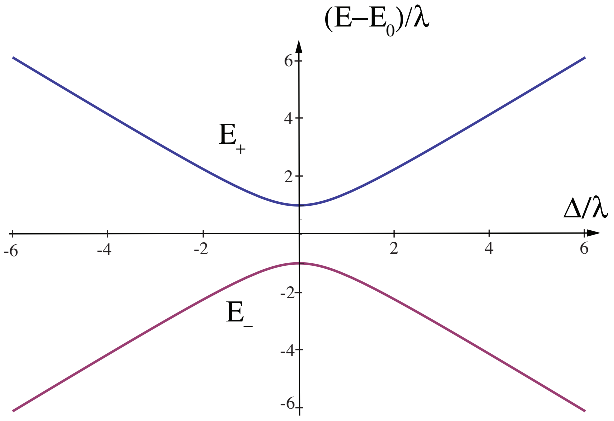
I braket-formulering

$$|\psi_0^{\pm}\rangle = \frac{1}{\sqrt{2}}(|\psi_L\rangle \pm |\psi_R\rangle) \tag{6}$$

Egenvektorene er den symmetriske og antisymmetriske superposisjon av $|\psi_L\rangle$ og $|\psi_R\rangle$. Den antisymmetriske superposisjon har lavest energi. Kan forstås ved at den har lavere sannsynlighet for at N-atomet befinner seg i potensialbarrieren hvor den potensielle energien er høyere.

b) Ny egenverdiligning

$$\begin{vmatrix} E_0 + \Delta - E & \lambda \\ \lambda & E_0 - \Delta - E \end{vmatrix} = 0 \quad \Rightarrow \quad (E - E_0)^2 = \lambda^2 + \Delta^2 \tag{7}$$

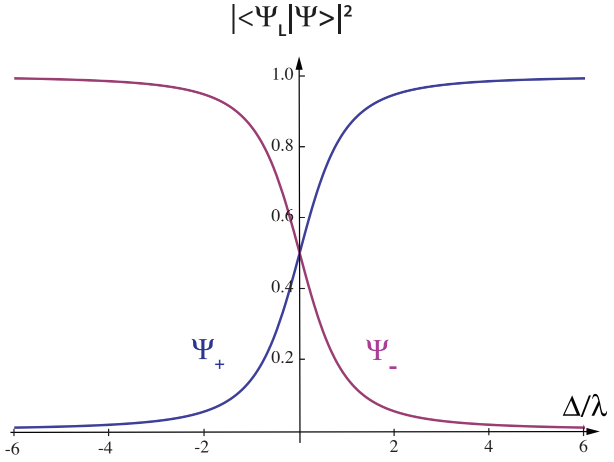


Løsninger

$$E_\pm = E_0 \pm \sqrt{\lambda^2 + \Delta^2} \tag{8}$$

c) Egenvektorer, matriseelementer

$$\begin{aligned} (E_0 + \Delta - E_\pm)\alpha_\pm + \lambda\beta_\pm &= 0 \quad \Rightarrow \\ (\Delta \mp \sqrt{\lambda^2 + \Delta^2})\alpha_\pm + \lambda\beta_\pm &= 0 \end{aligned} \tag{9}$$



Normerte løsninger

$$\begin{aligned} \alpha_\pm &= \frac{1}{\sqrt{2\sqrt{\lambda^2+\Delta^2}}}\sqrt{\sqrt{\lambda^2+\Delta^2} \pm \Delta} \\ \beta_\pm &= \pm\frac{1}{\sqrt{2\sqrt{\lambda^2+\Delta^2}}}\sqrt{\sqrt{\lambda^2+\Delta^2} \mp \Delta} \end{aligned} \tag{10}$$

Tilstander på braket-form

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2\sqrt{\lambda^2+\Delta^2}}}\left(\sqrt{\sqrt{\lambda^2+\Delta^2}\pm\Delta}\,|\psi_L\rangle \pm \sqrt{\sqrt{\lambda^2+\Delta^2}\mp\Delta}\,|\psi_R\rangle\right) \tag{11}$$

Overlapp

$$|\langle\psi_L|\psi_\pm\rangle|^2 = \frac{1}{2}\left(1\pm\frac{\Delta}{\sqrt{\lambda^2+\Delta^2}}\right) \tag{12}$$

Avoided crossing: Når Δ øker og passerer$\Delta = 0$ vil energinivåene nærme seg hverandre men unngår en direkte krysning ved en effektiv frastøtning mellom nivåene. Den minste avstanden er bestemt av λ. Tilstandsvektorene til de to nivåene byttes om nær dette punktet slik at grunntilstanden $|\psi_-\rangle$ svarer til $|\psi_L\rangle$ for stor negativ Δ og til $|\psi_R\rangle$ for stor positiv Δ.

d) Hamiltonoperator og tilstander i $\{|\psi_L\rangle, |\psi_R\rangle\}$ basis,

$$\hat{H} = \begin{pmatrix} E_0+\Delta & \lambda \\ \lambda & E_0-\Delta \end{pmatrix}, \quad \psi_0^\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{13}$$

Matriseelementer til $\hat{H}$ i $|\psi_0^\pm\rangle$ basis

$$\begin{aligned} \psi_0^{\pm\dagger}\hat{H}\psi_0^\pm &= \frac{1}{2}(1\ \ \pm 1)\begin{pmatrix} E_0+\Delta & \lambda \\ \lambda & E_0-\Delta \end{pmatrix}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = E_0 \pm \lambda \\ \psi_0^{\pm\dagger}\hat{H}\psi_0^\mp &= \frac{1}{2}(1\ \ \pm 1)\begin{pmatrix} E_0+\Delta & \lambda \\ \lambda & E_0-\Delta \end{pmatrix}\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} = \Delta \end{aligned} \tag{14}$$

Det gir følgende matriseform for H i $|\psi_\pm\rangle$ basis,

$$\hat{H} = \begin{pmatrix} E_0+\lambda & \Delta \\ \Delta & E_0-\lambda \end{pmatrix} = E_0\mathbb{1} + \lambda\sigma_z + \Delta\sigma_x \tag{15}$$

og i det oscillerende elektriske felt, hvor $\Delta = \Delta_0\cos\omega t$, blir Hamiltonoperatoren

$$\hat{H} = E_0\mathbb{1} + \lambda\sigma_z + \Delta_0\cos\omega t\sigma_x \tag{16}$$

e) I den roterende bølge-tilnærmelsen får H følgende form

$$\begin{aligned} \hat{H} &= E_0\mathbb{1} + \lambda\sigma_z + \frac{1}{2}\Delta_0(e^{i\omega t}\sigma_- + e^{-i\omega t}\sigma_+) \\ &= E_0\mathbb{1} + \lambda\sigma_z + \frac{1}{2}\Delta_0(\cos\omega t\sigma_x + \sin\omega t\sigma_y) \end{aligned} \tag{17}$$

Den har samme form som Hamiltonoperatoren for et spinn-1/2-system i et konstant magnetfelt langs z-aksen superponert med et roterende magnetfelt i xy-planet. I forelesningsnotatene er Hamiltonoperatoren

$$\hat{H} = \frac{1}{2}\omega_0\hbar\sigma_z + \frac{1}{2}\omega_1\hbar(\cos\omega t\sigma_x + \sin\omega t\sigma_y) \tag{18}$$

hvor ω_0 er proporsjonal med styrken på det konstante feltet og ω_1 er proporsjonal med styrken på det roterende feltet. Sammenligningen av uttrykkene gir relasjonene

$$\lambda = \frac{1}{2}\omega_0\hbar\,, \quad \Delta_0 = \omega_1\hbar \tag{19}$$

I det følgende benyttes disse identitetene. Hamiltonoperatoren (17) har også et konstantledd $E_0\mathbb{1}$, men dette er ikke av betydning for tidsutviklingen av systemet, siden den bare bidrar med en felles fasefaktor for alle tilstandene. I det følgende settes $E_0 = 0$.

Hamiltonoperatoren transformeres til tidsuavhengig form med den unitære, tidsavhengige transformasjonen

$$\hat{T}(t) = e^{\frac{i}{2}\omega t \sigma_z} \tag{20}$$

Den transformerte $\hat{H}$ blir

$$\begin{aligned} \hat{H}_{\hat{T}} &= \hat{T}(t)\hat{H}\hat{T}(t)^\dagger + i\hbar\frac{d\hat{T}}{dt}\hat{T}(t) \\ &= \frac{1}{2}\hbar\Omega(\cos\theta\sigma_z + \sin\theta\sigma_x) \end{aligned} \tag{21}$$

hvor

$$\Omega = \sqrt{(\omega-\omega_0)^2 + \omega_1^2} = \frac{1}{\hbar}\sqrt{(\omega\hbar - 2\lambda)^2 + \Delta_0^2} \tag{22}$$

er Rabifrekvensen og hvor θ er bestemt ved ligningene

$$\begin{aligned} \cos\theta &= \frac{\omega_0 - \omega}{\Omega} = \frac{2\lambda - \Delta_0}{\sqrt{(\omega\hbar - 2\lambda)^2 + \Delta_0^2}} \\ \sin\theta &= \frac{\omega_1}{\Omega} = \frac{\Delta_0}{\sqrt{(\omega\hbar - 2\lambda)^2 + \Delta_0^2}} \end{aligned} \tag{23}$$

Resonansfrekvensen er

$$\omega_0 = 2\lambda/\hbar \tag{24}$$

Tidsutviklingsoperatoren i det transformerte bildet er

$$\hat{\mathcal{U}}_T(t) = \cos(\frac{\Omega}{2}t)\mathbb{1} - i\sin(\frac{\Omega}{2}t)(\cos\theta\sigma_z + \sin\theta\sigma_x) \tag{25}$$

I Schrödingerbildet

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{2}\omega t\sigma_z}\hat{\mathcal{U}}_T(t) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{26}$$

med matriseelementer

$$\begin{aligned} A &= (\cos(\frac{\Omega}{2}t) - i\cos\theta\sin(\frac{\Omega}{2}t))e^{-\frac{i}{2}\omega t} \\ D &= (\cos(\frac{\Omega}{2}t) + i\cos\theta\sin(\frac{\Omega}{2}t))e^{\frac{i}{2}\omega t} \\ B &= -i\sin\theta\sin(\frac{\Omega}{2}t))e^{-\frac{i}{2}\omega t} \\ C &= -i\sin\theta\sin(\frac{\Omega}{2}t))e^{\frac{i}{2}\omega t} \end{aligned} \tag{27}$$

(For detaljerte mellomregninger refereres til forelesningsnotatene.)

f) Tilstander i $|\psi_0^\pm\rangle$ basis,

$$|\psi_L\rangle = \frac{1}{\sqrt{2}}(|\psi_0^+\rangle + |\psi_0^-\rangle)\,, \quad |\psi_R\rangle = \frac{1}{\sqrt{2}}(|\psi_0^+\rangle - |\psi_0^-\rangle) \tag{28}$$

På matriseform

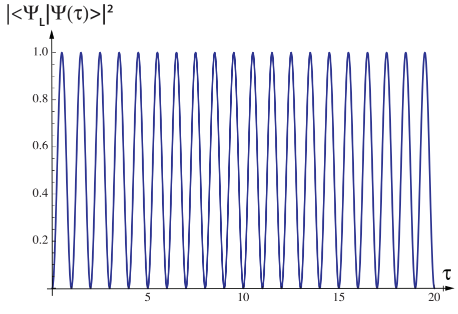
$$\psi_L = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\,, \quad \psi_R = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{29}$$

Overlapp

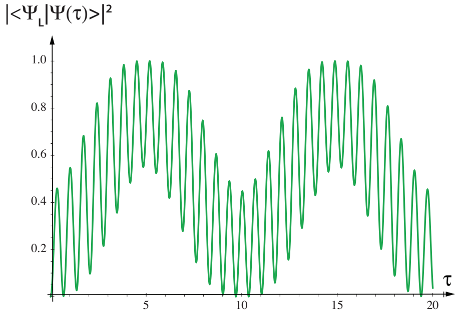
$$\begin{aligned}\langle\psi_R|\psi(t)\rangle &= \langle\psi_R|\hat{\mathcal{U}}(t)|\psi_L\rangle \\ &= \frac{1}{2}(1\ \ -1)\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \\ &= \frac{1}{2}((A-D)+(B-C))\end{aligned} \tag{30}$$

Innsatt for A, B, C, D,

$$\langle\psi_R|\psi(t)\rangle = -[\sin\theta\sin(\frac{\Omega}{2}t)\sin(\frac{\omega}{2}t) + i\{\cos(\frac{\Omega}{2}t)\sin(\frac{\omega}{2}t) + \cos\theta\sin(\frac{\Omega}{2}t)\cos(\frac{\omega}{2}t)\}] \tag{31}$$



ω = ω₀



ω = ω₀/10

g) Kvadrert uttrykk

$$
\begin{aligned}
|\langle\psi_R|\psi(t)\rangle|^2 &= [\sin\theta\sin(\frac{\Omega}{2}t)\sin(\frac{\omega}{2}t)]^2 + [\cos(\frac{\Omega}{2}t)\sin(\frac{\omega}{2}t) + \cos\theta\sin(\frac{\Omega}{2}t)\cos(\frac{\omega}{2}t)\}]^2 \\
&= \frac{1}{2}[1 - \cos\omega t + \cos^2\theta(1-\cos\Omega t)\cos\omega t + \cos\theta\sin\Omega t\sin\omega t] \qquad (32)
\end{aligned}
$$

Plot av funksjonen $|\langle\psi_R|\psi(t)\rangle|^2$ med $\tau = 2\pi\lambda t$ som tidskoordinat: De to figurene svarer til $\omega = \omega_0 = 2\lambda/\hbar$ og $\omega = \omega_0/10 = \lambda/5\hbar$. I begge tilfeller er $\omega_1 = \Delta_0/\hbar = 2\lambda/\hbar = \omega_0$.
Kommentar:
Ved resonans er oscillasjonene rene sinus-oscillasjonere med sirkelfrekvens ω_0. Det er det samme som når det periodiske feltet er slått av. Det er lett å sjekke av uttrykkene ovenfor at det oscillerende feltet ved resonans bare påvirker fasen til $\langle\psi_R|\psi(t)\rangle$. Ved $\omega = \omega_0/10$ er svingningene modulert av en langsommere oscillasjon som svarer omtrent til frekvensen ω. Den raskere frekvensen er også noe påvirket av oscillasjonene til det elektriske feltet. Uttrykket ovenfor viser at funksjonen $|\langle\psi_R|\psi(t)\rangle|^2$ er en lineær kombinasjon av tre periodiske funsjoner med frekvenser ω, $\Omega-\omega$ og $\Omega+\omega$.

OPPGAVE 2

a) Hamiltonoperator

$$
\begin{aligned}
\hat{H} &= \omega(\hat{S}_{1z} + \hat{S}_{2z}) + \frac{\alpha}{\hbar}[(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 - (\hat{\mathbf{S}}_1^2 + \hat{\mathbf{S}}_2^2)] \\
&= \omega\hat{S}_z + \frac{\alpha}{\hbar}[\hat{\mathbf{S}}^2 - \frac{3}{2}\hbar^2\mathbb{1}] \qquad (33)
\end{aligned}
$$

Egenverdier og egenvektorer

$$
\hat{H}|s,m\rangle = \left((s(s+1) - \frac{3}{2})\alpha + m\omega\right)\hbar\;|s,m\rangle \qquad (34)
$$

for de aktuelle tilstandene

$$
\begin{aligned}
\hat{H}|1,1\rangle &= (\frac{1}{2}\alpha + \omega)\hbar\;|1,1\rangle \\
\hat{H}|1,0\rangle &= \frac{1}{2}\alpha\hbar\;|1,0\rangle \\
\hat{H}|1,-1\rangle &= (\frac{1}{2}\alpha - \omega)\hbar\;|1,1\rangle \\
\hat{H}|1,1\rangle &= -\frac{3}{2}\alpha\hbar\;|1,1\rangle
\end{aligned}
\qquad (35)
$$

b) Initialtilstand

$$
\begin{aligned}
\hat{\rho}(0) &= |\psi(0)\rangle\langle\psi(0)| \\
&= \frac{1}{2}(|++\rangle\langle++| \;+\; |+-\rangle\langle+-| \;+\; |++\rangle\langle+-| \;+\; |+-\rangle\langle++|) \qquad (36)
\end{aligned}
$$

Tilstanden er ren siden kan uttrykkes ved en enkelt tilstandsvektor. Den er ukorrelert siden den kan skrives som en produktvektor,

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|+\rangle \otimes (|+\rangle + |-\rangle) \tag{37}$$

Det er derfor ingen klassisk korrelasjon eller kvantemekanisk sammenfiltring mellom delsystemene. Redusert tetthetsoperator for spinn 1

$$\begin{aligned}\hat{\rho}_1(0) &= \mathrm{Tr}_2\,\hat{\rho}(0) = |+\rangle\langle+| = \frac{1}{2}(\mathbb{1} + \sigma_z) \\ &\Rightarrow \quad \mathbf{r}_1 = \mathbf{k}\end{aligned} \tag{38}$$

Redusert tetthetsoperator for spinn 2

$$\begin{aligned}\hat{\rho}_2(0) &= \mathrm{Tr}_1\,\hat{\rho}(0) \\ &= \frac{1}{2}(|+\rangle\langle+| \;+\; |-\rangle\langle-| \;+\; |+\rangle\langle-| \;+\; |-\rangle\langle+|) \\ &= \frac{1}{2}(\mathbb{1} + \sigma_x) \\ &\Rightarrow \quad \mathbf{r}_2 = \mathbf{i}\end{aligned} \tag{39}$$

c) Initialtilstand

$$\begin{aligned}|\psi(0)\rangle &= \frac{1}{\sqrt{2}}(|++\rangle + \frac{1}{2}(|+-\rangle + |-+\rangle) + \frac{1}{2}(|+-\rangle - |-+\rangle)) \\ &= \frac{1}{\sqrt{2}}(|1,1\rangle + \frac{1}{2}|1,0\rangle + \frac{1}{2}|0,0\rangle)\end{aligned} \tag{40}$$

Tidsutvikling

$$\begin{aligned}|\psi(t)\rangle &= \frac{1}{\sqrt{2}}(e^{-i(\frac{1}{2}\alpha+\omega)t}\,|1,1\rangle + \frac{1}{2}e^{-i\frac{1}{2}\alpha t}\,|+-\rangle + \frac{1}{2}e^{i\frac{3}{2}\alpha t}\,|0,0\rangle) \\ &= \frac{1}{\sqrt{2}}(e^{-i(\frac{1}{2}\alpha+\omega)t}\,|++\rangle + \frac{1}{2}(e^{-i\frac{1}{2}\alpha t} + e^{i\frac{3}{2}\alpha t})\,|+-\rangle + \frac{1}{2}(e^{-i\frac{1}{2}\alpha t} - e^{i\frac{3}{2}\alpha t})\,|-+\rangle) \\ &= \frac{1}{\sqrt{2}}(e^{-i(\frac{1}{2}\alpha+\omega)t}\,|++\rangle + e^{i\frac{1}{2}\alpha t}\cos\alpha t\,|+-\rangle - ie^{i\frac{1}{2}\alpha t}\sin\alpha t\,|-+\rangle) \\ &\equiv A\,|++\rangle + B\,|+-\rangle + C\,|-+\rangle)\end{aligned} \tag{41}$$

Tetthetsoperator

$$\begin{aligned}\hat{\rho}(t) &= |A|^2\,|++\rangle\langle++| + |B|^2\,|+-\rangle\langle+-| + |C|^2\,|-+\rangle\langle-+| \\ &\quad +AB^*\,|++\rangle\langle+-| + A^*B\,|+-\rangle\langle++| \\ &\quad +AC^*\,|++\rangle\langle-+| + A^*C\,|-+\rangle\langle++| \\ &\quad +BC^*\,|+-\rangle\langle-+| + B^*C\,|-+\rangle\langle+-|\end{aligned} \tag{42}$$

Koeffisienter

$$\begin{aligned}|A|^2 = \frac{1}{2}\,, \quad |B|^2 = \frac{1}{2}\cos^2\alpha t\,, \quad |C|^2 = \frac{1}{2}\sin^2\alpha t \\ AB^* = \frac{1}{2}e^{-i(\alpha+\omega)t}\cos\alpha t\,, \quad A^*B = \frac{1}{2}e^{i(\alpha+\omega)t}\cos\alpha t \\ AC^* = \frac{i}{2}e^{-i(\alpha+\omega)t}\sin\alpha t\,, \quad A^*C = -\frac{i}{2}e^{i(\alpha+\omega)t}\sin\alpha t \\ BC^* = \frac{i}{4}\sin 2\alpha t\,, \quad B^*C = -\frac{i}{4}\sin 2\alpha t\end{aligned} \tag{43}$$

d) Redusert tetthetsoperator for spinn 1

$$\begin{aligned}\hat{\rho}_1(t) &= (|A|^2 + |B|^2)\,|+\rangle\langle+| + |C|^2\,|-\rangle\langle-| + AC^*\,|+\rangle\langle-| + A^*C\,|-\rangle\langle+| \\ &= \frac{1}{2}(1+\cos^2\alpha t)\,|+\rangle\langle+| + \frac{1}{2}\sin^2\alpha t\,|-\rangle\langle-| \\ &\quad + \frac{i}{2}e^{-i(\alpha+\omega)t}\sin\alpha t\,|+\rangle\langle-| - \frac{i}{2}e^{i(\alpha+\omega)t}\sin\alpha\,|-\rangle\langle+| \end{aligned} \tag{44}$$

Benytter

$$|\pm\rangle\langle\pm| = \frac{1}{2}(\mathbb{1} \pm \sigma_z)\,, \quad |\pm\rangle\langle\mp| = \frac{1}{2}(\sigma_x \pm i\sigma_y) \tag{45}$$

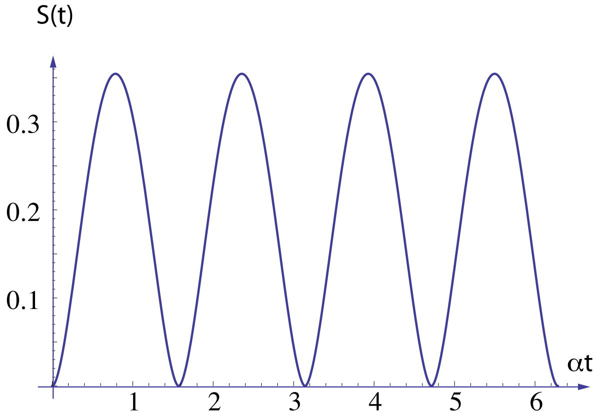
Det gir

$$\hat{\rho}_1(t) = \frac{1}{2}(\mathbb{1} + \cos^2\alpha t\,\sigma_z + \sin[(\alpha+\omega)t]\sin\alpha t\,\sigma_x - \cos[(\alpha+\omega)t]\sin\alpha t\,\sigma_y \tag{46}$$

og

$$\mathbf{r}_1(t) = \sin\alpha t\{\sin[(\alpha+\omega)t]\mathbf{i} - \cos[(\alpha+\omega)t]\mathbf{j}\} + \cos^2\alpha t\,\mathbf{k} \tag{47}$$

Når $\omega >> \alpha$ presseserer vektoren raskt rundt z-aksen, mens vinkelen mellom vektoren og z-aksen gjennomfører en mer langsom periodisk variasjon.



e) Tetthetsoperatoren $\hat{\rho}_1 = \frac{1}{2}(\mathbb{1} + \mathbf{r}_1 \cdot \boldsymbol{\sigma})$ har egenverdier

$$p_\pm = \frac{1}{2}(1 \pm r_1) \tag{48}$$

Sammenfiltringsentropien

$$S = -\left[\frac{1+r_1}{2}\log\frac{1+r_1}{2} + \frac{1-r_1}{2}\log\frac{1-r_1}{2}\right] \tag{49}$$

Tidsavhengighet til r_1,

$$r_1 = [\cos^4\alpha t + \sin^2\alpha t]^{1/2} = [1 - \frac{1}{4}\sin^2 2\alpha t]^{1/2} \tag{50}$$

Lign. (49) og (50) benyttes til å plotte tidsavhengigheten til $S(t)$. Basis-2 logaritme brukes.

Spinn 1 har maksimal blanding når r_1 er minst. Det svarer til størst sammenfiltringsentropi. Den størst mulige verdien for S svarer til $r_1 = 0$, som gir $S_{totmax} = \log 2 = 1.0$. Den maksimale verdi under tidsutviklingen oppnås når $\sin^2 2\alpha t = 1$, som gir $r_1 = \sqrt{3}/2$. Den tilsvarende sammenfiltringsentropien er $S_{max} = 2 - (\sqrt{3}/2)\log[2+\sqrt{3}] = 0.35$.

f) Heisenbergs ligning for det totale spinn er

$$\frac{d}{dt}\hat{\boldsymbol{S}} = \omega\,\mathbf{k} \times \hat{\mathbf{S}} \tag{51}$$

Forventningsverdien er

$$\left\langle \hat{\boldsymbol{S}} \right\rangle = \left\langle \hat{\boldsymbol{S}}_1 \right\rangle + \left\langle \hat{\boldsymbol{S}}_2 \right\rangle = \frac{\hbar}{2}(\mathbf{r}_1 + \mathbf{r}_2) = \frac{\hbar}{2}\mathbf{r} \tag{52}$$

Det gir bevegelsesligning

$$\frac{d\mathbf{r}}{dt} = \omega\,\mathbf{k} \times \mathbf{r} \tag{53}$$

Vektoren $\mathbf{r}$ presseserer om z-aksen med sirkelfrekvens ω. Ved $t = 0$ er vinkelen mellom $\mathbf{r}$ og z-aksen 45^o. Denne vinkelen er konstant under bevegelsen.

Midterm Exam FYS4110, fall semester 2011

Solutions

Problem 1

a) Total spin $\vec{S} = \frac{\hbar}{2}(\vec{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{\sigma}) \equiv \frac{\hbar}{2}(\vec{\Sigma}_A + \vec{\Sigma}_B)$

$$\vec{S}^2 = \frac{\hbar^2}{2}(3\mathbb{1}\otimes\mathbb{1} + \vec{\Sigma}_A\cdot\vec{\Sigma}_B)$$

$$= \frac{\hbar^2}{2}(3\mathbb{1} + \sum_{k=1}^{3}\sigma_k\otimes\sigma_k)$$

$$\sigma_k\otimes\sigma_k\,|\psi_a\rangle = -|\psi_a\rangle \qquad k = 1,2,3$$

$$\sigma_z\otimes\sigma_z\,|\psi_s\rangle = -|\psi_s\rangle$$

$$\sigma_x\otimes\sigma_x\,|\psi_s\rangle = +|\psi_s\rangle$$

$$\sigma_x\otimes\sigma_x\,|\psi_0\rangle = +|\psi_s\rangle$$

The three cases

$$\text{I}:\ \langle\vec{S}^2\rangle_1 = \langle\psi_a|\frac{\hbar^2}{2}(3\mathbb{1} + \sum_{k=1}^{3}\sigma_k\otimes\sigma_k)|\psi_a\rangle = \underline{0}$$

$$\text{II}:\ \langle\vec{S}^2\rangle_2 = \langle\psi_s|\frac{\hbar^2}{2}(3\mathbb{1} + \sum_{k=1}^{3}\sigma_k\otimes\sigma_k)|\psi_s\rangle = \underline{2\hbar^2}$$

$$\text{III}:\ \langle\vec{S}^2\rangle_3 = \frac{1}{2}(\langle\vec{S}^2\rangle_1 + \langle\vec{S}^2\rangle_2) = \underline{\hbar^2}$$

$\hat{\rho}_1$ is a spin 0 state, $\hat{\rho}_2$ is a spin 1 state

$\hat{\rho}_3$ is a mixture (incoherent) of spin 0 and spin 1

This means: only $\hat{\rho}_1$ is rotationally invariant.

b) Reduced ~~opera~~ density operators

$$\hat{\rho}_1^A = \mathrm{Tr}_B\left[\tfrac{1}{2}(|+-\rangle\langle+-| + |-+\rangle\langle-+| - \underset{\nwarrow}{|+-\rangle\langle-+|} - \underset{\nearrow}{|-+\rangle\langle+-|})\right] \quad (1)$$

$$= \tfrac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|)_A = \underline{\tfrac{1}{2}\mathbb{1}_A} \qquad \text{cross terms}$$

$\hat{\rho}_2^A = \hat{\rho}_3^A = \hat{\rho}_1^A = \underline{\frac{1}{2}\mathbb{1}_A}$ since the cross terms in (1) do not contribute.

Similarly $\hat{\rho}_1^B = \hat{\rho}_2^B = \hat{\rho}_3^B = \underline{\frac{1}{2}\mathbb{1}_B}$ maximally mixed

$\hat{\rho}_1$ and $\hat{\rho}_2$ are pure states

$\Rightarrow$ entropies $\underline{S_1 = S_2 = 0}$

$\hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2)$ is mixed, with probabilities $p_1 = p_2 = \frac{1}{2}$

$$S_3 = -p_1 \log p_1 - p_2 \log p_2 = \underline{\log 2}$$

Entropies of subsystems

$$S_1^A = S_2^A = S_3^A = \underline{\log 2} = S_1^B = S_2^B = S_3^B$$

Inequality : $S \geq \max\{S_A, S_B\}$

I and II : not satisfied

III : satisfied as equality

Degree of entanglement

I and II are pure states, degree of entanglement measured by the entanglement entropy

$$S_1^A = S_1^B = \underline{\log 2}\,; \quad S_2^A = S_2^B = \underline{\log 2}$$

Case III

$$\hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2) = \frac{1}{2}(|+-\rangle\langle +-| + |-+\rangle\langle -+|)$$

$$= \frac{1}{2}(|+\rangle\langle +| \otimes |-\rangle\langle -| + |-\rangle\langle -| \otimes |+\rangle\langle +|)$$

It is a mixture of product states, which means that it is separable (non-entangled)

Degree of entanglement = $\underline{0}$

c) $|\theta\rangle = \cos\theta\,|+\rangle + \sin\theta\,|-\rangle \Rightarrow$

$$S_\theta|\theta\rangle = (\cos\theta\, S_z + \sin\theta\, S_x)|\theta\rangle$$

$$= \frac{\hbar}{2}\left[\left(\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}\right)|+\rangle + \left(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}\right)|-\rangle\right]$$

$$= \frac{\hbar}{2}\left(\cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle\right) = \underline{\left(+\frac{\hbar}{2}\right)|\theta\rangle} \quad \text{spin up state}$$

$$P_A = \langle \hat{P}(\theta) \rangle_A = \mathrm{Tr}_A\,(\hat{P}(\theta)\,\hat{\rho}_A)$$

$$= \langle \theta |\, \tfrac{1}{2} \mathbb{1}_A \,| \theta \rangle = \underline{\tfrac{1}{2}}$$

This is valid for all three cases I, II, III.

Means that there is equal probability for spin up and spin down in any direction θ.

d) $$P(\theta,\theta') = \mathrm{Tr}\,(\hat{P}(\theta) \otimes \hat{P}(\theta')\; \hat{\rho})$$

$$= \langle \theta,\theta' |\, \hat{\rho}\, | \theta,\theta' \rangle \qquad |\theta,\theta'\rangle = |\theta\rangle \otimes |\theta'\rangle$$

$$\langle + - | \theta,\theta' \rangle = \langle + | \theta \rangle \langle - | \theta' \rangle = \cos\tfrac{\theta}{2} \sin\tfrac{\theta'}{2}$$

$$\langle - + | \theta,\theta' \rangle = \langle - | \theta \rangle \langle + | \theta' \rangle = \sin\tfrac{\theta}{2} \cos\tfrac{\theta'}{2}$$

implies

case I : $$P_1(\theta,\theta') = \tfrac{1}{2}\big[\langle \theta\theta' | +- \rangle \langle +- | \theta\theta' \rangle + \langle \theta\theta' | -+ \rangle \langle -+ | \theta\theta' \rangle - \langle \theta\theta' | +- \rangle \langle -+ | \theta\theta' \rangle - \langle \theta\theta' | -+ \rangle \langle +- | \theta\theta' \rangle\big]$$

$$= \tfrac{1}{2}\big[\cos^2\tfrac{\theta}{2} \sin^2\tfrac{\theta'}{2} + \sin^2\tfrac{\theta}{2} \cos^2\tfrac{\theta'}{2} - 2\cos\tfrac{\theta}{2} \sin\tfrac{\theta}{2} \cos\tfrac{\theta'}{2} \sin\tfrac{\theta'}{2}\big]$$

$$= \tfrac{1}{2}\left(\cos\tfrac{\theta}{2} \sin\tfrac{\theta'}{2} - \sin\tfrac{\theta}{2} \cos\tfrac{\theta'}{2}\right)^2$$

$$= \underline{\tfrac{1}{2} \sin^2 \tfrac{\theta-\theta'}{2}}$$

case II and III :

similar evaluations give

$$P_2(\theta,\theta') = \underline{\tfrac{1}{2} \sin^2 \tfrac{\theta+\theta'}{2}} \qquad P_3(\theta,\theta') = \underline{\tfrac{1}{4}\left(\sin^2 \tfrac{\theta-\theta'}{2} + \sin^2 \tfrac{\theta+\theta'}{2}\right)}$$

e) Plots of the function $F(\theta,\theta')$ for $\theta' = 0.5\,\theta$ (to the left), 3D plots for variable θ and θ' also included (to the right).

Cases I and II show Bell inequality broken (negative F, colored red in 3D plot).

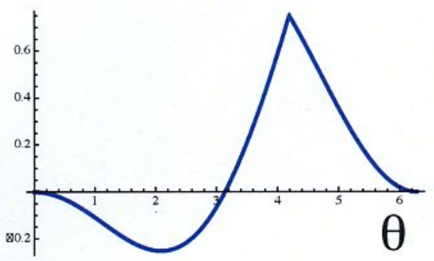
Case III shows no breaking of Bell inequality.

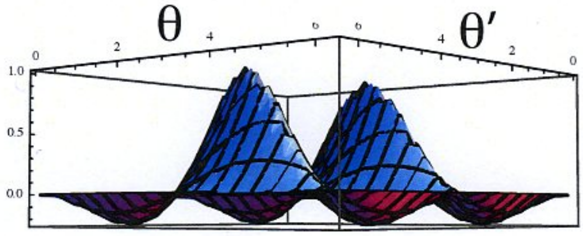
Results consistent with b), I and II being entangled, III being non-entangled.

$\theta' = 0.5\,\theta$

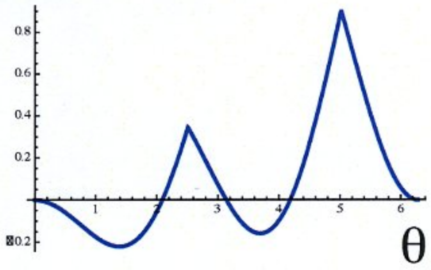
3D plot

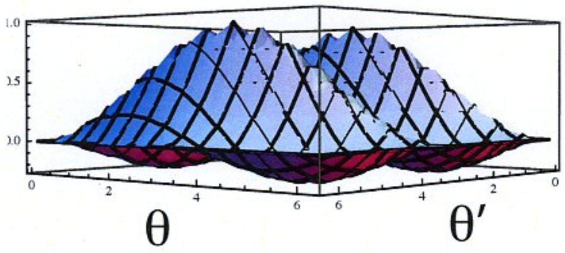
Case I



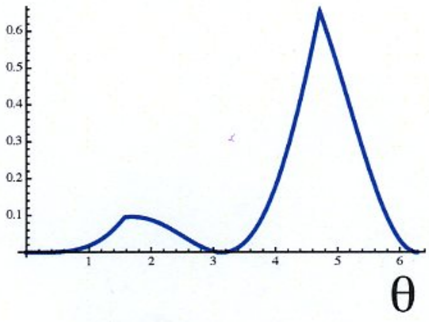


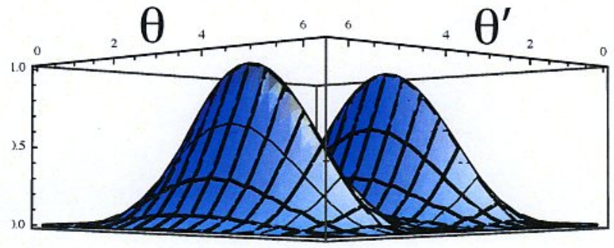
Case II





Case III





f) Experimental quantities

$$P^{A}_{exp}(\theta) = \frac{n_{++}+n_{+-}}{N} \qquad P^{B}_{exp}(\theta) = \frac{n_{++}+n_{-+}}{N}$$

$$\underline{P^{*}_{exp}(\theta,\theta') = \frac{n_{++}}{N}}$$

Problem 2

a) $\hat{H}_1 |g,n\rangle = -i\hbar\lambda\overset{\sqrt{n}}{\vee}|e,n-1\rangle$

$\hat{H}_1 |e,n-1\rangle = i\hbar\lambda\sqrt{n}\,|g,n\rangle$

mixes only these two levels

$\Rightarrow$

$\langle g,n|\hat{H}|g,n\rangle = \hbar(-\frac{1}{2}\omega_0 + n\omega)$

$\langle e,n-1|\hat{H}|e,n-1\rangle = \hbar(\frac{1}{2}\omega_0 + (n-1)\omega)$

$\langle g,n|\hat{H}|e,n-1\rangle = i\hbar\lambda\sqrt{n}$

$\langle e,n-1|\hat{H}|g,n\rangle = -i\hbar\lambda\sqrt{n}$

In matrix form

$$H_n = \frac{1}{2}\hbar\begin{pmatrix} -\omega_0 + 2n\omega & -2i\lambda\sqrt{n} \\ -2i\lambda\sqrt{n} & \omega_0 + 2(n-1)\omega \end{pmatrix}$$

$$= \frac{1}{2}\hbar\begin{pmatrix} \omega-\omega_0 & 2i\lambda\sqrt{n} \\ -2i\lambda\sqrt{n} & \omega_0-\omega \end{pmatrix} + \hbar\overset{\omega}{\vee}(n\mp\frac{1}{2})\,\mathbb{1}$$

$$\Rightarrow \underline{\Delta = \omega - \omega_0}, \quad \underline{\varepsilon_n = \hbar\overset{\omega}{\vee}(n-\frac{1}{2})} \quad \underline{\omega_n = 2\lambda\sqrt{n}}$$

$\hat{H}|g0\rangle = \hat{H}_0|g,0\rangle = -\frac{1}{2}\hbar\omega_0|g,0\rangle$

time evolution $|\psi(0)\rangle = |g,0\rangle \Rightarrow \underline{|\psi(t)\rangle = e^{\frac{i}{2}\omega_0 t}|g,0\rangle}$

b) $\omega = \omega_0 \Rightarrow \Delta = 0$

$$H_n = \tfrac{1}{2}\hbar\omega_n \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + \varepsilon_n \mathbb{1}$$

$$\underline{= \varepsilon_n \mathbb{1} - \tfrac{1}{2}\hbar\omega_n \sigma_y}$$

Eigenstates and eigenvalues

$$\sigma_y \phi_n^\pm = \mp \phi_n^\pm \Rightarrow E_n^\pm = \varepsilon_n \pm \tfrac{1}{2}\hbar\omega_n$$

$$\underline{= \hbar\left[(n-\tfrac{1}{2})\omega \pm \lambda\sqrt{n}\right]}$$

$$\phi_n^\pm = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\Rightarrow \mp\alpha = -i\beta \quad , \quad \beta = \mp i\alpha \qquad \underline{\phi_n^\pm = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp i \end{pmatrix}}$$

General state

$$\psi_n = d_n^+ \phi_n^+ + d_n^- \phi_n^- = \frac{1}{\sqrt{2}}\begin{pmatrix} d_n^+ + d_n^- \\ -i(d_n^+ - d_n^-) \end{pmatrix}$$

$$\Rightarrow c_{n1} = \tfrac{1}{\sqrt{2}}(d_n^+ + d_n^-) \qquad\qquad d_{n}^+ = \tfrac{1}{\sqrt{2}}(c_{n1} + i c_{n2})$$

$$\Rightarrow$$

$$c_{n2} = -\tfrac{i}{\sqrt{2}}(d_n^+ - d_n^-) \qquad\qquad d_n^- = \tfrac{1}{\sqrt{2}}(c_{n1} - i c_{n2})$$

Time evolution $d_n^\pm(t) = e^{-\frac{i}{\hbar}E_n^\pm t} d_n^\pm(0)$

$$\Rightarrow c_{n1}(t) = \tfrac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E_n^+ t} d_n^+(0) + e^{-\frac{i}{\hbar}E_n^- t} d_n^-(0)\right)$$

$$= \tfrac{1}{2}\left(\left(e^{-\frac{i}{\hbar}E_n^+ t} + e^{-\frac{i}{\hbar}E_n^- t}\right) c_{n1}(0)\right.$$

$$\left. + \tfrac{i}{2}\left(\left(e^{-\frac{i}{\hbar}E_n^+ t} - e^{-\frac{i}{\hbar}E_n^- t}\right) c_{n2}(0)\right)\right.$$

$$\Rightarrow c_{n1}(t) = e^{-\frac{i}{\hbar}\varepsilon_n t}\left(\cos\frac{\omega_n t}{2}\, c_{n1}(0) + \sin\frac{\omega_n t}{2}\, c_{n2}(0)\right)$$

equiv. derivation:

$$\underline{c_{n2}(t) = e^{-\frac{i}{\hbar}\varepsilon_n t}\left(\cos\frac{\omega_n t}{2}\, c_{n2}(0) - \sin\frac{\omega_n t}{2}\, c_{n1}(0)\right)}$$

c) General state

$$|\psi\rangle = \sum_{ni} c_{ni}\,|ni\rangle$$

Density operator

$$\hat{\rho} = |\psi\rangle\langle\psi| = \sum_{ni}\sum_{n'j} c_{ni}\,c^*_{n'j}\,|ni\rangle\langle n'j|$$

matrix elements

$$\underline{\rho_{ni,n'j} = c_{ni}\,c^*_{n'j}}$$

Reduced density operator of the atom

$$\hat{\rho}_{atom} = Tr_{photon}\,\hat{\rho} = \sum_n \langle n|\hat{\rho}|n\rangle$$ photon states

$$= \sum_n \sum_{n'i} \sum_{n''j} c_{n'i}\,c^*_{n''j}\,\langle n|n'i\rangle\langle n''j|n\rangle$$

$$\langle n|n'1\rangle = |g\rangle\,\delta_{nn'} \equiv |1\rangle\,\delta_{nn'}$$

$$\langle n|n'2\rangle = |e\rangle\,\delta_{n,n'-1} \equiv |2\rangle\,\delta_{n,n'-1}$$

$$\Rightarrow \langle n|n'i\rangle = |i\rangle\,\delta_{(n+i-1),n'}$$

matrix elements

$$\rho_{ij} = \langle i|\hat{\rho}_{atom}|j\rangle = \sum_n \sum_{n'} \sum_{n''} c_{n'i}\,c^*_{n''j}\,\delta_{(n+i-1),n'}\,\delta_{(n+j-1),n''}$$

$$= \underline{\sum_n c_{(n+i-1)i}\,c^*_{(n+j-1)j}}$$

Diagonal elements

$\rho_{11} = \sum_n |c_{n1}|^2$ prob. for atom to be in the gnd. state

$\rho_{22} = \sum_n |c_{n2}|^2$ — " — excited —"—

Initial state $(t=0)$

Case I $\quad \hat{\rho} = |\psi(0)\rangle\langle\psi(0)| = |e\rangle\langle e| \otimes |m-1\rangle\langle m-1|$

$$\hat{\rho}_{atom} = Tr_{photon}\,\hat{\rho} = |e\rangle\langle e|\;\langle m-1|m-1\rangle = |e\rangle\langle e|$$

$$\Rightarrow \underline{\rho_{ij} = \delta_{i2}\,\delta_{j2}}$$

Case II $\quad \hat{\rho} = |$[illegible]$\rangle\langle e| \otimes |\alpha\rangle\langle\alpha\rangle$

$$\Rightarrow \underline{\rho_{ij} = \delta_{i2}\,\delta_{j2}}$$

$c_{ni}(0) = \delta_{nm}\,\delta_{i2}$

d) From b):

Case I: $\quad c_{n1}(t) = e^{-\frac{i}{\hbar}\varepsilon_m t}\,\sin\frac{\omega_m t}{2}\,\delta_{nm}$

$$c_{n2}(t) = e^{-\frac{i}{\hbar}\varepsilon_m t}\,\cos\frac{\omega_m t}{2}\,\delta_{nm}$$

density matrix

$$\underline{\rho_{11}(t) = \sin^2\frac{\omega_m t}{2}} \qquad \underline{\rho_{22}(t) = \cos^2\frac{\omega_m t}{2}}$$

$$\rho_{12} = \sum_n \sin\frac{\omega_m t}{2}\cos\frac{\omega_m t}{2}\,\delta_{n,m}\,\delta_{n+1,m} \;\; \underline{=0}$$

$$\rho_{21} = \rho_{12}^{*} \;\underline{=0}$$

Case II : $\quad c_{n1}(t) = e^{-\frac{i}{\hbar}\varepsilon_n t}\,\sin\frac{\omega_n t}{2}\,\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\,e^{-|\alpha|^2/2}$

$$c_{n2}(t) = e^{-\frac{i}{\hbar}\varepsilon_n t}\,\cos\frac{\omega_n t}{2}\,\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\,e^{-|\alpha|^2/2}$$

$$\Rightarrow \underline{\rho_{11}(t) = \sum_{n=1}^{\infty}\sin^2\frac{\omega_n t}{2}\,\frac{|\alpha|^{2(n-1)}}{(n-1)!}\,e^{-|\alpha|^2}}$$

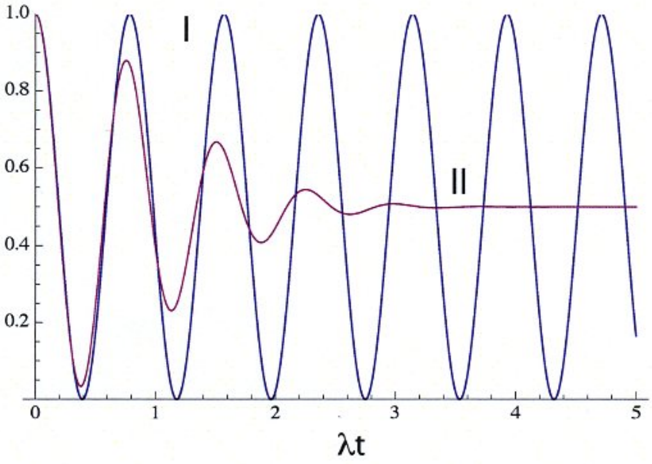
$$\underline{\rho_{22}(t) = \sum_{n=1}^{\infty}\cos^2\frac{\omega_n t}{2}\,\frac{|\alpha|^{2(n-1)}}{(n-1)!}\,e^{-|\alpha|^2}}$$

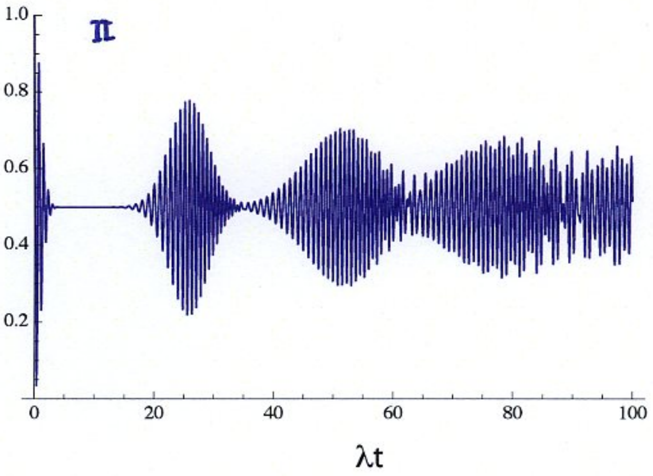
$$\underline{\rho_{12}(t) = e^{-i\omega t}\sum_{n=1}^{\infty}\sin\frac{\omega_n t}{2}\cos\frac{\omega_{n+1} t}{2}\,\frac{\alpha^{(n-1)}\alpha^{*n}}{\sqrt{(n-1)!\,n!}}\,e^{-|\alpha|^2}}$$

$\rho_{21}(t) = \rho_{12}(t)^{*} \qquad$ with $\omega_n = 2\lambda\sqrt{n}$

e) Plots of $p_{11}(t)$ for cases I and II,
probability for the atom to be in the excited state.

Case I with the photon number initially defined as $n = 4$ shows regular Rabi oscillations between $|e\rangle$ and $|g\rangle$.





Case II, with the e.m. field initially in a coherent state seems first to show damped Rabi oscillations, but the oscillations recover~~s~~ and show some irregular "quantum beats".

When the atom is excited and de-excited by a classical ~~Ra~~ e.m. field the Rabi oscillations are regular, like in I.

Midterm Exam FYS4110, 2012

Solutions

Problem 1

a) Total spin

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 \quad \Rightarrow \quad S_z = S_{1z} + S_{2z} + S_{3z}$$

$$\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + \vec{S}_3^2 + 2(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1)$$

$$= \frac{9}{4}\hbar^2 \mathbb{1} + 2\,(\text{--} \,''\, \text{--})$$

$$\Rightarrow \quad \vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1 = \frac{1}{2}\vec{S}^2 - \frac{9}{8}\hbar^2 \mathbb{1}$$

$$\underline{H = \frac{a}{2}\vec{S}^2 + b S_z - \frac{9}{8} a \hbar^2 \mathbb{1}}$$

Spin compositions

spin ½ × spin ½ = spin 0 + spin 1

$\Rightarrow$ spin ½ × (spin ½ × spin ½)

= spin ½ × spin 0 + spin ½ × spin 1

= <u>spin ½ + spin ½ + spin 3/2</u>

b) Lowest energy of the spin ½ subspaces, for $S_z = -\frac{1}{2}\hbar$, is

$$E_0^{1/2} = \frac{a}{2}\frac{3}{4}\hbar^2 - \frac{b}{2}\hbar - \frac{9}{8}a\hbar^2$$

$$= \underline{-\frac{3}{4}a\hbar^2 - \frac{1}{2}b\hbar}$$

Lowest energy for spin 3/2, with $S_z = -\frac{3}{2}\hbar$, is

$$E_0^{3/2} = \frac{a}{2}\frac{15}{4}\hbar^2 - 3\frac{b}{2}\hbar - \frac{9}{8}a\hbar^2$$

$$= \underline{\frac{3}{4}a\hbar^2 - \frac{3}{2}b\hbar}$$

Energy difference

$$E_0^{3/2} - E_0^{1/2} = \frac{3}{2} a\hbar^2 - b\hbar$$

this is positive when $\underline{b < \frac{3}{2} a\hbar}$

This is the condition for the ground state to have spin ½. It is doubly degenerate since the Hamiltonians in the two spin ½ subspaces are identical

c) We examine $|\psi_a\rangle$

$$|\psi_a\rangle = |-\rangle_1 \otimes |\psi_a\rangle_{23}$$

$$|\psi_a\rangle_{23} = \frac{1}{\sqrt{2}}\left(|+-\rangle_{23} - |-+\rangle_{23}\right)$$

This is a spin singlet state (spin 0)
(Is demonstrated by applying $(\vec{S}_2+\vec{S}_3)^2 = 2\vec{S}_2\cdot\vec{S}_3 + \frac{3}{2}\hbar^2\mathbb{1}$ to the state $|\psi_a\rangle_{23}$)

1: The composition of any spin ½ state with a spin 0 state is a spin ½ state.

2: z-component $S_z|\psi_a\rangle = \frac{\hbar}{2}(-1+1-1)|\psi_a\rangle = \underline{-\frac{\hbar}{2}|\psi_a\rangle}$

⇒ The state lies in the subspace of the ground state.

The states $|\psi_b\rangle$ and $|\psi_c\rangle$:

They are derived from $|\psi_a\rangle$ by cyclic permutations of the three spins: 123 → 231 → 312

The total spin $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$ is invariant under permutations ⇒ the three states have the same spin quantum numbers ⇒ they all lie in the subspace of the degenerate ground state.

d) Partition 1 + (23) for $|\psi_a\rangle$:

$$\rho_a = |\psi_a\rangle\langle\psi_a| = (|-\rangle\langle-|)_1 \otimes (|\psi_a\rangle\langle\psi_a|)_{23}$$

$\Rightarrow \rho_a = \rho_{a1} \otimes \rho_{a23}$ product state

There is no correlation $\Rightarrow$ no entanglement with respect to this partition

Partition 2 + (31):

$$\rho_{a2} = \mathrm{Tr}_{13}\, \rho_a = \mathrm{Tr}_1\, \rho_{a1}\; \mathrm{Tr}_3\, \rho_{a23} \qquad \mathrm{Tr}_1\, \rho_{a1} = 1$$

$$= \frac{1}{2} \mathrm{Tr}_3 \left(|+-\rangle\langle+-| + |-+\rangle\langle-+| - |+-\rangle\langle-+| - |-+\rangle\langle+-| \right)_{23}$$

$$= \frac{1}{2} \left(|+\rangle\langle+| + |-\rangle\langle-| \right)_2$$

$$= \frac{1}{2} \mathbb{1}_2$$

Entropy: $S_{a2} = \log 2$

This is the maximal entropy, since the spin space of particle 2 is of dimension 2.

It is the entanglement entropy of the composite system 1 + (23)

Partition 3 + (12)

The density operator is symmetric with respect to the permutation $1 \leftrightarrow 2$ $\Rightarrow$ $\rho_{a3} = \frac{1}{2}\mathbb{1}_3$

$\Rightarrow S_{a3} = \log 2$: maximally mixed

Since $|\psi_b\rangle$ and $|\psi_c\rangle$ are derived from $|\psi_a\rangle$ by permutations, the conclusions are the same up to permutation of spin labels:

$|\psi_b\rangle$ 123 → 231

$|\psi_c\rangle$ 123 → 312

e)

$$\langle \Phi_I | \Phi_{II} \rangle = \frac{1}{3}\left(1 + e^{4\pi i/3} + e^{-4\pi i/3}\right)$$

$$= \frac{1}{3}\left(1 + e^{-2\pi i/3} + e^{2\pi i/3}\right)$$

$$e^{\pm 2\pi i/3} = \cos(2\pi/3) \pm i \sin(2\pi/3)$$

$$= -\frac{1}{2} \pm i\frac{1}{2}\sqrt{3}$$

$$\Rightarrow e^{2\pi i/3} + e^{-2\pi i/3} = -1$$

$$\Rightarrow \langle \Phi_I | \psi_{II} \rangle = \frac{1}{3}(1-1) = \underline{0}$$ orthogonal

If $|\psi_I\rangle$ belongs to the subspace:

$$|\Phi_I\rangle = \alpha|\psi_a\rangle + \beta|\psi_b\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\alpha|-+-\rangle - (\alpha-\beta)|--+\rangle - \beta|+--\rangle\right)$$

$$\Rightarrow \quad \frac{\alpha}{\sqrt{2}} = \frac{1}{\sqrt{3}}\left(-\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right) \quad (1)$$

$$\frac{\beta}{\sqrt{2}} = -\frac{1}{\sqrt{3}} \quad (2)$$

$$\frac{\alpha-\beta}{\sqrt{2}} = -\frac{1}{\sqrt{3}}\left(\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right) \quad (3)$$

Consistency check:

$$(1)-(2): \quad \frac{\alpha-\beta}{\sqrt{2}} = \frac{1}{\sqrt{3}}\left(-\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right) + \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right)$$

the same as (3)

$$\Rightarrow |\Phi_I\rangle = \sqrt{\frac{2}{3}}\left(-\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right)|\psi_a\rangle - \sqrt{\frac{2}{3}}|\psi_b\rangle$$

With $|\psi_{II}\rangle$:

$$e^{\pm 2\pi i/3} \rightarrow e^{\mp 2\pi i/3}$$

$$\Rightarrow \underline{|\psi_{II}\rangle = \sqrt{\frac{2}{3}}\left(-\frac{1}{2} - i\frac{1}{2}\sqrt{3}\right)|\psi_a\rangle - \sqrt{\frac{2}{3}}|\psi_b\rangle}$$

Both belong to the subspace

f) Density operator

$$\rho_I = \frac{1}{3}\Big(|{+}{-}{-}\rangle\langle{+}{-}{-}| + |{-}{+}{-}\rangle\langle{-}{+}{-}| + |{-}{-}{+}\rangle\langle{-}{-}{+}|$$

$$+ e^{-2\pi i/3} |{+}{-}{-}\rangle\langle{-}{+}{-}| + e^{2\pi i/3} |{+}{-}{-}\rangle\langle{-}{-}{+}|$$

$$+ e^{2\pi i/3} |{-}{+}{-}\rangle\langle{+}{-}{-}| + e^{-2\pi i/3} |{-}{-}{+}\rangle\langle{+}{-}{-}|$$

$$+ e^{-2\pi i/3} |{-}{+}{-}\rangle\langle{-}{-}{+}| + e^{2\pi i/3} |{-}{-}{+}\rangle\langle{-}{+}{-}| \Big)$$

Reduced density operators

$$\rho_{I1} = \frac{1}{3}\left(|+\rangle\langle+| + |-\rangle\langle-| + |-\rangle\langle-| \right)_1$$

$$= \frac{1}{3}(|+\rangle\langle+|)_1 + \frac{2}{3}(|-\rangle\langle-|)_1$$

Entropy $S_{I1} = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = \log 3 - \frac{2}{3}\log 2$

$$\rho_{I2} = \frac{1}{3}\left(|-\rangle\langle-| + |+\rangle\langle+| + |-\rangle\langle-| \right)_2$$

$$= \frac{1}{3}(|+\rangle\langle+|)_2 + \frac{2}{3}(|-\rangle\langle-|)_2$$

$$\rho_{I3} = \frac{1}{3}(|+\rangle\langle+|)_3 + \frac{2}{3}(|-\rangle\langle-|)_3$$

$$\Rightarrow \quad S_{I1} = S_{I2} = S_{I3} = \underline{\underline{\log 3 - \frac{2}{3}\log 2}}$$

The results are precisely the same for $|\Psi_{II}\rangle$

Comparison with the average entanglement entropy of $|\psi_a\rangle$ ($|\psi_b\rangle$ and $|\psi_c\rangle$):

$$\bar{S}_a = \frac{2}{3}\log 2$$

Difference $S_I - \bar{S}_a = \log 3 - \frac{4}{3}\log 2$

$\log_2$: $\quad S_I - \bar{S}_a = \log_2 3 - \frac{4}{3} = 0.25 > 0$

g) Measurement of S_{1z} in the state $|\psi_I\rangle$

If measured result is $S_{1z} = +\frac{\hbar}{2}$, the spin of particle 1 is projected into the state $|+\rangle_1$

$\Rightarrow$ The full state is changed to:

$$|\psi_I\rangle \to |+--\rangle = |+\rangle_1 \otimes |-\rangle_2 \otimes |-\rangle_3$$

This is a pure product state, with no entanglement

If measured result is $S_{1z} = -\frac{\hbar}{2}$, the spin of particle 1 is projected into the state $|-\rangle_1$.

$\Rightarrow$ The full state is changed to

$$|\psi_I\rangle \to \frac{1}{\sqrt{2}} |-\rangle_1 \otimes \left(e^{2\pi i/3} |+-\rangle_{23} + e^{-2\pi i/3} |-+\rangle_{23} \right)$$

for the (23) subsystem

$$\rho_{23} = \frac{1}{2} \left(|+-\rangle\langle +-| + |-+\rangle\langle -+| \right)_{23}$$

and the reduced density operators are

$$\rho_2 = \frac{1}{2} \left(|+\rangle\langle +| + |-\rangle\langle -| \right)_2 = \frac{1}{2} \mathbb{1}_2$$

similarly

$$\rho_3 = \frac{1}{2} \mathbb{1}_3$$

The entanglement entropy of subsystem 23 then is $\underline{S = \log 2}$

Problem 2

$$\vec{A} = -\tfrac{1}{2}\vec{r}\times\vec{B} = -\tfrac{B}{2}\vec{r}\times\vec{k}$$

$$\Rightarrow A_x = -\tfrac{1}{2}By, \quad A_y = \tfrac{1}{2}Bx$$

Introduce $\vec{\pi} = \vec{p} - e\vec{A} \Rightarrow \pi_x = p_x + \tfrac{1}{2}eBy; \; \pi_y = p_y - \tfrac{1}{2}eBx$

$$H = \frac{1}{2m}\vec{\pi}^2 = \frac{1}{2m}(\pi_x^2 + \pi_y^2)$$

a) $L = (\vec{r}\times\vec{p})_z = xp_y - yp_x$

$$\begin{aligned}[L,\pi_x] &= [xp_y - yp_x, p_x + \tfrac{1}{2}eBy] \\ &= [x,p_x]p_y + \tfrac{1}{2}eBx[p_y,y] \\ &= i\hbar(p_y - \tfrac{1}{2}eBx) \\ &= i\hbar\pi_y\end{aligned}$$

$$\begin{aligned}[L,\pi_y] &= [xp_y - yp_x, p_y - \tfrac{1}{2}eBx] \\ &= -[y,p_y]p_x + \tfrac{1}{2}eBy[p_x,x] \\ &= -i\hbar(p_x + \tfrac{1}{2}eBy) \\ &= -i\hbar\pi_x\end{aligned}$$

$$\begin{aligned}[L,H] &= \frac{1}{2m}[L,\pi_x^2+\pi_y^2] \\ &= \frac{1}{2m}\left([L,\pi_x]\pi_x + \pi_x[L,\pi_x] + [L,\pi_y]\pi_y + \pi_y[L,\pi_y]\right) \\ &= \frac{i\hbar}{2m}(\pi_y\pi_x + \pi_x\pi_y - \pi_x\pi_y - \pi_y\pi_x) = \underline{0}\end{aligned}$$

L commutes with H $\Rightarrow$ L is a constant of motion b)

$$X = x + \frac{1}{m\omega}\pi_y \qquad m\omega = eB$$

$$= x + \frac{1}{eB}\left(p_y - \tfrac{1}{2}eBx\right)$$

$$= \frac{1}{2}x + \frac{1}{eB}p_y$$

$$Y = y - \frac{1}{m\omega}\pi_x$$

$$= y - \frac{1}{eB}\left(p_x + \tfrac{1}{2}eBy\right)$$

$$= \frac{1}{2}y - \frac{1}{eB}p_x$$

$$\Rightarrow [X,Y] = \left[\tfrac{1}{2}x, -\tfrac{1}{eB}p_x\right] + \left[\tfrac{1}{eB}p_y, \tfrac{1}{2}y\right] = -\frac{i\hbar}{eB} = -i\,l_B^2$$

$$[a,a^+] = \frac{1}{2l_B^2}\left([X,+iY] + [-iY,X]\right)$$

$$= +\frac{i}{l_B^2}[X,Y] = \underline{1}$$

Similarly

$$\eta_x = \frac{1}{eB}\pi_y = -\frac{1}{2}x + \frac{1}{eB}p_y$$

$$\eta_y = -\frac{1}{eB}\pi_x = -\frac{1}{2}y - \frac{1}{eB}p_x$$

$$[\eta_x, \eta_y] = \frac{1}{2eB}\left\{[x,p_x] - [p_y,y]\right\} = i\,l_B^2$$

$$[b,b^+] = \frac{1}{2l_B^2}(-2i)[\eta_x,\eta_y] = \underline{1}$$

$$[X,\eta_x] = [Y,\eta_y] = 0$$

$$[X,\eta_y] = \left[\tfrac{1}{2}x + \tfrac{1}{eB}p_y, -\tfrac{1}{2}y - \tfrac{1}{eB}p_x\right] = 0$$

$$[Y,\eta_x] = \left[\tfrac{1}{2}y - \tfrac{1}{eB}p_x, -\tfrac{1}{2}x + \tfrac{1}{eB}p_y\right] = 0$$

$$\Rightarrow [a,b] = [a^+,b] = [a,b^+] = 0$$

Commut. relations as for two independent harm. oscillators

c) $$H = \frac{(eB)^2}{2m}(\eta_x^2+\eta_y^2)$$

$$= \frac{(eB)^2}{2m}\frac{\ell_B^2}{2}\left((b+b^\dagger)^2-(b-b^\dagger)^2\right)$$

$$= \frac{1}{2}\hbar\omega(b^\dagger b + bb^\dagger) = \underline{\hbar\omega(b^\dagger b+\tfrac{1}{2})}$$

Constant energy splitting $\hbar\omega$, as for harmonic oscillator Independent of $a, a^\dagger$, implies all energy eigenstates reached by a and $a^\dagger$ have the same energy

Lowest energy states $b|\psi\rangle = 0 \Rightarrow E_0 = \frac{1}{2}\hbar\omega$

Define $a|0\rangle = 0$ and $b|0\rangle = 0$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \Rightarrow b|n\rangle = 0$$

all have the same energy E_0

Angular momentum

$$x = X-\eta_x, \quad y = Y-\eta_y$$

$$p_x = -\frac{eB}{2}(Y+\eta_y) \; , \; p_y = \frac{eB}{2}(X+\eta_x)$$

$$\Rightarrow L = [(X-\eta_x)(X+\eta_x)+(Y-\eta_y)(Y+\eta_y)]\frac{eB}{2}$$

$$= \frac{eB}{2}(X^2+Y^2-\eta_x^2-\eta_y^2)$$

$$= \frac{eB}{2}\frac{\ell_B^2}{2}\left((a+a^\dagger)^2-(a-a^\dagger)^2-(b+b^\dagger)^2+(b-b^\dagger)^2\right)$$

$$= \frac{1}{2}\hbar(aa^\dagger+a^\dagger a-bb^\dagger-b^\dagger b)$$

$$= \underline{\hbar(a^\dagger a-b^\dagger b)}$$

$$\Rightarrow L|n\rangle = \hbar a^\dagger a|n\rangle = n\hbar|n\rangle$$

anglular momentum $l_n = n\hbar$

d) $|z,-z\rangle_a = N(z)(|z\rangle\otimes|-z\rangle - |-z\rangle\otimes|z\rangle)$

$(a_1+a_2)|z,-z\rangle_a = (z-z)|z,-z\rangle_a = 0$ eigenvalue 0

$a_1 a_2 |z,-z\rangle_a = z(-z)|z,-z\rangle_a = \underline{-z^2|z,-z_a\rangle}$

Normalization

$\langle z,-z|z,-z\rangle_a = 1$

$\Rightarrow |N(z)|^2 \big(\langle z|z\rangle\langle -z|-z\rangle + \langle -z|-z\rangle\langle z|z\rangle$

$- \langle z|-z\rangle\langle -z|z\rangle - \langle -z|z\rangle\langle z|-z\rangle\big)$

$= 2|N(z)|^2 (1 - |\langle z|-z\rangle|^2) \overset{!}{=} 1$

$\langle z|-z\rangle = e^{-\frac{1}{2}(|z|^2+|-z|^2) + z^*(-z)} = e^{-2|z|^2}$

$\Rightarrow 2|N(z)|^2(1-e^{-4|z|^2})$

$\Rightarrow \underline{N(z) = \frac{1}{\sqrt{2(1-e^{-4|z|^2})}}}$

Density operators

$\rho = |z,-z\rangle_a\langle z,-z|_a = |N(z)|^2$

$*\big(|z\rangle\langle z|\otimes|-z\rangle\langle -z| + |-z\rangle\langle -z|\otimes|z\rangle\langle z|$

$- |z\rangle\langle -z|\otimes|-z\rangle\langle z| - |-z\rangle\langle z|\otimes|z\rangle\langle -z|\big)$

Reduced density operators

$\rho_1 = |N(z)|^2 \big(|z\rangle\langle z| + |-z\rangle\langle -z|$

$- |z\rangle\langle -z|\,\langle z|-z\rangle - |-z\rangle\langle +z|\,\langle -z|z\rangle\big)_1$

$\underline{= \frac{1}{2(1-e^{4|z|^2})}\big(|z\rangle\langle z| + |-z\rangle\langle -z| - e^{-2|z|^2}(|z\rangle\langle -z| + |-z\rangle\langle z|)\big)}$

same expression for ρ_2

e) Density matrix in the coherent state representation

$$\rho_1(z,z') = \langle z|\hat{\rho}_1|z'\rangle$$

$$= |N(Z)|^2 \big(\langle z|Z\rangle\langle Z|z'\rangle + \langle z|-Z\rangle\langle -Z|z'\rangle$$

$$- e^{-2|Z|^2}\big(\langle z|Z\rangle\langle -Z|z'\rangle + \langle z|-Z\rangle\langle Z|z'\rangle\big)\big)$$

$$\langle z|Z\rangle = e^{-\frac{1}{2}(|z|^2+|Z|^2)+z^*Z} \Rightarrow$$

$$\rho_1(z,z') = |N(Z)|^2 e^{-|Z|^2} e^{-\frac{1}{2}(|z|^2+|z'|^2)}$$

$$\times \big(\big(e^{z^*Z+Z^*z'} + e^{-(z^*Z+Z^*z')}\big) - e^{-2|Z|^2}\big(e^{z^*Z-Z^*z'} + e^{-z^*Z+Z^*z'}\big)\big)$$

$$= \frac{e^{-|Z|^2}}{1-e^{-4|Z|^2}} e^{-\frac{1}{2}(|z|^2+|z'|^2)}\big(\cosh(z^*Z+Z^*z')$$

$$- e^{-2|Z|^2}\cosh(z^*Z-Z^*z')\big)$$

One-particle density

$$\rho(z) = 2\rho_1(z,z)$$

$$= 2\,\frac{e^{-(|Z|^2+|z|^2)}}{1-e^{-4|Z|^2}}\big(\cosh(2\mathrm{Re}(z^*Z)) - e^{-2|Z|^2}\cos(2\,\mathrm{Im}(z^*Z))\big)$$

Assume Z real

$$\rho(z) = 2\,\frac{e^{-(Z^2+|z|^2)}}{1-e^{-4Z^2}}\big(\cosh(2Z\,\mathrm{Re}\,z) - e^{-2Z^2}\cos(2Z\,\mathrm{Im}\,z)\big)$$

Plots for $Z = 2, 1, 0.1$

$Z = 2$ two particles far apart, two gaussians

$Z = 1$ the two parts begin to merge

$Z = 0.1$ the two parts on the top of each others, not a fully gaussian form, flattened on the top, due to Pauli exclusion

f) $\hat{\rho}_1 |z\rangle = |N(z)|^2 \{ |z\rangle (1 - e^{-4|z|^2}) + |-z\rangle (e^{-2|z|^2} - e^{-2|z|^2}))$

$= \frac{1}{2} |z\rangle$

$\hat{\rho}_1 |-z\rangle = |N(z)|^2 \{ |-z\rangle (1 - e^{-4|z|^2}) + |z\rangle (e^{-2|z|^2} - e^{-2|z|^2}) \}$

$= \frac{1}{2} |-z\rangle$

$\Rightarrow \hat{\rho}_1 = \frac{1}{2} \hat{P}$ $\quad \hat{P}$ projection on subspace spanned by $|z\rangle$ and $|-z\rangle$

(note $\hat{\rho}_1 |\psi\rangle = 0$ for any state orthogonal to both $|z\rangle$ and $|-z\rangle$)

$\Rightarrow \hat{\rho}_1 = \frac{1}{2} (|1\rangle\langle 1| + |2\rangle\langle 2|)$

with $|1\rangle$ and $|2\rangle$ as orthonormalized states in this subspace

Entropy $S_1 = -\frac{1}{2} \log \frac{1}{2} - \frac{1}{2} \log \frac{1}{2} = \log 2$

$\Rightarrow$ entanglement entropy of two-particle system.

Entanglement is due do antisymmetrization, Fermi-Dirac statistics

g) N particles in the lowest angular momentum states

Antisymmetric state

$$|\psi\rangle = \mathcal{N}_1 \left(|0,1,2, \cdots (N-1)\rangle - |1,0,2, \dots, (N-1)\rangle + \cdots \right)$$

N! permutations, sign change for odd number of interchange of pair of particle indices.

Normalization : $\langle\psi|\psi\rangle = |\mathcal{N}_1|^2 \cdot N!$ $\quad \mathcal{N}_1 = \frac{1}{\sqrt{N!}}$

Density operator

$$\rho = |\psi\rangle\langle\psi| = |\mathcal{N}_1|^2 \Big(|0,1 \cdots (N-1)\rangle\langle 0,1 \cdots (N-1)|$$
$$+ |1,0, \dots, (N-1)\rangle\langle 1,0, \dots (N-1)| + \cdots \Big)$$

N! terms, all with weight +1

Particle 1 (first position) all angular momenta appear with the same weight

Reduced density operator

$$\hat{\rho}_1 = \mathrm{Tr}_{2,3,N-1}\, \hat{\rho} = N_2 \left(|0\rangle\langle 0| + |1\rangle\langle 1| + \cdots |N-1\rangle\langle N-1| \right)$$

Normalization $\mathrm{Tr}\,\hat{\rho}_1 = 1 \Rightarrow |N_2|^2 N = 1 \quad N_2 = \frac{1}{\sqrt{N}}$

$$\Rightarrow \quad \underline{\hat{\rho}_1 = \frac{1}{N} \sum_{n=0}^{N-1} |n\rangle\langle n|}$$

One-particle density

$$\rho_1(z) = N \rho_1(z,z) = \sum_{n=0}^{N-1} |\langle z|n\rangle|^2$$
$$= \underline{\sum_{n=0}^{N-1} \frac{|z|^{2n}}{n!} e^{-|z|^2}}$$

Plot of $\rho(z)$ for $N=10$:

Almost constant density $\rho(z) \simeq 1$ for $|z|^2 \lesssim \sqrt{10}$

Increase in the density prohibited by Pauli exclusion principle the lowest angular momenta occupy the area with lowest $|z|^2$. This means that the density of the inner part cannot be increased by adding particles

h) Plot of $\rho(z)$ for $N=2$

Looks precisely the same as the two-particle coherent state for $Z=0.01$.

Limit $Z \to 0$:

Two-particle coherent state, Z real

One particle density:

$$\rho(z) = \frac{2e^{-Z^2}}{1-e^{-4Z^2}} e^{-|z|^2}\left(\cosh(2Z\,\mathrm{Re}z) - e^{-2Z^2}\cos(2Z\,\mathrm{Im}z)\right)$$

$Z \to 0$, expand in Z^2 to first order

$$e^{-Z^2} \simeq 1-Z^2, \quad 1-e^{-4Z^2} \simeq 4Z^2$$

$$\cosh(2Z\,\mathrm{Re}z) \simeq 1+2Z^2(\mathrm{Re}z)^2; \quad \cos(2Z\,\mathrm{Im}z) \simeq 1-2Z^2(\mathrm{Im}z)^2$$

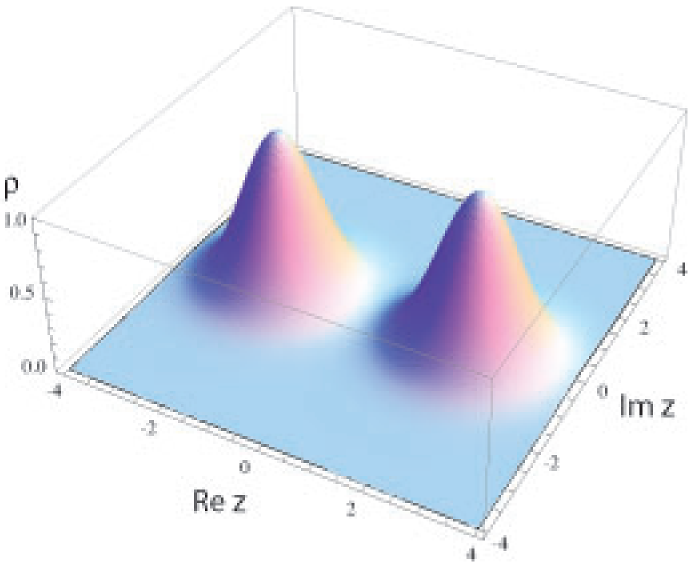
$$\cosh(2Z\,\mathrm{Re}z) - e^{-2Z^2}\cos(2Z\,\mathrm{Im}z) \simeq 2Z^2(1+|z|^2)$$

$$\rho(z) \simeq \frac{2(1-Z^2)}{4Z^2} e^{-|z|^2}\, 2Z^2(1+|z|^2) \simeq e^{-|z|^2}(1+|z|^2) + \mathcal{O}(Z^2)$$

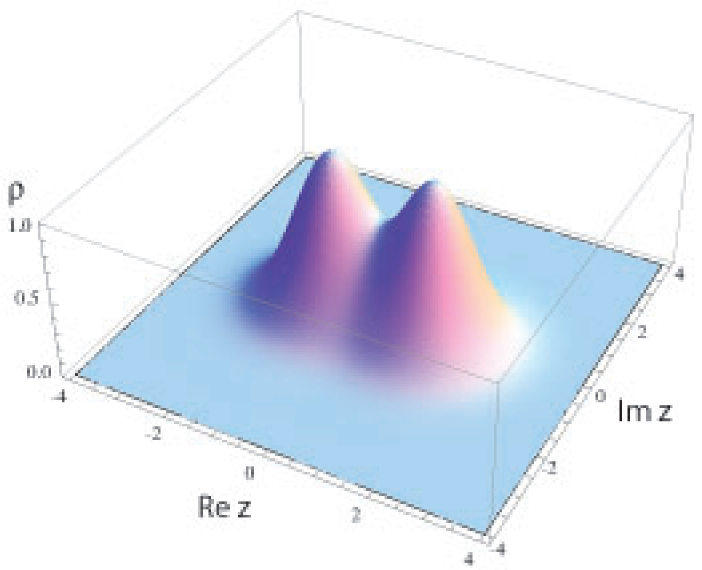
$$\lim_{Z\to 0} \rho(z) = e^{-|z|^2}(1+|z|^2) = \underline{\sum_{n=0}^{1} \frac{|z|^{2n}}{n!} e^{-|z|^2}}$$

same as when ang. mom $l=0$ and $l=1$ are occupied

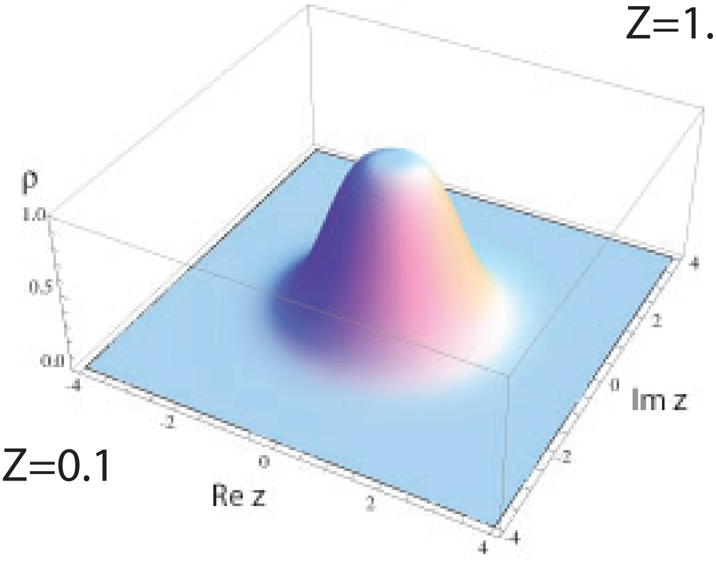
Antisymmetrized coherent states



Z=2.0

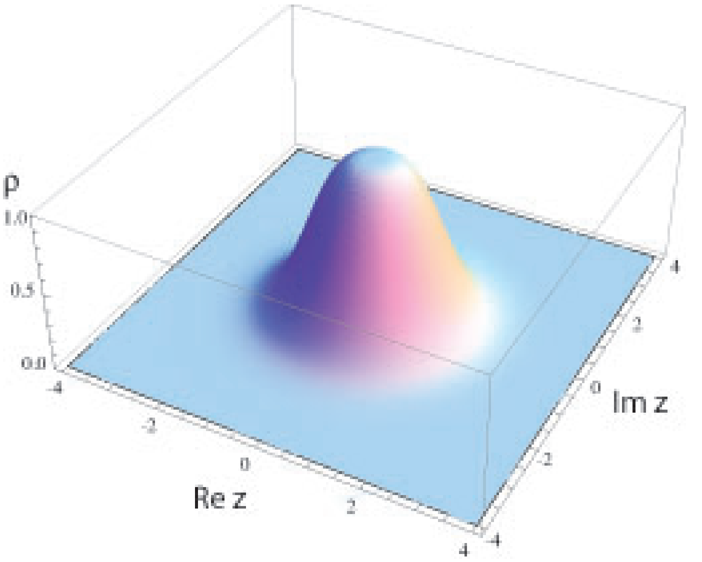


Z=1.0

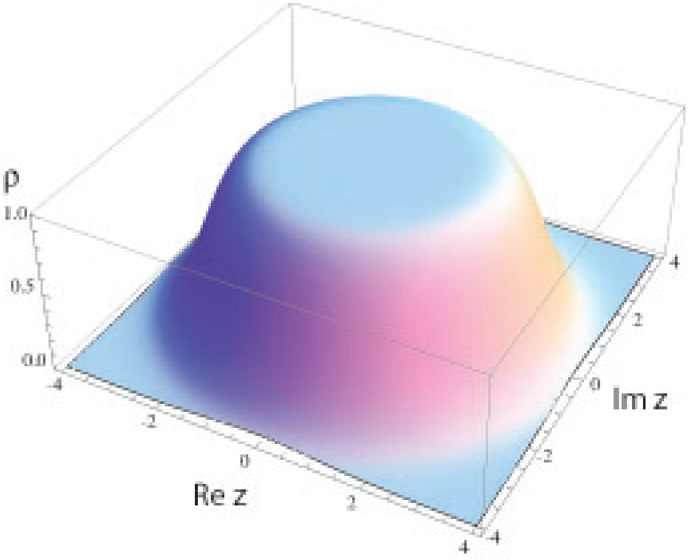


Z=0.1

Angular momentum states



N=2



N=10

Midttermineksamen FYS4110, høsten 2013

Løsninger

Oppgave 1

a) Benytter produktregelen for Paulimatriser:

$$\hat{\rho}^2 = \frac{1}{16}\Big[\big(1+\vec{a}^2+\vec{b}^2+\sum_{ij} c_{ij}^2\big)\,\mathbb{1}\otimes\mathbb{1}$$

$$+\,2\sum_i\big(a_i+\sum_j c_{ij}b_j\big)\,\sigma_i\otimes\mathbb{1}$$

$$+\,2\sum_j\big(b_j+\sum_i a_i c_{ij}\big)\,\mathbb{1}\otimes\sigma_j$$

$$+\sum_{ij}\big(2c_{ij}+2a_i b_j-\sum_{klmn}\varepsilon_{kmi}\varepsilon_{lnj}c_{kl}c_{mn}\big)\,\sigma_i\otimes\sigma_j\Big]$$

Reduserte tetthetsmatriser,

benytter $\mathrm{Tr}\,\sigma_i = 0 \quad i=1,2,3$: , $\mathrm{Tr}\,\mathbb{1} = 2$ for hvert delsystem

$$\hat{\rho}_A = \tfrac{1}{2}(\mathbb{1}+\vec{a}\cdot\vec{\sigma}), \quad \hat{\rho}_B = \tfrac{1}{2}(\mathbb{1}+\vec{b}\cdot\vec{\sigma})$$

$$\hat{\rho}_A^2 = \tfrac{1}{4}\big((1+\vec{a}^2)\mathbb{1}+2\vec{a}\cdot\vec{\sigma}\big), \quad \hat{\rho}_B^2 = \tfrac{1}{4}\big((1+\vec{b}^2)\mathbb{1}+2\vec{b}\cdot\vec{\sigma}\big)$$

b) Spektralutvikling av $\hat{\rho}$

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \quad \text{med } 0 \le p_k \le 1, \ \sum_k p_k = 1$$

og $\langle\psi_k|\psi_l\rangle = \delta_{kl}$

$$\Rightarrow \hat{\rho}^2 = \sum_k p_k^2 |\psi_k\rangle\langle\psi_k|$$

med $p_k^2 \le p_k$

$$\Rightarrow \mathrm{Tr}\,\hat{\rho}^2 \le \mathrm{Tr}\,\hat{\rho}$$

Likhet $p_k^2 = p_k \Rightarrow p_k = 1$ eller 0,

kan bare oppnås med $p_k = 1$ for én k-verdi

$$\Rightarrow \hat{\rho} = |\psi\rangle\langle\psi|$$, dvs ren tilstand

Betingelse på koeffisienter

$$\mathrm{Tr}\,\hat{\rho}^2 = \frac{1}{4}\Big(1 + \vec{a}^2 + \vec{b}^2 + \sum_{ij} c_{ij}^2\Big) \leq 1$$

$$\Leftrightarrow \quad \underline{\vec{a}^2 + \vec{b}^2 + \sum_{ij} c_{ij}^2 \leq 3}$$

Tilsvarende $\mathrm{Tr}_A\,\hat{\rho}_A^2 \leq 1 \Rightarrow \underline{\vec{a}^2 \leq 1}$

$\mathrm{Tr}_B\,\hat{\rho}_B^2 \leq 1 \Rightarrow \underline{\vec{b}^2 \leq 1}$

c)

Anta $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$ tensorprodukttilstand

med $\hat{\rho}_A = \frac{1}{2}(\mathbb{1} + \vec{a}\cdot\vec{\sigma})$; $\hat{\rho}_B = \frac{1}{2}(\mathbb{1} + \vec{b}\cdot\vec{\sigma})$

$$\Rightarrow \quad \hat{\rho} = \frac{1}{4}(\mathbb{1} + \vec{a}\cdot\vec{\sigma}) \otimes (\mathbb{1} + \vec{b}\cdot\vec{\sigma})$$

$$= \frac{1}{4}\Big(\mathbb{1}\otimes\mathbb{1} + \vec{a}\cdot\vec{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{b}\cdot\vec{\sigma} + \sum_{ij} a_i b_j\, \sigma_i \otimes \sigma_j\Big)$$

$$\Rightarrow \quad \underline{c_{ij} = a_i b_j}$$

Anta $\hat{\rho}$ ren og maksimalt sammenfiltret,

dvs $\hat{\rho}_A$ og $\hat{\rho}_B$ er maksimalt blandet:

$$\hat{\rho}^2 = \hat{\rho}, \quad \hat{\rho}_A = \frac{1}{2}\mathbb{1}_A, \quad \hat{\rho}_B = \frac{1}{2}\mathbb{1}_B$$

$$\Rightarrow \quad \vec{a} = \vec{b} = 0$$

$$\hat{\rho}^2 = \frac{1}{16}\Big[\Big(1 + \sum_{ij} c_{ij}^2\Big)\mathbb{1}\otimes\mathbb{1} + 2\sum_{ij}\Big(c_{ij} - \frac{1}{2}\sum_{klmn} \varepsilon_{kmi}\,\varepsilon_{lnj}\, c_{kl}\, c_{mn}\Big)\sigma_i \otimes \sigma_j$$

$$\hat{\rho}^2 = \hat{\rho} \quad \Rightarrow$$

$$\underline{\sum_{ij} c_{ij}^2 = 3} \quad \& \quad \underline{\frac{1}{2}\sum_{klmn} \varepsilon_{kmi}\,\varepsilon_{lnj}\, c_{kl}\, c_{mn} = -c_{ij}}$$

d) Oversettelse fra bra-ket-notasjon

$$|\pm\rangle\langle\pm| = \frac{1}{2}(\mathbb{1} \pm \sigma_z)$$

$$|\pm\rangle\langle\mp| = \frac{1}{2}(\sigma_x \pm i\sigma_y)$$

$$\Rightarrow \quad |++\rangle\langle++| + |--\rangle\langle--| = \frac{1}{2}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\sigma_z)$$

$$|++\rangle\langle--| + |--\rangle\langle++| = \frac{1}{2}(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y)$$

$$\Rightarrow \quad \underline{\hat{\rho}_{B1} = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z)}$$

$$\underline{\hat{\rho}_{B2} = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} - \sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z)}$$

B1 & B2 : $\vec{a} = \vec{b} = 0 \quad C_{ij} = C_i\delta_{ij}$

B1 : $C_x = +1, \ C_y = -1, \ C_z = +1$

B2 : $C_x = -1, \ C_y = +1, \ C_z = +1$

$C_{ij} = C_i\delta_{ij} \quad \Rightarrow$

$$\sum_{ij} C_{ij}^2 = \sum_i C_i^2 = \underline{3} \quad \text{for B1 \& B2}$$

$$\frac{1}{2}\sum_{klmn} \varepsilon_{kmi}\varepsilon_{lnj} C_{kl} C_{mn} = \frac{1}{2}\sum_{km} \varepsilon_{kmi}\varepsilon_{kmj} C_k C_m$$

$$= \frac{1}{2}\delta_{ij}\sum_{km} \varepsilon_{kmi}^2 C_k C_m$$

$i=j=1: \quad = \frac{1}{2}(\varepsilon_{231}^2 + \varepsilon_{321}^2) C_2 C_3 = C_2 C_3 = \mp 1$

$i=j=2: \quad = \frac{1}{2}(\varepsilon_{312}^2 + \varepsilon_{132}^2) C_1 C_3 = C_1 C_3 = \pm 1$

$i=j=3: \quad = \frac{1}{2}(\varepsilon_{123}^2 + \varepsilon_{213}^2) C_1 C_2 = C_1 C_2 = -1$

likhet med $-C_{ij} = -C_i\delta_{ij}$:

$i=j=1: \quad = -C_1 = \mp 1$

$i=j=2: \quad = -C_2 = \pm 1$

$i=j=3 \quad = -C_3 = -1$

dvs : betingelse $\frac{1}{2}\sum_{klmn} \varepsilon_{kmi}\varepsilon_{lnj} C_{kl} C_{mn} = -C_{ij}$ <u>er oppfylt</u>

e) $\hat{\rho}_1(t) = \cos^2\omega t\, \hat{\rho}_{B1} + \sin^2\omega t\, \hat{\rho}_{B2}$

$+ \cos\omega t \sin\omega t \left(|B1\rangle\langle B2| + |B2\rangle\langle B1| \right)$

$|B1\rangle\langle B2| + |B2\rangle\langle B1| = \frac{1}{2}\left(|++\rangle\langle ++| - |--\rangle\langle --| \right)$

$= \frac{1}{2}\left(\mathbb{1}\otimes\sigma_z + \sigma_z\otimes\mathbb{1} \right)$

$$\Rightarrow \hat{\rho}_1(t) = \underline{\frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sin(2\omega t)(\mathbb{1}\otimes\sigma_z + \sigma_z\otimes\mathbb{1}) + \cos(2\omega t)(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y) \right.}$$
$$\underline{\left. + \sigma_z\otimes\sigma_z \right)}$$

$\hat{\rho}_A(t) = \frac{1}{2}\left(\mathbb{1} + \sin(2\omega t)\,\sigma_z \right)$
$\hat{\rho}_B(t) = $ — " — $\Big\}$ egenverdier $\underline{p_\pm = \frac{1}{2}(1 \pm \sin(2\omega t))}$

Sammenfiltringsentropi $S_e(t) = \underline{-p_+ \log p_+ - p_- \log p_-}$

f) $\hat{\rho}_2(t) = \cos^2\omega t\, \hat{\rho}_{B1} + \sin^2\omega t\, \hat{\rho}_{B2}$

$\hat{\rho}_2(t)|B1\rangle = \cos^2\omega t\, |B1\rangle$
$\hat{\rho}_2(t)|B2\rangle = \sin^2\omega t\, |B2\rangle$

Entropi $S(t) = \underline{-\cos^2\omega t \log(\cos^2\omega t) - \sin^2\omega t \log(\sin^2\omega t)}$

$\hat{\rho}_A = \rho_B = \frac{1}{2}\mathbb{1} \Rightarrow \underline{S_A = S_B = \log 2}$

g) $\omega t = \frac{\pi}{4}$

$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|B1\rangle + |B2\rangle \right) = |++\rangle = |+\rangle\otimes|+\rangle$

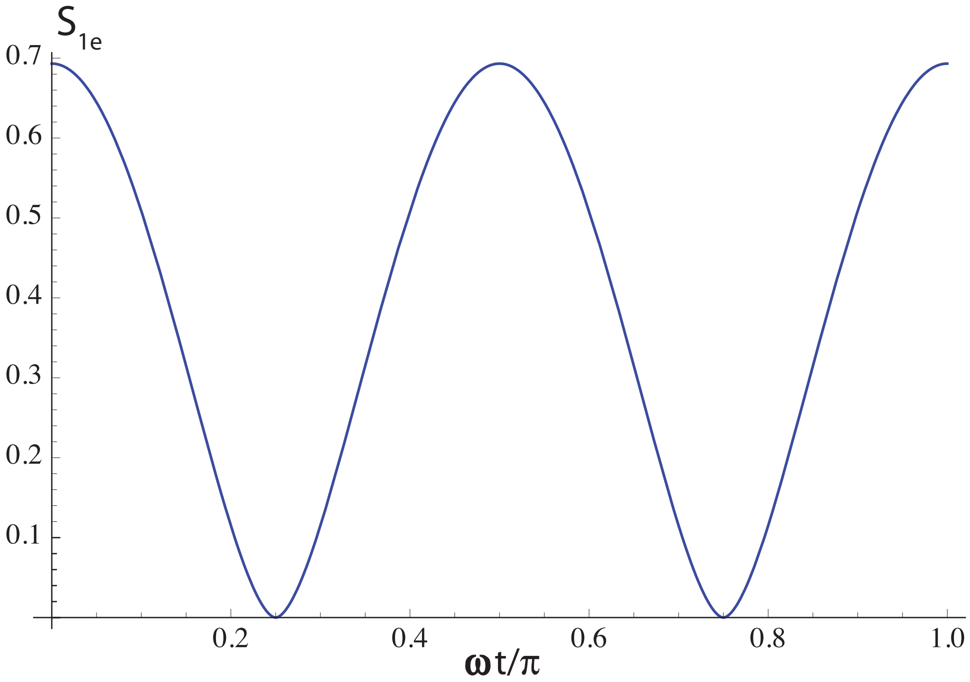
ren produkttilstand $\Rightarrow$ separabel $\quad \underline{\hat{\rho}_1 = |+\rangle\langle +| \otimes |+\rangle\langle +|}$

$\hat{\rho}_2 = \frac{1}{2}\left(\hat{\rho}_{B1} + \hat{\rho}_{B2} \right) = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\sigma_z \right)$

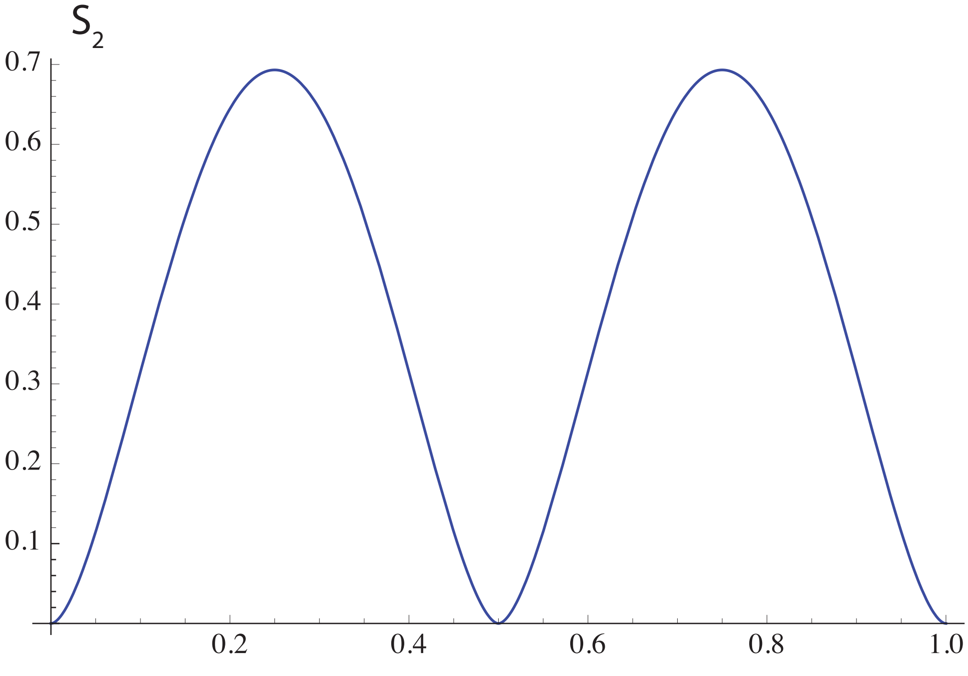
$= \underline{\frac{1}{2}\left\{ \left[\frac{1}{2}(\mathbb{1}+\sigma_z)\right]\otimes\left[\frac{1}{2}(\mathbb{1}+\sigma_z)\right] + \left[\frac{1}{2}(\mathbb{1}-\sigma_z)\right]\otimes\left[\frac{1}{2}(\mathbb{1}-\sigma_z)\right] \right\}}$

sum av to produkttilstander $\Rightarrow$ separabel

Oppgave 1 e)
Sammenfiltringsentropi
(målt i naturlig logaritme)



Oppgave 1 f)
Von Neumann-entropi



Oppgave 2

a) $\hat{H}|g,1\rangle = (\frac{1}{2}\hbar\omega - i\gamma\hbar)|g,1\rangle + \frac{1}{2}\hbar\lambda|e,0\rangle$

$\hat{H}|e,0\rangle = \frac{1}{2}\hbar\omega|e,0\rangle + \frac{1}{2}\hbar\lambda|g,1\rangle$

($\hat{H}|g,0\rangle = -\frac{1}{2}\hbar\omega|g,0\rangle$ frakoblet de andre)

I 2-dim. underrom,

$$\hat{H} = \begin{pmatrix} \frac{1}{2}\hbar\omega & \frac{1}{2}\hbar\lambda \\ \frac{1}{2}\hbar\lambda & \frac{1}{2}\hbar(\omega-2i\gamma) \end{pmatrix} = \frac{1}{2}\hbar(\omega-i\gamma)\mathbb{1} + \frac{1}{2}\hbar\begin{pmatrix} i\gamma & \lambda \\ \lambda & -i\gamma \end{pmatrix}$$

b) Tidsutviklingsoperatoren kan skrives som

$$\hat{U}(t) = e^{-\frac{1}{2}(\omega-i\gamma)t}\, e^{-i\vec{\Omega}\cdot\vec{\sigma}t}$$

med $\vec{\Omega} = \frac{1}{2}(\lambda\vec{i} + i\gamma\vec{k})$

$$e^{-i\vec{\Omega}\cdot\vec{\sigma}t} = 1 - i\vec{\Omega}\cdot\vec{\sigma}t + \frac{1}{2!}(-i\vec{\Omega}\cdot\vec{\sigma}t)^2 + \dots + \frac{1}{n!}(-i\vec{\Omega}\cdot\vec{\sigma}t)^n + \dots$$

Utnytter $(\vec{\Omega}\cdot\vec{\sigma})^2 = \vec{\Omega}^2 \equiv \Omega^2$

$\Rightarrow \quad (\vec{\Omega}\cdot\vec{\sigma})^3 = \Omega^2\,\vec{\Omega}\cdot\vec{\sigma}$ etc

Skiller mellom like og odde potenser

$$e^{-i\vec{\Omega}\cdot\vec{\sigma}t} = 1 - \frac{1}{2}\Omega^2t^2 + \frac{1}{4!}\Omega^4t^4 + \dots$$
$$- i\frac{\vec{\Omega}}{\Omega}\cdot\vec{\sigma}\left(\Omega t - \frac{1}{3!}\Omega^3t^3 + \dots\right)$$
$$= \cos(\Omega t) - i\frac{\vec{\Omega}}{\Omega}\cdot\vec{\sigma}\sin(\Omega t)$$

$$\Rightarrow \quad \hat{U}(t) = e^{-\frac{1}{2}(\omega-i\gamma)t}\left(\cos\Omega t - i\frac{\vec{\Omega}}{\Omega}\cdot\vec{\sigma}\sin\Omega t\right)$$

korrekt form med $\vec{\Omega} = \frac{1}{2}(\lambda\vec{i} + i\gamma\vec{k})$

$\Rightarrow \quad \vec{\Omega}^2 = \frac{1}{4}(\lambda^2 - \gamma^2)$ reell og positiv når $\lambda > \gamma$

$\Rightarrow \quad \Omega = \frac{1}{2}\sqrt{\lambda^2 - \gamma^2}$

c) På matriseform

$$\psi(t) = \hat{u}(t)\,\psi(0)$$

$$= e^{-\frac{1}{2}(i\omega+\gamma)t}\begin{pmatrix} \cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t & -i\frac{\lambda}{2\Omega}\sin\Omega t \\ -i\frac{\lambda}{2\Omega}\sin\Omega t & \cos\Omega t - \frac{\gamma}{2\Omega}\sin\Omega t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= e^{-\frac{1}{2}(i\omega+\gamma)t}\begin{pmatrix} \cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t \\ -i\frac{\lambda}{2\Omega}\sin\Omega t \end{pmatrix}$$

$$\Rightarrow \quad |\psi(t)\rangle = \underline{e^{-\frac{1}{2}(i\omega+\gamma)t}\left[\left(\cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t\right)|e,0\rangle\right.}$$
$$\underline{\left. -i\frac{\lambda}{2\Omega}\sin\Omega t\,|g,1\rangle\right]}$$

d) $$\mathrm{Tr}\,\hat{\rho}(t) = \langle\psi(t)|\psi(t)\rangle$$

$$= e^{-\gamma t}\left(\left(\cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t\right)^2 + \frac{\lambda^2}{4\Omega^2}\sin^2\Omega t\right)$$

$$= \underline{e^{-\gamma t}\left(\frac{\lambda^2}{4\Omega^2} - \frac{\gamma^2}{4\Omega^2}\cos 2\Omega t + \frac{\gamma}{2\Omega}\sin 2\Omega t\right)}$$

$$\mathrm{Tr}\,\hat{\rho}_{cav} = 1 \quad \Rightarrow$$

$$f(t) = \underline{1 - \mathrm{Tr}\,\hat{\rho}(t)} = 1 - \langle\psi(t)|\psi(t)\rangle$$

Ved utsendelse av fotonet gjennom kavitetsveggen vil systemet ende opp i tilstand $|g,0\rangle$. Tillegget til $\hat{\rho}$ sørger for at det skjer slik at den samlete sannsynlighet for at atomet er i en av tilstandene $|e\rangle$ og $|g\rangle$ er konstant, lik 1.

e) Besetningssannsynligheter for atomet

$$p_e(t) = \langle e,0|\hat{\rho}_{tot}(t)|e,0\rangle$$
$$= \langle e,0|\hat{\rho}(t)|e,0\rangle$$
$$= |\langle\psi(t)|e,0\rangle|^2$$
$$= e^{-\gamma t}\left(\cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t\right)^2$$
$$= \underline{e^{-\gamma t}\left(\frac{\lambda^2}{8\Omega^2} + \frac{\lambda^2-2\gamma^2}{8\Omega^2}\cos 2\Omega t + \frac{\gamma}{2\Omega}\sin 2\Omega t\right)}$$

$$p_g(t) = \underline{1 - p_e(t)}$$

Sannsynlighet for et foton i kaviteten

$$p_f(t) = \langle g,1|\hat{\rho}(t)|g,1\rangle$$
$$= |\langle\psi(t)|g,1\rangle|^2$$
$$= \underline{\frac{\lambda^2}{8\Omega^2}e^{-\gamma t}(1-\cos 2\Omega t)}$$

f) $$\hat{\rho}_{cav}(t) = |\psi(t)\rangle\langle\psi(t)| + f(t)|g,0\rangle\langle g,0|$$
$$= \langle\psi(t)|\psi(t)\rangle\,|\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)| + \text{—"—}$$
$$= \underline{(1-f(t))|\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)| + f(t)|g,0\rangle\langle g,0|}$$

hvor $\langle\tilde{\psi}(t)|\tilde{\psi}(t)\rangle = 1$

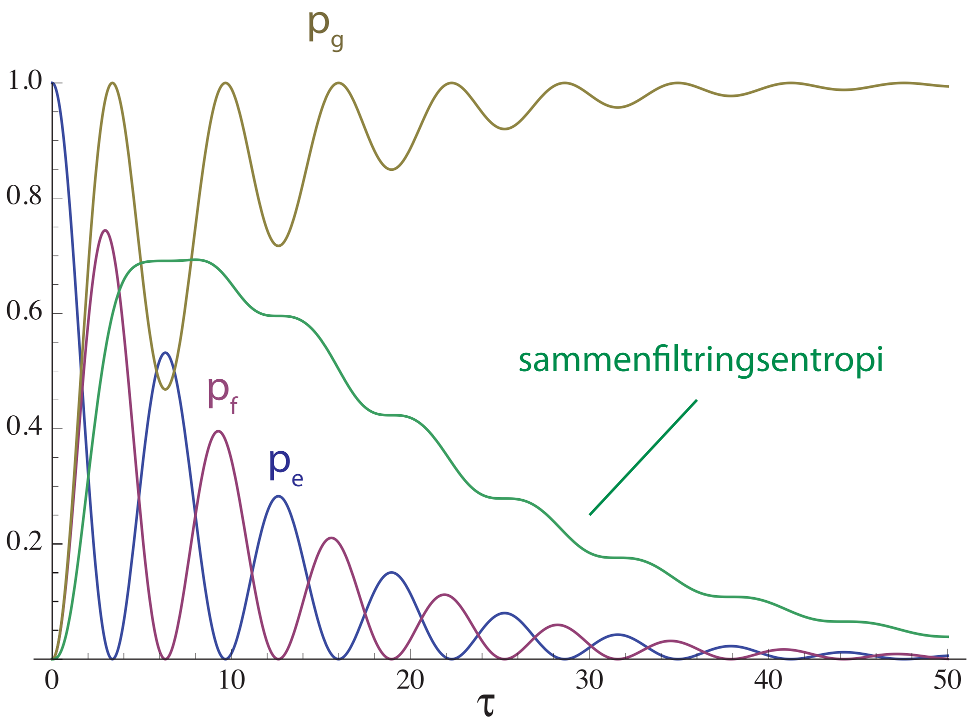
Dette er en spektralutvikling av $\hat{\rho}_{tot}$ siden $\langle\tilde{\psi}|g,0\rangle = 0$

Egenverdiene er $f(t)$ og $1-f(t)$.

Entropi $S = \underline{-f\log f - (1-f)\log(1-f)}$

er lik sammenfiltringsentropien til det sammensatte systemet.

Oppgave 2 e) og f)
Besetningssannsynligheter og sammenfiltringsentropi



FYS4110 Midterm Exam 2014

Solutions

Problem 1 Spin splitting in positronium

a) $\langle ij|\vec{\Sigma}_e\cdot\vec{\Sigma}_p|kl\rangle$

$$= \sum_{m,n} \langle ij|\vec{\sigma}_e\otimes\mathbb{1}_p|mn\rangle\cdot\langle mn|\mathbb{1}_e\otimes\vec{\sigma}_p|kl\rangle$$

$$= \sum_{m,n} (\langle i|\vec{\sigma}_e|m\rangle\delta_{jn})\cdot(\delta_{mk}\langle n|\vec{\sigma}_p|l\rangle)$$

$$= \underline{\langle i|\vec{\sigma}_e|k\rangle\cdot\langle j|\vec{\sigma}_p|l\rangle}$$

b) Matrix elements

$$\vec{\sigma} = \sigma_x\vec{i} + \sigma_y\vec{j} + \sigma_z\vec{k} \quad\Rightarrow$$

$$\langle +|\vec{\sigma}|+\rangle = \vec{k}\ ;\quad \langle -|\vec{\sigma}|-\rangle = -\vec{k}$$

$$\langle +|\vec{\sigma}|-\rangle = \vec{i}-i\vec{j}\ ,\quad \langle -|\vec{\sigma}|+\rangle = \vec{i}+i\vec{j}$$

$$\Rightarrow \langle ++|\vec{\Sigma}_e\cdot\vec{\Sigma}_p|++\rangle = \vec{k}\cdot\vec{k} = 1$$

$$\langle ++|\ \text{—''—}\ |+-\rangle = \vec{k}\cdot(\vec{i}-i\vec{j}) = 0$$

$$\langle ++|\ \text{—''—}\ |-+\rangle = \ \text{—''—}\ = 0$$

$$\langle ++|\ \text{—''—}\ |--\rangle = (\vec{i}-i\vec{j})^2 = 0$$

$$\langle +-|\ \text{—''—}\ |+-\rangle = -\vec{k}\cdot\vec{k} = -1$$

$$\langle +-|\ \text{—''—}\ |-+\rangle = (\vec{i}-i\vec{j})\cdot(\vec{i}+i\vec{j}) = 2$$

$$\langle +-|\ \text{—''—}\ |--\rangle = (\vec{i}-i\vec{j})\cdot(-\vec{k}) = 0$$

$$\langle -+|\ \text{—''—}\ |-+\rangle = (-\vec{k})\cdot\vec{k} = -1$$

$$\langle -+|\ \text{—''—}\ |--\rangle = (-\vec{k})\cdot(\vec{i}-i\vec{j}) = 0$$

$$\langle --|\ \text{—''—}\ |--\rangle = (-\vec{k})^2 = 1$$

other terms determined by hermiticity of $\vec{\Sigma}_e\cdot\vec{\Sigma}_p$

Matrix representation

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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c) From b) follows

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |0,0\rangle = \frac{1}{\sqrt{2}} \left(\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |+-\rangle - \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |-+\rangle \right)$$

$$= -\frac{3}{4} \hbar^2 |0,0\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,1\rangle = \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |++\rangle = \frac{\hbar^2}{4} |1,1\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,0\rangle = \frac{\hbar^2}{4} |1,0\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,-1\rangle = \frac{\hbar^2}{4} |1,-1\rangle$$

In the spin basis,

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p = \frac{\hbar^2}{4} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Total spin $\hat{\vec{S}}^2 = (\hat{\vec{S}}_e + \hat{\vec{S}}_p)^2 = \hat{\vec{S}}_e^2 + \hat{\vec{S}}_p^2 + 2 \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p$

$$= \frac{\hbar^2}{4} \left[(\vec{\sigma}_e \otimes \mathbb{1}_p)^2 + (\mathbb{1}_e \otimes \vec{\sigma}_p)^2 \right] + 2 \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p$$

$$= \frac{3}{2} \hbar^2 \mathbb{1} + 2 \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p$$

$\Rightarrow$ in spin basis

$$\hat{\vec{S}}^2 = 2\hbar^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S_z = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\vec{S}^2 = s(s+1)\hbar^2 \Rightarrow$ $s = 0$ for $|0,0\rangle$ singlet

$s = 1$ for $|1,m\rangle$ $m = 0, \pm 1$ triplet

d) Need to find the matrix elements of $(S_e)_z - (S_p)_z \equiv D$

$$D|1,1\rangle = D|1,-1\rangle = 0$$

$$D|0,0\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(2|+-\rangle - (-2)|-+\rangle\right) = \hbar\,|1,0\rangle$$

$$D|1,0\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(2|+-\rangle + (-2)|-+\rangle\right) = \hbar\,|0,0\rangle$$

mixes only $|0,0\rangle$ and $|1,0\rangle$

Hamiltonian in the spin basis

$$H = \begin{pmatrix} E_0 - \frac{3}{4}\hbar^2\kappa & 0 & \lambda\hbar^2 & 0 \\ 0 & E_0 + \frac{1}{4}\hbar^2\kappa & 0 & 0 \\ \lambda\hbar^2 & 0 & E_0 + \frac{1}{4}\hbar^2\kappa & 0 \\ 0 & 0 & 0 & E_0 + \frac{1}{4}\hbar^2\kappa \end{pmatrix}$$

e) · $|1,1\rangle$ and $|1,-1\rangle$ are eigenvectors with eigenvalues $\underline{E = E_0 + \frac{1}{4}\hbar^2\kappa}$ (indep. of λ)

Eigenvalue problem for the remaining two states

$$\begin{pmatrix} E_0 - \frac{3}{4}\hbar^2\kappa & \lambda\hbar^2 \\ \lambda\hbar^2 & E_0 + \frac{1}{4}\hbar^2\kappa \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = E \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

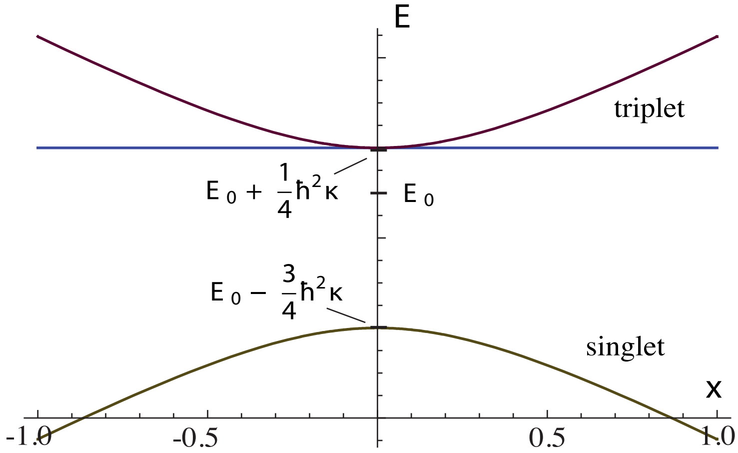
write this as $\left(E_0 - \frac{1}{4}\hbar^2\kappa\right)\mathbb{1} + \frac{1}{2}\hbar^2\kappa \begin{pmatrix} -1 & 2x \\ 2x & 1 \end{pmatrix}$ $\quad x = \lambda/\kappa$

$$\Rightarrow \begin{pmatrix} -1 & 2x \\ 2x & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \mu \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \quad \text{with} \quad E = E_0 - \frac{1}{4}\hbar^2\kappa + \frac{1}{2}\hbar^2\kappa\mu$$

eigenvalues $\begin{vmatrix} -1-\mu & 2x \\ 2x & 1-\mu \end{vmatrix} = 0 \Rightarrow \mu^2 = 4x^2 + 1$

$$E_\pm = E_0 - \frac{1}{4}\hbar^2\kappa \pm \frac{1}{2}\hbar^2\kappa\sqrt{4x^2+1}$$

$$= \underline{E_0 - \frac{1}{4}\hbar^2\kappa \pm \frac{1}{2}\hbar^2\sqrt{\kappa^2 + 4\lambda^2}}$$



f) $$\hat{\rho}_a = |a\rangle\langle a| = |\alpha|^2 |+-\rangle\langle +-| + |\beta|^2 |-+\rangle\langle -+|$$
$$+ \alpha\beta^* |+-\rangle\langle -+| + \alpha^*\beta |-+\rangle\langle +-|$$

$$\hat{\rho}_b = |b\rangle\langle b| = |\beta|^2 |+-\rangle\langle +-| + |\alpha|^2 |-+\rangle\langle -+|$$
$$- \alpha\beta^* |+-\rangle\langle -+| - \alpha^*\beta |-+\rangle\langle +-|$$

Reduced density operators

$$\hat{\rho}_{ae} = \mathrm{Tr}_p\, \hat{\rho}_a = |\alpha|^2 |+\rangle\langle +| + |\beta|^2 |-\rangle\langle -|$$

$$\hat{\rho}_{ap} = \mathrm{Tr}_e\, \hat{\rho}_a = |\alpha|^2 |-\rangle\langle -| + |\beta|^2 |+\rangle\langle +|$$

$$\hat{\rho}_{be} = \mathrm{Tr}_p\, \hat{\rho}_B = |\beta|^2 |+\rangle\langle +| + |\alpha|^2 |-\rangle\langle -|$$

$$\hat{\rho}_{bp} = \mathrm{Tr}_e\, \hat{\rho}_B = |\beta|^2 |-\rangle\langle -| + |\alpha|^2 |+\rangle\langle +|$$

g. Entropy

$$S_{ae} = S_{ap} = S_{be} = S_{bp} = -\left(|\alpha|^2 \log|\alpha|^2 + |\beta|^2 \log|\beta|^2\right)$$
$$= \underline{-\left(|\alpha|^2 \log|\alpha|^2 + (1-|\alpha|^2)\log(1-|\alpha|^2)\right)}$$

g) Eigenstates

$$|a\rangle = \gamma|0,0\rangle + \delta|1,0\rangle = \alpha|+-\rangle + \beta|-+\rangle$$

$$\Rightarrow \quad \alpha = \frac{\gamma+\delta}{\sqrt{2}} \quad , \quad \beta = \frac{\gamma-\delta}{\sqrt{2}}$$

γ, δ determined by eigenvalue eq. in e):

$$-\gamma + 2x\delta = \mu\gamma \quad \Rightarrow \quad \delta = \frac{\mu+1}{2x}\gamma$$

$\mu = \pm\sqrt{4x^2+1}$; choose $\mu = -\sqrt{4x^2+1}$ (+ gives $|b\rangle$)

gives $\delta \to 0$ for $x \to 0$

Note γ, δ real.

Normalization : $\gamma^2 + \delta^2 = \left(1 + \left(\frac{\mu+1}{2x}\right)^2\right)\gamma^2 = 1$

$$\Rightarrow \quad \gamma^2 = \frac{4x^2}{4x^2 + (\mu+1)^2}$$

α

$$\alpha^2 = \frac{1}{2}\left(1 + \frac{\mu+1}{2x}\right)^2 \gamma^2 = \frac{1}{2}\,\frac{(2x+\mu+1)^2}{4x^2+(\mu+1)^2}$$

$$(2x+\mu+1)^2 = 4x^2 + 1 + 4x + \mu^2 + 2(2x+1)\mu$$

$$= 2\left(\mu^2 + 2x(\mu+1) + \mu\right) = 2(\mu+1)(\mu+2x)$$

$$4x^2 + (\mu+1)^2 = 4x^2 + 1 + \mu^2 + 2\mu = 2(\mu^2+\mu) = 2\mu(\mu+1)$$

$$\Rightarrow \quad \alpha^2 = \frac{1}{2}\,\frac{2(\mu+2x)(\mu+1)}{2\mu(\mu+1)} = \underline{\frac{1}{2}\left(1 + \frac{2x}{\sqrt{4x^2+1}}\right)}$$

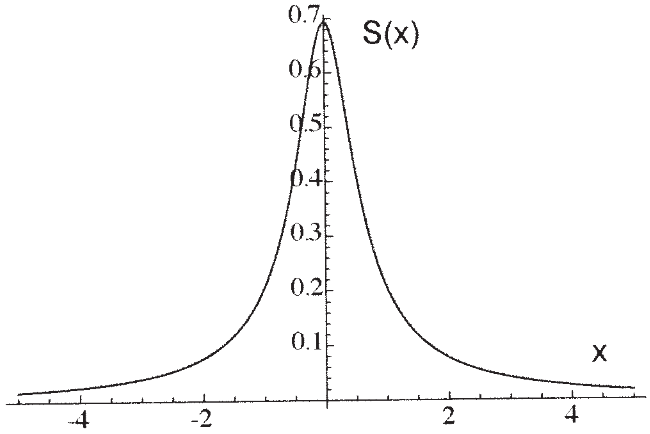
$$\beta^2 = 1 - \alpha^2 = \frac{1}{2}\left(1 - \frac{2x}{\sqrt{4x^2+1}}\right)$$

Entropy of reduced density matrices

$$\underline{S(x) = -\left[\alpha(x)^2 \log \alpha(x)^2 + \beta(x)^2 \log \beta(x)^2\right]}$$

For $x = 0$: $\hat{\rho}_{ae} = \hat{\rho}_{be} = \frac{1}{2}\mathbb{1}_e$, $\hat{\rho}_{ap} = \hat{\rho}_{bp} = \frac{1}{2}\mathbb{1}_p$

maximal entanglement $S(0) = \log 2$



Entanglement of states $|a\rangle$ and $|b\rangle$ as functions of $x = \lambda/\kappa$

Problem 2, Spin-coherent states

a) Eigenvalue equation

$$\hat{J}_-|\psi\rangle = \lambda|\psi\rangle, \quad |\psi\rangle = \sum_m c_m |j,m\rangle$$

Since $m \le j$, there must be a maximum value, $m \le m_{max}$ in the expansion. Application of $\hat{J}_-$ reduces $m \Rightarrow m_{max} \to m_{max}-1$.

Repeated application $\Rightarrow \hat{J}_-^{2j+1}|\psi\rangle = 0 = \lambda^{2j+1}|\psi\rangle$

This implies $\lambda = 0$, which is satisfied only for $|\psi\rangle = |j,-j\rangle$

Similar argument for $\hat{J}_+$ gives eigenvalue $= 0$ also for this operator. This is satisfied only for $|\psi\rangle = |j,j\rangle$.

b) $\hat{\vec{J}}^2 = j(j+1)\hbar^2 \Rightarrow (\Delta\vec{J})^2 = j(j+1)\hbar^2 - \langle\hat{\vec{J}}\rangle^2$

Implies: min. value for $(\Delta\vec{J})^2 \Leftrightarrow$ max value for $\langle\hat{\vec{J}}\rangle^2$.

For general state, define unit vector $\vec{n}$ by

$$\langle\vec{J}\rangle = J\vec{n}, \quad J^2 = \langle\vec{J}\rangle^2$$

This gives $\langle\vec{J}\rangle^2 = \langle J_{\vec{n}}\rangle^2 \quad \hat{J}_{\vec{n}} = \vec{n}\cdot\hat{\vec{J}}$

Rotational invariance $\Rightarrow$

all directions equivalent, may choose z-axis with $\vec{k} = \vec{n}$

For $\vec{n} = \vec{k}$:

$$\langle\vec{J}\rangle^2 = \langle J_z\rangle^2, \quad \langle J_x\rangle = \langle J_y\rangle = 0$$

$$\Rightarrow \langle\vec{J}\rangle^2 \le j(j+1)\hbar^2 \quad \text{since} \quad -j\hbar \le \langle J_z\rangle \le j\hbar$$

Inequality valid for all directions $\vec{n}$.

For $\vec{n} = \vec{k}$:

max. value for $\langle \vec{J} \rangle^2$ for $\langle \hat{J}_z \rangle^2 = j^2 \hbar^2$,

which is the case for the states $|j, -j\rangle$ and $|j, j\rangle$

For general $\vec{n}$ this corresponds to

$$\underline{\hat{J}_{\vec{n}} |j, \vec{n}\rangle = j\hbar |j, \vec{n}\rangle}$$

with $|j, \vec{n}\rangle$ denoting the eigenstate of $\hat{J}_{\vec{n}}$ with maximal eigenvalue. Note: all min. uncertainty states are then included, since $j \to -j$ is equivalent to $\vec{n} \to -\vec{n}$.

Minimum uncertainty value

$$(\Delta \vec{J})^2 = j(j+1)\hbar^2 - j^2\hbar^2 = \underline{j\hbar^2}$$

c) Spin $j = 1/2$

$\vec{J} = \frac{\hbar}{2} \vec{\sigma}$, use standard representation of Pauli matrices

$$\Rightarrow \vec{\sigma} = \sigma_x \vec{i} + \sigma_y \vec{j} + \sigma_z \vec{k} = \begin{pmatrix} \vec{k} & \vec{i} - i\vec{j} \\ \vec{i} + i\vec{j} & -\vec{k} \end{pmatrix}$$

General spin state $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\quad |\alpha|^2 + |\beta|^2 = 1$

$$\langle \vec{J} \rangle = \frac{\hbar}{2} \psi^\dagger \vec{\sigma} \psi = \frac{\hbar}{2} (\alpha^* \beta^*) \begin{pmatrix} \vec{k} & \vec{i} - i\vec{j} \\ \vec{i} + i\vec{j} & -\vec{k} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \frac{\hbar}{2} \left((\alpha^*\beta + \alpha\beta^*)\vec{i} + i(\alpha\beta^* - \alpha^*\beta)\vec{j} + (|\alpha|^2 - |\beta|^2)\vec{k} \right)$$

$$\Rightarrow \langle \vec{J} \rangle^2 = \frac{\hbar^2}{4} \left((\alpha^*\beta + \alpha\beta^*)^2 + (\alpha\beta^* - \alpha^*\beta)^2 + (|\alpha|^2 - |\beta|^2)^2 \right)$$

$$= \frac{\hbar^2}{4} (|\alpha|^2 + |\beta|^2)^2 = \frac{\hbar^2}{4} = \underline{j^2\hbar^2} \quad \text{for } j = \frac{1}{2}$$

$\langle \vec{J} \rangle^2$ maximal $\Rightarrow (\Delta \vec{J})^2$ minimal, valid for all ψ.

d) Coherent state, $j = 1/2$

$$\vec{\sigma}\cdot\vec{n}\,|z\rangle = |z\rangle \Rightarrow \sum_{m'} \langle m|\vec{\sigma}\cdot\vec{n}|m'\rangle\langle m'|z\rangle = \langle m|z\rangle$$

Matrix form

$$\begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

with $|z\rangle = \alpha|+\tfrac{1}{2}\rangle + \beta|-\tfrac{1}{2}\rangle$

$n_x = \vec{n}\cdot\vec{i}$ etc

$$\Rightarrow (n_z - 1)\,\alpha + (n_x - i n_y)\,\beta = 0$$

$$\Rightarrow (1-\cos\theta)\,\alpha = e^{-i\varphi}\sin\theta\,\beta$$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{\sin\theta}{1-\cos\theta}\,e^{-i\varphi} = \cot\tfrac{\theta}{2}\,e^{-i\varphi} = z$$

Normalized : $|\alpha|^2 + |\beta|^2 = 1$

$$\Rightarrow \alpha = \frac{z}{\sqrt{1+|z|^2}},\quad \beta = \frac{1}{\sqrt{1+|z|^2}}$$ up to common phase factor

$$\Rightarrow \underline{\langle m|z\rangle = \frac{z^{m+1/2}}{\sqrt{1+|z|^2}}}$$

e) $$\langle z|z_0\rangle = \sum_m \langle z|m\rangle\langle m|z_0\rangle = \frac{1+z^* z_0}{\sqrt{(1+|z|^2)(1+|z_0|^2)}}$$

$$\Rightarrow \underline{|\langle z|z_0\rangle|^2 = \frac{1 + z^* z_0 + z z_0^* + |z|^2|z_0|^2}{(1+|z|^2)(1+|z_0|^2)}}$$

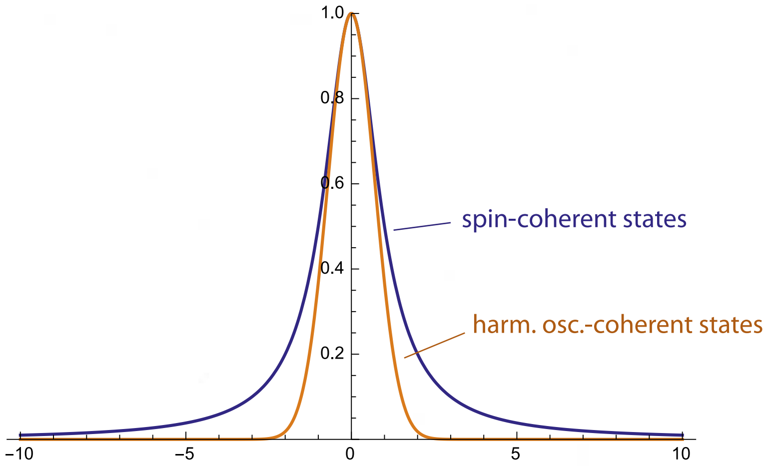
$$z_0 = 0 \quad |\langle z|0\rangle|^2 = \frac{1}{1+|z|^2} = \underline{\frac{1}{1+r^2}} \qquad z = r e^{-i\varphi}$$

Harmonic oscillator coherent states

$$|\langle z|z_0\rangle|^2 = e^{-|z-z_0|^2}$$

$$z_0 = 0,\ z = r e^{-i\varphi} \Rightarrow |\langle z|0\rangle|^2 = e^{-|z|^2} = \underline{e^{-r^2}}$$

Coherent states, overlap functions $|\langle z|0\rangle|^2$



f) $$I \equiv \int d^2z \, \frac{1}{(1+|z|^2)^2} \, |z\rangle\langle z|$$

$$= \sum_{m,m'} \int d^2z \, \frac{\langle m|z\rangle\langle z|m'\rangle}{(1+|z|^2)^2} \, |m\rangle\langle m'|$$

$$= \sum_{m,m'} \int d^2z \, \frac{z^{m+\frac{1}{2}} \, z^{*\,m'+\frac{1}{2}}}{(1+|z|^2)^3} \, |m\rangle\langle m'|$$

Change to polar coordinates $z = re^{i\varphi}$, $d^2z = r\,d\varphi\,dr$

$$I = \sum_{m,m'} \int_0^\infty dr \, \frac{r^{m+m'+2}}{(1+r^2)^3} \underbrace{\int_0^{2\pi} d\varphi \, e^{i(m-m')\varphi}}_{= 2\pi\delta_{mm'}} |m\rangle\langle m'|$$

$$= 2\pi \sum_m \int_0^\infty \frac{r^{2m+2}}{(1+r^2)^3} \, |m\rangle\langle m|$$

$$m = -\tfrac{1}{2}: \quad \frac{r^{2m+2}}{(1+r^2)^3} = \frac{r}{(1+r^2)^3} = -\frac{1}{4}\frac{d}{dr}\frac{1}{(1+r^2)^2}$$

$$\Rightarrow \int_0^\infty dr \, \frac{r}{(1+r^2)^3} = -\frac{1}{4}\left[\frac{1}{(1+r^2)^2}\right]_0^\infty = \underline{\frac{1}{4}}$$

$$m = +\tfrac{1}{2}: \quad \frac{r^{2m+2}}{(1+r^2)^3} = \frac{r^3}{(1+r^2)^3} = r\left(\frac{1}{(1+r^2)^2} - \frac{1}{(1+r^2)^3}\right)$$

$$= \frac{d}{dr}\left[-\frac{1}{2}\frac{1}{1+r^2} + \frac{1}{4}\frac{1}{(1+r^2)^2}\right]$$

$$\Rightarrow \int_0^\infty dr \, \frac{r^3}{(1+r^2)^3} = \Big[\; - \text{"} - \;\Big]_0^\infty = \underline{\frac{1}{4}}$$

$$I = 2\pi \sum_m \frac{1}{4} |m\rangle\langle m| = \frac{\pi}{2} \mathbb{1}$$

This gives

$$\underline{\int \frac{d^2z}{\pi} \frac{2}{(1+|z|^2)^2} |z\rangle\langle z| = \mathbb{1}}$$

completeness relation for the $j=\frac{1}{2}$ spin coherent states

g) $\hat{H} = \frac{1}{2}\hbar\omega\sigma_z$

$\Rightarrow \hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{2}\omega\sigma_z t}$ time evolution operator

$$\begin{aligned}
\hat{U}(t)|z_0\rangle &= \sum_m e^{-\frac{i}{2}\omega\sigma_z t}|m\rangle\langle m|z_0\rangle \\
&= \sum_m e^{-i\omega m t}|m\rangle\langle m|z_0\rangle \qquad \sigma_z|m\rangle = 2m|m\rangle \\
&= e^{\frac{i}{2}\omega t} \sum_m \frac{(e^{-i\omega t}z_0)^{m+\frac{1}{2}}}{\sqrt{1+|z_0|^2}} |m\rangle \\
&= \underline{e^{\frac{i}{2}\omega t} | e^{-i\omega t} z_0\rangle}
\end{aligned}$$

$= e^{i\alpha(t)}|z(t)\rangle$ with $\underline{\alpha = \frac{1}{2}\omega t}$ and $z(t) = \underline{e^{-i\omega t} z_0}$

Midterm Exam FYS4110/9110, 2015

Solutions

Problem 1

a) Spin compositions

spin ½ x spin ½ = spin 0 + spin 1

with spin 0 and spin 1 defining orthogonal subspaces in the composite Hilbert space

Repeated

spin ½ x (spin ½ x spin ½) = spin ½ x spin 0 + spin ½ x spin 1

= spin ½ + spin ½ + spin 3/2

defining three orthogonal subspaces in the full Hilbert space.

b) Scalar products

$$\langle\psi_n|\psi_{n'}\rangle = \frac{1}{3}\left(1 + e^{2\pi i(n'-n)/3} + e^{-2\pi i(n'-n)/3}\right)$$

$$= \frac{1}{3}\left(1 + 2\cos\left(\frac{2\pi}{3}(n'-n)\right)\right)$$

$$n' = n \Rightarrow \cos\left(\frac{2\pi}{3}(n'-n)\right) = \cos 0 = 1$$

$$n' = \pm n \Rightarrow \cos\left(\frac{2\pi}{3}(n'-n)\right) = \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$$

$$\Rightarrow \underline{\langle\psi_n|\psi_{n'}\rangle = \delta_{nn'}}$$ orthogonal for $n \neq n'$

$$\hat{S}_z|\psi_n\rangle = \frac{\hbar}{2}(1-1-1)|\psi_n\rangle = \underline{-\frac{\hbar}{2}|\psi_n\rangle}$$

Use lowering operator in the spectrum of $\hat{S}_z$

$$\hat{S}_- = \hat{S}_x - i\hat{S}_y = \hat{S}_{-1} + \hat{S}_{-2} + \hat{S}_{-3}$$

For single spin $\hat{S}_-|u\rangle = |d\rangle$, $\hat{S}_-|d\rangle = 0$

For the three spins

$$\hat{S}_-|udd\rangle = \hat{S}_-|dud\rangle = \hat{S}_-|ddu\rangle = |ddd\rangle$$

$$\Rightarrow \hat{S}_-|\psi_n\rangle = \frac{1}{\sqrt{3}}\left(1+e^{2\pi i n/3}+e^{-2\pi i n/3}\right)|ddd\rangle$$

$$= \frac{1}{\sqrt{3}}\left(1+2\cos\left(\frac{2\pi n}{3}\right)\right)|ddd\rangle$$

$$\cos\left(\pm\frac{2\pi}{3}\right) = -\frac{1}{2} \Rightarrow$$

$$\underline{\hat{S}_-|\psi_0\rangle = \sqrt{3}\,|ddd\rangle} \qquad \underline{\hat{S}_-|\psi_{\pm 1}\rangle = 0}$$

This shows that $|\psi_\pm\rangle$ have no component with $s=\frac{3}{2}$

$\Rightarrow$ they are $s=\frac{1}{2}$ states $(\vec{S}^2 = \frac{3}{4}\hbar^2)$

This implies that $|\psi_0\rangle$ is the $s=\frac{3}{2}$ state $(\vec{S}^2 = \frac{15}{4}\hbar^2)$

c) Reduced density operator of spin 1

$$\hat{\rho}_1 = \mathrm{Tr}_{23}\Big(\frac{1}{3}\big(|udd\rangle\langle udd| + |dud\rangle\langle dud| + |ddu\rangle\langle ddu|$$
$$+\, e^{2\pi i n/3}\left(|dud\rangle\langle udd| + |udd\rangle\langle ddu|\right)$$
$$+\, e^{-2\pi i n/3}\left(|udd\rangle\langle dud| + |ddu\rangle\langle udd|\right)$$
$$+\, e^{4\pi i n/3}\,|dud\rangle\langle ddu| + e^{-4\pi i n/3}\,|ddu\rangle\langle dud|\big)$$

$$= \frac{1}{3}|u\rangle\langle u| + \frac{2}{3}|d\rangle\langle d|$$

Entanglement entropy for the 1(23) bipartite system

$$S_1 = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = \log 3 - \frac{2}{3}\log 2 = \underline{0.918}$$

max value $S_{1max} = \log 2 = \underline{1}$ (both $\log = \log_2$)

The entanglement entropy is the same for all n, close to but somewhat smaller than the max. value

The symmetry with respect to permuting the spins implies that the other partitions give the same value

d) Measurement of $\hat{S}_{1z}$

The state of spin 1 is projected to $|u\rangle$ or $|d\rangle$ depending on the result

A Result: spin up

$$|\psi_n\rangle \to |udd\rangle = |u\rangle \otimes |d\rangle \otimes |d\rangle$$

product state : no entanglement

B Result: spin down

$$|\psi_n\rangle \to |d\rangle \otimes |\phi_n\rangle$$

$$|\phi_n\rangle = \frac{1}{\sqrt{2}}\left(e^{2\pi i n/3}|ud\rangle + e^{-2\pi i n/3}|du\rangle\right)$$

$$\hat{\rho}_n = |\phi_n\rangle\langle\phi_n| = \frac{1}{2}\left(|ud\rangle\langle ud| + |du\rangle\langle du| + \text{cross terms}\right)$$

Reduced density operators

$$\hat{\rho}_{n1} = \hat{\rho}_{n2} = \frac{1}{2}\left(|u\rangle\langle u| + |d\rangle\langle d|\right) = \frac{1}{2}\mathbb{1}$$

Spin 2 and 3 are now in a maximally mixed state

e) New state

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|uuu\rangle - |ddd\rangle\right)$$

Reduced density operator

$$\hat{\rho}_1 = \mathrm{Tr}_{23}(|\phi\rangle\langle\phi|) = \frac{1}{2}\mathrm{Tr}_{23}\left(|uuu\rangle\langle uuu| + |ddd\rangle\langle ddd| + \text{cross terms}\right)$$

$$= \frac{1}{2}(|u\rangle\langle u| + |d\rangle\langle d|$$

$$= \frac{1}{2}\mathbb{1}$$

Entanglement entropy of partition 1(23)

$$S_1 = \underline{\log 2} = 1 \quad \text{maximal entanglement}$$

The same for the other partitions due to the symmetry of $|\phi\rangle$ under permutation of the spins

f) $|f\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)\,, \quad |b\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$

$|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle)\,, \quad |\ell\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|d\rangle)$

$$\Rightarrow \quad |u\rangle = \frac{1}{\sqrt{2}}(|f\rangle + |b\rangle) = \frac{1}{\sqrt{2}}(|r\rangle + |\ell\rangle)$$

$$|d\rangle = \frac{1}{\sqrt{2}}(|f\rangle - |b\rangle) = -\frac{i}{\sqrt{2}}(|r\rangle - |\ell\rangle)$$

$$\Rightarrow \quad |\phi\rangle = \frac{1}{\sqrt{2}}(|uuu\rangle - |ddd\rangle)$$

$$= \frac{1}{2}(|bbb\rangle + |f^2b\rangle + |fbf\rangle + |bff\rangle)$$

$$= \underline{\frac{1}{2}(|rrf\rangle + |\ell\ell f\rangle + |r\ell b\rangle + |\ell r b\rangle)}$$

Measurement of S_{2z} <u>or</u> S_{3z} determines S_{1z}

Measurement of S_{2x} and S_{3x}:

outcomes
$(bb)_{23} \Rightarrow b_1$
$(fb)_{23} \Rightarrow f_1$
$(bf)_{23} \Rightarrow f_1$
$(ff)_{23} \Rightarrow b_1$

} determines <u>uniquely</u> S_{x1}

Measurement of S_{y2} and S_{3x}

outcomes:
$(rf)_{23} \Rightarrow r_1$
$(\ell f)_{23} \Rightarrow \ell_1$
$(\ell b)_{23} \Rightarrow r_1$
$(rb)_{23} \Rightarrow \ell_1$

} determines <u>uniquely</u> S_{y1}

Problem 2

a) Total spin $\vec{S} = \frac{\hbar}{2}(\vec{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{\sigma}) \equiv \frac{\hbar}{2}(\vec{\Sigma}_A + \vec{\Sigma}_B)$

$$\vec{S}^2 = \frac{\hbar^2}{2}(3\,\mathbb{1}\otimes\mathbb{1} + \vec{\Sigma}_A\cdot\vec{\Sigma}_B)$$

$$= \frac{\hbar^2}{2}\left(3\,\mathbb{1} + \sum_{k=1}^{3}\sigma_k\otimes\sigma_k\right)$$

$$\sigma_k\otimes\sigma_k\,|\psi_a\rangle = -|\psi_a\rangle$$

$$\sigma_z\otimes\sigma_z\,|\psi_s\rangle = -|\psi_s\rangle$$

$$\sigma_x\otimes\sigma_x\,|\psi_s\rangle = \sigma_y\otimes\sigma_y\,|\psi_s\rangle = |\psi_s\rangle$$

The three cases

$$\text{I}\quad \langle\vec{S}^2\rangle_1 = \frac{\hbar^2}{2}(3-3) = \underline{0}$$

$$\text{II}\quad \langle\vec{S}^2\rangle_2 = \frac{\hbar^2}{2}(3+1) = \underline{2\hbar^2}$$

$$\text{III}\quad \langle\vec{S}^2\rangle_3 = \frac{1}{2}(\langle\vec{S}^2\rangle_1 + \langle\vec{S}^2\rangle_2) = \underline{\hbar^2}$$

$\hat{\rho}_1$ is a spin 0 state, $\hat{\rho}_2$ is a spin 1 state and $\hat{\rho}_3$ is a mixed state composed of spin 0 and 1

$\Rightarrow$ Only $\hat{\rho}_1$ is rotationally invariant

b) Reduced density operators

cross terms

$$\hat{\rho}_1^A = \mathrm{Tr}_B\left[\frac{1}{2}(|+-\rangle\langle+-| + |-+\rangle\langle-+| - |+-\rangle\langle-+| - |-+\rangle\langle+-|)\right]$$

$$= \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|)_A = \underline{\frac{1}{2}\mathbb{1}_A}$$

Since the cross terms do not contribute:

$$\hat{\rho}_2^A = \hat{\rho}_3 = \hat{\rho}_1 = \underline{\frac{1}{2}\mathbb{1}_A}$$

Similarly $\hat{\rho}_1^B = \hat{\rho}_2^B = \hat{\rho}_3^B = \underline{\frac{1}{2}\mathbb{1}_B}$

} maximally mixed

$\hat{\rho}_1$ and $\hat{\rho}_2$ are pure states $\Rightarrow$ entropies $\underline{S_1 = S_2 = 0}$

$\hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2)$ is mixed with probabilities $p_1 = p_2 = \frac{1}{2}$

$\Rightarrow$ entropy $S_3 = -p_1 \log p_1 - p_2 \log p_2 = \underline{\log 2}$

Entropies of subsystems

$S_1^A = S_2^A = S_3^A = \underline{\log 2}$, same for B

Inequality : $S_{max} \geq \max\{S_A, S_B\}$

I and II : not satisfied

III : satisfied as equality

Degree of entanglement

I and II are pure states,

entanglement entropies $\underline{S_1^A = S_2^A = \log 2}$, same for B

maximally entangled

III : $\hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2) = \frac{1}{2}(|+-\rangle\langle+-| + |-+\rangle\langle-+|)$

$= \frac{1}{2}(|+\rangle\langle+| \otimes |-\rangle\langle-| + |-\rangle\langle-| \otimes |+\rangle\langle+|)$

mixture of product states $\Rightarrow$ separable

no entanglement

c) $|\theta\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle \Rightarrow$

$$\hat{S}_\theta|\theta\rangle = (\cos\theta\, S_z + \sin\theta\, S_x)(\cos\tfrac{\theta}{2}|+\rangle + \sin\tfrac{\theta}{2}|-\rangle)$$

$$= \frac{\hbar}{2}\left[(\cos\theta\cos\tfrac{\theta}{2} + \sin\theta\sin\tfrac{\theta}{2})|+\rangle + (\sin\theta\cos\tfrac{\theta}{2} - \cos\theta\sin\tfrac{\theta}{2})|-\rangle\right]$$

$$= \underline{\frac{\hbar}{2}(\cos\tfrac{\theta}{2}|+\rangle + \sin\tfrac{\theta}{2}|-\rangle) = |\theta\rangle}$$

$$P_A = \mathrm{Tr}_A(\hat{\rho}_A \hat{P}(\theta)) = \langle\theta|\tfrac{1}{2}\mathbb{1}_A|\theta\rangle = \underline{\tfrac{1}{2}}$$

This is valid for all three cases I, II and III, it means that the probabilities for spin up and down are equal for any direction θ.

d) Joint probabilities

$$P(\theta,\theta') = \mathrm{Tr}\,(\hat{\rho}\,\hat{P}(\theta)\otimes\hat{P}(\theta'))$$

$$= \langle\theta,\theta'|\hat{\rho}|\theta,\theta'\rangle \qquad |\theta;\theta'\rangle = |\theta\rangle\otimes|\theta'\rangle$$

$$\langle +-|\theta,\theta'\rangle = \langle +|\theta\rangle\langle -|\theta'\rangle = \cos\tfrac{\theta}{2}\sin\tfrac{\theta'}{2}$$

$$\langle -+|\theta,\theta'\rangle = \langle -|\theta\rangle\langle +|\theta'\rangle = \sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2}$$

Case I :

$$P_1(\theta,\theta') = \tfrac{1}{2}\big[\langle\theta\theta'|+-\rangle\langle +-|\theta\theta'\rangle + \langle\theta\theta'|-+\rangle\langle -+|\theta\theta'\rangle$$

$$- \langle\theta\theta'|+-\rangle\langle -+|\theta\theta'\rangle - \langle\theta\theta'|-+\rangle\langle +-|\theta\theta'\rangle\big]$$

$$= \tfrac{1}{2}\left[\cos^2\tfrac{\theta}{2}\sin^2\tfrac{\theta'}{2} + \sin^2\tfrac{\theta}{2}\cos^2\tfrac{\theta'}{2} - 2\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2}\sin\tfrac{\theta'}{2}\right]$$

$$= \tfrac{1}{2}\left(\cos\tfrac{\theta}{2}\sin\tfrac{\theta'}{2} - \sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2}\right)^2$$

$$= \underline{\tfrac{1}{2}\sin^2\tfrac{\theta-\theta'}{2}}$$

Similar evaluations for case II and III

$$\underline{P_2(\theta,\theta') = \tfrac{1}{2}\sin^2\tfrac{\theta+\theta'}{2}}\,, \quad \underline{P_3(\theta,\theta') = \tfrac{1}{4}\left(\sin^2\tfrac{\theta-\theta'}{2} + \sin^2\tfrac{\theta+\theta'}{2}\right)}$$

f) Experimental quantities

$$P^A_{exp}(\theta) = \underline{\frac{n_{++}+n_{+-}}{N}}\,, \quad P^B_{exp}(\theta) = \underline{\frac{n_{++}+n_{-+}}{N}}$$

$$P_{exp}(\theta,\theta') = \underline{\frac{n_{++}}{N}}$$

e) Plots of the function $F(\theta,\theta')$

Left: Plot of the curves $F(\theta, \theta/2)$ for cases I, II, III

Right: 3D plots of $F(\theta,\theta')$

Cases I and II: Bell's inequality broken (negative F, colered red in 3D plot

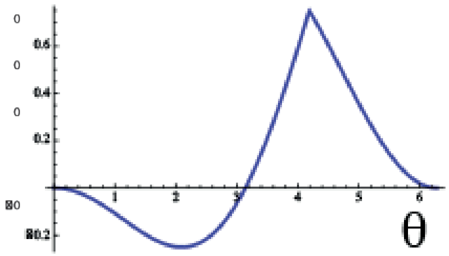
Case III: Bell's inequality unbroken (F positive)

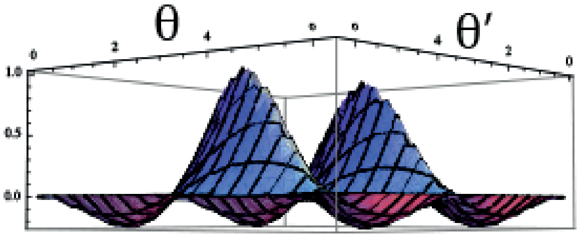
Results consistent with b): I an II entangled state,
III non-entangled

$\theta' = 0.5\ \theta$

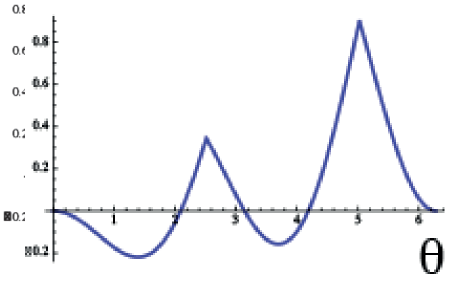
3D plot

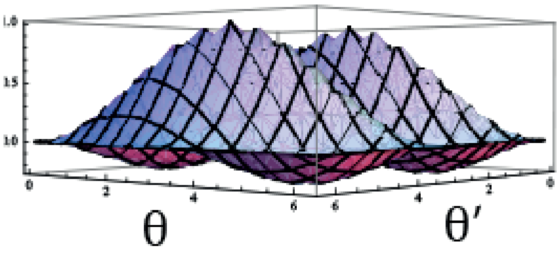
Case I



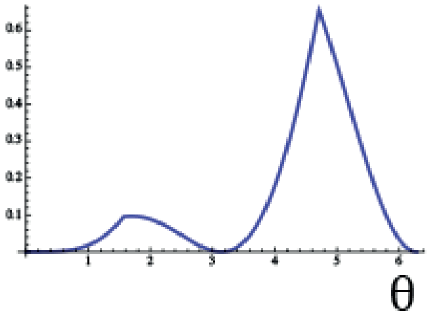


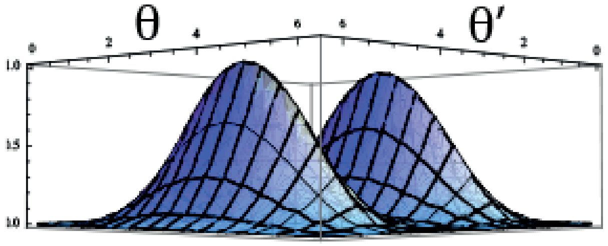
Case II





Case III





Midterm Exam FYS4110, 2016
Solutions

Problem 1

a) The expectation values P_1, P_2 and P_{12} determine the probabilities of detecting photons with the given polarization, respectively at detector 1, detector 2 and at both detectors. This implies the following correspondences, $\frac{n_1}{N} \approx P_1$ for large N, similarly $\frac{n_2}{N} \approx P_2$ and $\frac{n_{12}}{N} \approx P_{12}$.

b) Density operator, two-photon system

$$\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}(|HV\rangle\langle HV| + |VH\rangle\langle VH| + e^{i\chi}|VH\rangle\langle HV| + e^{-i\chi}|HV\rangle\langle VH|) \tag{1}$$

Reduced density operators

$$\begin{aligned}\hat{\rho}_1 = Tr_2\hat{\rho} = \langle H_2|\hat{\rho}|H_2\rangle + \langle V_2|\hat{\rho}|V_2\rangle = \frac{1}{2}(|H\rangle\langle H| + |V\rangle\langle V|)_1 = \frac{1}{2}\mathbb{1}_1 \\ \hat{\rho}_2 = Tr_1\hat{\rho} = \langle H_1|\hat{\rho}|H_1\rangle + \langle V_1|\hat{\rho}|V_1\rangle = \frac{1}{2}(|H\rangle\langle H| + |V\rangle\langle V|)_2 = \frac{1}{2}\mathbb{1}_2\end{aligned} \tag{2}$$

Both reduced density operators have maximum von Neuman entropy $S_{1/2} = -Tr\hat{\rho}_{1/2}\log\hat{\rho}_{1/2} = \log 2$. Since the two-photon system is in a pure state, $S_{1/2}$ is equal to the entanglement entropy, which gives the measure of the degree of entanglement between the two photons. Thus, the photon pairs have maximum entanglement for all values of the phase angle χ.

c) Since the reduced density operators are independent of χ, the results for P_1 and P_2 are the same in the three cases,

$$\begin{aligned}P_1(\theta_1) &= \mathrm{Tr}(\hat{\rho}\hat{P}_1(\theta_1)) = \mathrm{Tr}_1(\hat{\rho}_1\hat{P}_1(\theta_1)) = \frac{1}{2}\,\mathrm{Tr}\,\hat{P}_1(\theta_1) = \frac{1}{2}\langle\theta_1|\theta_1\rangle = \frac{1}{2} \\ P_2(\theta_2) &= \mathrm{Tr}(\hat{\rho}\hat{P}_2(\theta_2)) = \mathrm{Tr}_2(\hat{\rho}_2\hat{P}_2(\theta_2)) = \frac{1}{2}\,\mathrm{Tr}\,\hat{P}_2(\theta_2) = \frac{1}{2}\langle\theta_2|\theta_2\rangle = \frac{1}{2}\end{aligned} \tag{3}$$

The probabilities P_1 and P_2 are independent of the polarization angles.

The joint probability is given by

$$P_{12}(\theta_1,\theta_2) = \mathrm{Tr}(\hat{\rho}|\theta_1\theta_2\rangle\langle\theta_1\theta_2|) = |\langle\psi|\theta_1\theta_2\rangle|^2\,, \quad |\theta_1\theta_2\rangle = |\theta_1\rangle\otimes|\theta_2\rangle \tag{4}$$

case I: $\chi = \pi$

$$\begin{aligned}|\psi_I\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) \\ \Rightarrow \quad \langle\psi_I|\theta_1\theta_2\rangle &= \frac{1}{\sqrt{2}}(\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2)) \\ &= -\frac{1}{\sqrt{2}}\sin(\theta_1-\theta_2) \\ \Rightarrow \quad P_{12}(\theta_1,\theta_2) &= \frac{1}{2}\sin^2(\theta_1-\theta_2)\end{aligned} \tag{5}$$

case II: $\chi = 0$

$$
\begin{aligned}
|\psi_{II}\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle) \\
\Rightarrow \quad \langle\psi_{II}|\theta_1\theta_2\rangle &= \frac{1}{\sqrt{2}}(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)) \\
&= -\frac{1}{\sqrt{2}}\sin(\theta_1+\theta_2) \\
\Rightarrow \quad P_{12}(\theta_1,\theta_2) &= \frac{1}{2}\sin^2(\theta_1+\theta_2) \qquad (6)
\end{aligned}
$$

case III: $\chi = \pi/2$

$$
\begin{aligned}
|\psi_{III}\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle + i|VH\rangle) \\
\Rightarrow \quad \langle\psi_{III}|\theta_1\theta_2\rangle &= \frac{1}{\sqrt{2}}(\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2)) \\
\Rightarrow \quad P_{12}(\theta_1,\theta_2) &= \frac{1}{2}(\cos^2(\theta_1)\sin^2(\theta_2) + \sin^2(\theta_1)\cos^2(\theta_2)) \\
&= \frac{1}{4}(\sin^2(\theta_1-\theta_2) + \sin^2(\theta_1+\theta_2)) \qquad (7)
\end{aligned}
$$

d) The result (7) is the same as half the sum of the corresponding results for the cases I and II. This means that the expression for P_{12} in case III is the same as for the density operator

$$
\hat{\rho}'_{III} = \frac{1}{2}(\hat{\rho}_I + \hat{\rho}_{II}) = \frac{1}{2}(|\psi_I\rangle\langle\psi_I| + |\psi_{II}\rangle\langle\psi_{II}|) = \frac{1}{2}(|H\rangle\langle H| \otimes |V\rangle\langle V| + |V\rangle\langle V| \otimes |H\rangle\langle H|) \quad (8)
$$

which is a separable (unentangled) state.

e) Define the function

$$
F(\theta) = F(0,\theta,2\theta) = P_{12}(\theta,2\theta) - |P_{12}(0,\theta) - P_{12}(0,2\theta)| \qquad (9)
$$

This function should be non-negative if Bell's inequality is satisfied. Three plots are shown of this function, corresponding to the three cases I, I, III. In case I and II the curves do not satisfy the inequality, in accordance with the expectation that when the two-photon state is entangled Bell's inequality is nor respected. In case III the function is non-negative, which means that the Bell inequality is unbroken. This can be understood as due to the fact that the same expression for $F(\theta)$ can be found for a separable (unentangled) two-photon state. Since also in case III the state is maximally entangled, the Bell inequality studied here can not be sufficient general to register entanglement for all values of χ.

f) Results with detector 2 projecting on the new polarization states with $\phi = \pm\pi/4$.
The two-photon polarization state corresponds to case III ($\chi = \pi/2$).

Polarization state of the two projectors,

$$
|\theta_1\theta_{\phi 2}\rangle = \cos\theta_1\sin\theta_2 e^{-i\phi}|HV\rangle + \sin\theta_1\cos\theta_2 e^{i\phi}|VH\rangle + (terms\ |HH\rangle, |VV\rangle) \qquad (10)
$$

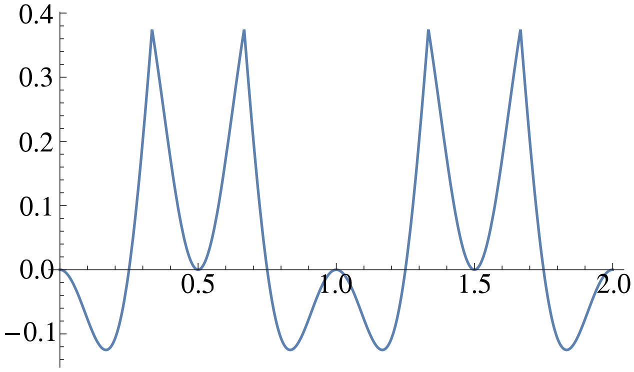
The joint probability is now

$$
\begin{aligned}
P_{12}(\theta_1,\theta_2) &= |\langle\psi_{III}|\theta_1\theta_{\phi 2}\rangle|^2 \\
&= \left|\frac{1}{\sqrt{2}}(e^{-i\phi}\cos\theta_1\sin\theta_2 - ie^{i\phi}\sin\theta_1\cos\theta)\right|^2 \\
&= \frac{1}{4}\left((1+\sin 2\phi)\sin^2(\theta_1+\theta_2) + (1-\sin 2\phi)\sin^2(\theta_1-\theta_2)\right) \qquad (11)
\end{aligned}
$$

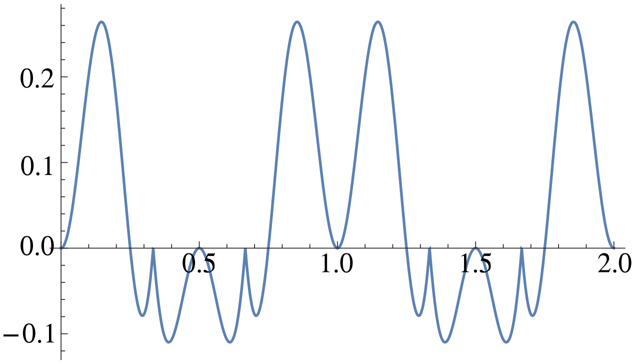
Problem1e)

Bell's inequality: Plots of F(0,θ,2θ)

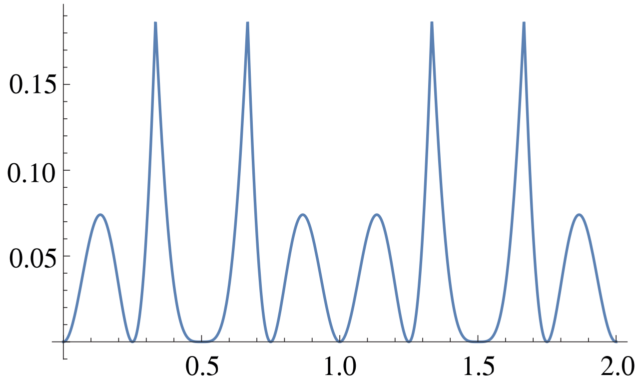
Case I: $\chi=\pi$



Case II: $\chi=0$



Case III: $\chi=\pi/2$



Case A:

$$\phi = \pi/4 \Rightarrow \sin 2\phi = 1 \Rightarrow P_{12} = \frac{1}{2}\sin^2(\theta_1 + \theta_2) \tag{12}$$

Case B:

$$\phi = -\pi/4 \Rightarrow \sin 2\phi = -1 \Rightarrow P_{12} = \frac{1}{2}\sin^2(\theta_1 - \theta_2) \tag{13}$$

We note that $P_{12}(\theta_1, \theta_2)$ in case A is the the same function of θ_1 and θ_2 as earlier found in case I (Eq. (5)). Similarly $P_{12}(\theta_1, \theta_2)$ in case B is the the same function as earlier found in case II (Eq. (6)). In both cases Bell's inequality is broken, and similarly this will be true in cases A and B. Consequently breaking of Bell's inequality is found also for the state $|\psi_{III}\rangle$, but only if one of the detectors register non-linear photon polarization.

2 Atom-photon interactions in a microcavity

a) Action of $\hat{H}$ on the basis states

$$\begin{aligned}
\hat{H}|g,1\rangle &= (\frac{1}{2}\hbar\omega - i\gamma\hbar)|g,1\rangle + \frac{1}{2}\lambda|e,0\rangle \\
\hat{H}|e,0\rangle &= \frac{1}{2}\hbar\omega|e,0\rangle + \frac{1}{2}\lambda|g,1\rangle \\
\hat{H}|g,0\rangle &= -\frac{1}{2}\hbar\omega|g,0\rangle
\end{aligned} \tag{14}$$

The ground state $|g,0\rangle$ is disconnected from the other states and can be disregarded. Extracting the matrix elements of $\hat{H}$ from (14) we find that the Hamiltonian, restricted to the subspace spanned by the vectors $|g,1\rangle$ and $|e,0\rangle$, takes the matrix form

$$H = \frac{1}{2}\hbar(\omega - i\gamma)\mathbb{1} + \frac{1}{2}\hbar\begin{pmatrix} i\gamma & \lambda \\ \lambda & -i\gamma \end{pmatrix} \tag{15}$$

b) The time evolution operator is

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{2}(\omega - i\gamma)t}e^{-i\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}t} \tag{16}$$

with $\boldsymbol{\Omega} = \frac{1}{2}(\lambda\mathbf{i} + i\gamma\mathbf{k})$. The second term can be expanded in powers of the Pauli matrix $\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}/\Omega$,

$$\begin{aligned}
e^{-i\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}t} &= (1 - \frac{1}{2}\Omega^2t^2 + \frac{1}{4!}\Omega^4t^4 ...)\mathbb{1} \\
&\quad -i\frac{\boldsymbol{\omega}}{\Omega}\cdot\boldsymbol{\sigma}(\Omega t - \frac{1}{3!}\Omega^3t^3 + ...) \\
&= \cos(\Omega t)\mathbb{1} - i\frac{\boldsymbol{\Omega}}{\Omega}\cdot\boldsymbol{\sigma}\sin(\Omega t)
\end{aligned} \tag{17}$$

where we have exploited the property of Pauli matrices that even powers are proportional to the identity and odd order are proportional to the Pauli matrix. From this follows the result

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{2}(\omega - i\gamma)t}(\cos(\Omega t)\mathbb{1} - i\sin(\Omega t)\,\frac{\boldsymbol{\Omega}}{\Omega}\cdot\boldsymbol{\sigma}) \tag{18}$$

$\boldsymbol{\Omega} = \frac{1}{2}(\lambda\mathbf{i} + i\gamma\mathbf{k})$ gives $\Omega^2 = \frac{1}{4}(\lambda^2 - \gamma^2)$ and $\Omega = \frac{1}{2}\sqrt{\lambda^2 - \gamma^2}$, which is real and positive when $\lambda > \gamma$.

c) In matrix form the time dependent wave function is

$$\begin{aligned}
\psi(t) &= \hat{\mathcal{U}}(t)\psi(0) \\
&= e^{-\frac{1}{2}(i\omega+\gamma)t}\begin{pmatrix} \cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t & -i\frac{\lambda}{2\Omega}\sin\Omega t \\ -i\frac{\lambda}{2\Omega}\sin\Omega t & \cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= e^{-\frac{1}{2}(i\omega+\gamma)t}\begin{pmatrix} \cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t \\ -i\frac{\lambda}{2\Omega}\sin\Omega t \end{pmatrix} \qquad (19)
\end{aligned}$$

In bra-ket form this gives

$$|\psi(t)\rangle = e^{-\frac{1}{2}(i\omega+\gamma)t}\left((\cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t)|e,0\rangle - i\frac{\lambda}{2\Omega}\sin\Omega t|g,1\rangle\right) \qquad (20)$$

d) Assuming Tr $\hat{\rho}_{cav} = 1$ we find

$$\begin{aligned}
f(t) &= 1 - \mathrm{Tr}\,\hat{\rho}(t) \\
&= 1 - \langle\psi(t)|\psi(t)\rangle \\
&= 1 - e^{-\gamma t}\left(\frac{\lambda^2}{4\Omega^2} - \frac{\gamma^2}{4\Omega^2}\cos(2\Omega t) + \frac{\gamma}{2\Omega}\sin(2\Omega t)\right) \qquad (21)
\end{aligned}$$

When the photon escapes through the walls, the system inside the cavity ends up in the state $|g,0\rangle$. The term added to the density matrix $\hat{\rho}$ takes care of this in such a way that the sum of the probabilities for the atom to be in one of the states $|e\rangle$ and $|g\rangle$ is constant, equal to 1.

e) Occupation probabilities for the atom; the excited state

$$\begin{aligned}
p_e(t) &= \langle e,0|\hat{\rho}_{tot}(t)|e,0\rangle \\
&= \langle e,0|\hat{\rho}(t)|e,0\rangle \\
&= |\langle\psi(t)|e,0\rangle|^2 \\
&= e^{-\gamma t}(\cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t)^2 \\
&= e^{-\gamma t}(\frac{\lambda^2}{8\Omega^2} + \frac{\lambda^2 - 2\gamma^2}{8\Omega^2}\cos(2\Omega t) + \frac{\gamma}{2\Omega}\sin(2\Omega t) \qquad (22)
\end{aligned}$$

and the ground state

$$p_g(t) = 1 - p_e(t) \qquad (23)$$

The probability for one photon being present in the cavity is

$$\begin{aligned}
p_{ph}(t) &= \langle g,1|\hat{\rho}(t)|g,1\rangle \\
&= |\langle\psi(t)|g,1\rangle|^2 \\
&= \frac{\lambda^2}{8\Omega^2}e^{-\gamma t}(1 - \cos(2\Omega t)) \qquad (24)
\end{aligned}$$

f) Eigenvalues of $\hat{\rho}_{cav}(t)$,

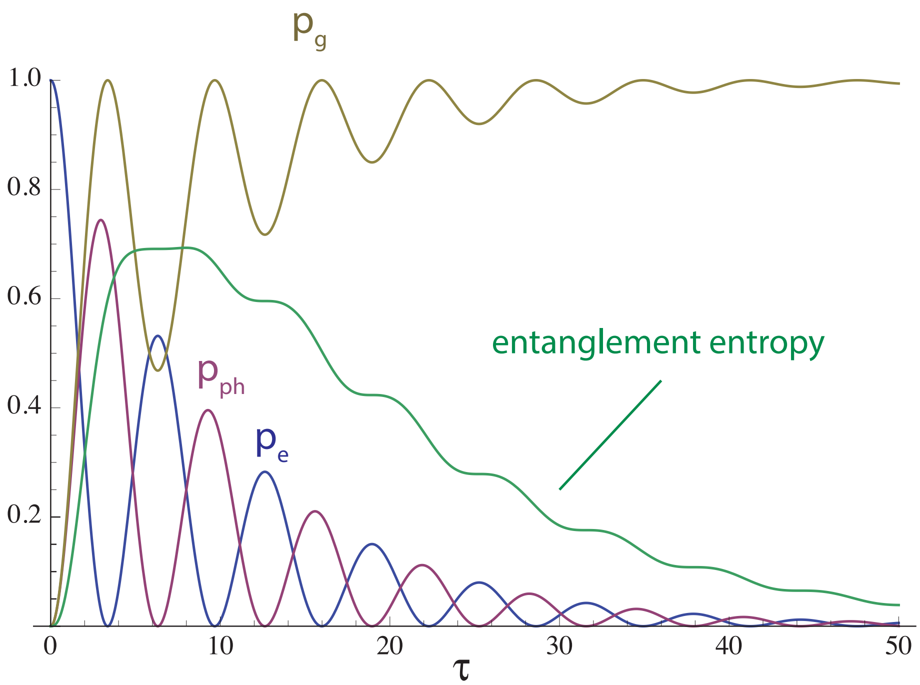
$$\begin{aligned}
\hat{\rho}_{cav}(t) &= |\psi(t)\rangle\langle\psi(t)| + f(t)|g,0\rangle\langle g,0| \\
&= \langle\psi(t)|\psi(t)\rangle|\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)| + f(t)|g,0\rangle\langle g,0| \\
&= (1 - f(t))|\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)| + f(t)|g,0\rangle\langle g,0| \qquad (25)
\end{aligned}$$

where $|\tilde{\psi}(t)\rangle$ is normalized to 1. Since this state is orthogonal to the normalized state $|g,0\rangle$, the above expression gives the spectral decomposition of $\hat{\rho}_{cav}$, with eigenvalues $f(t)$ and $1-f(t)$. The corresponding von Neuman entropy is

$$S = -f\log f - (1-f)\log(1-f) \tag{26}$$

with f given by (21). With the cavity system viewed as a part of a larger system in a pure state, which includes also the photon states of the escaped photon, the above expression for S can be identified as the entanglement entropy of the larger, composite system.

Problem 2 e) og f)
Occupation probabilities and entanglement entropy



Fys 4110 Midterm exam 2017 Solutions

Problem 1.

a) Hamiltonian: $H = -\frac{\hbar\omega_0}{2}\sigma_z - \frac{\hbar\omega_1}{2}(\cos\omega t\,\sigma_x - \sin\omega t\,\sigma_y)$

We transform to a rotating frame with angular velocity ω (same as driving field).

Time dependent unitary transform $T(t) = e^{-\frac{i\omega}{2}t\sigma_z}$

Transformed state $|\psi'\rangle = T(t)|\psi\rangle$

Hamiltonian $H' = THT^\dagger + i\hbar\frac{dT}{dt}T^\dagger$

Using the relations

$$e^{-\frac{i\omega}{2}t\sigma_z}\sigma_x e^{i\frac{\omega}{2}t\sigma_z} = \cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y$$

$$e^{-i\frac{\omega}{2}t\sigma_z}\sigma_y e^{i\frac{\omega}{2}t\sigma_z} = \cos\omega t\,\sigma_y - \sin\omega t\,\sigma_x$$

we get a time-independent Hamiltonian

$$H' = \frac{\hbar}{2}(\omega-\omega_0)\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

Define: $\Omega = \sqrt{(\omega-\omega_0)^2+\omega_1^2}$

$$\cos\theta = \frac{\omega_0-\omega}{\Omega} \qquad \sin\theta = \frac{\omega_1}{\Omega}$$

$$H' = -\frac{1}{2}\hbar\Omega(\cos\theta\,\sigma_z + \sin\theta\,\sigma_x)$$

This gives the time evolution

$$U'(t) = e^{-\frac{i}{\hbar}H't} = \cos\frac{\Omega t}{2}\mathbb{1} + i\sin\frac{\Omega t}{2}(\cos\theta\,\sigma_z + \sin\theta\,\sigma_x)$$

Transform back: $U(t) = T(t)^{\dagger}\, U'(t)\, \underbrace{T(0)}_{1}$

If $|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$ with $\overbrace{c_0(0)=1 \text{ and } c_1(0)=0}^{\text{Ground state}}$

we get

$$c_0(t) = \left(\cos\frac{\Omega t}{2} + i\sin\frac{\Omega t}{2}\cos\theta\right)e^{-\frac{i\omega t}{2}}$$

$$c_1(t) = i\sin\frac{\Omega t}{2}\sin\theta\, e^{\frac{i\omega t}{2}}$$

The probability to find the excited state is

$$P_1(t) = |c_1(t)|^2 = \sin^2\frac{\Omega t}{2}\sin^2\theta$$

b) Hamiltonian: $H = \underbrace{\frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega a^{\dagger}a}_{H_0} + \underbrace{i\hbar\lambda(a^{\dagger}\sigma_- - a\sigma_+)}_{H_1}$

The eigenstates of H_0: $H_0|\pm,n\rangle = \underbrace{\hbar(n\omega \pm \frac{1}{2}\omega_0)}_{E_{\pm,n}}|\pm,n\rangle$

The ground state is unaffected by interaction: $H_1|-,0\rangle = 0$

For the excited states we have:

$$H_1|+,n\rangle = i\hbar\lambda\sqrt{n+1}\,|-,n+1\rangle$$

$$H_1|-,n+1\rangle = -i\hbar\lambda\sqrt{n+1}\,|+,n\rangle$$

⇒ H_1 mixes only pairs of states and the full H consists of 2×2 blocks on the diagonal.

In the space $\{|+,n\rangle, |-,n+1\rangle\}$ we have

$$H_n = \frac{1}{2}\hbar\begin{pmatrix}\Delta & -ig_n \\ ig_n & -\Delta\end{pmatrix} + \epsilon_n \mathbb{1}$$

$\Delta = \omega_0 - \omega$ $\quad g_n = 2\lambda\sqrt{n+1}$ $\quad \epsilon_n = (n+\frac{1}{2})\hbar\omega$

Defining $\Omega_n = \sqrt{\Delta^2 + g_n^2}$ $\cos\theta_n = \frac{\Delta}{\Omega_n}$ $\sin\theta_n = \frac{g_n}{\Omega_n}$

$$H_n = \frac{1}{2}\hbar\Omega_n(\cos\theta_n\sigma_z + \sin\theta_n\sigma_y) + \epsilon_n \mathbb{1}$$

The eigenstates are $|\psi_n^+\rangle = \cos\frac{\theta_n}{2}|+,n\rangle + i\sin\frac{\theta_n}{2}|-,n+1\rangle$

$$|\psi_n^-\rangle = i\sin\frac{\theta_n}{2}|+,n\rangle + \cos\frac{\theta_n}{2}|-,n+\rangle$$

with eigenvalues $E_n^\pm = \epsilon_n \pm \frac{1}{2}\hbar\Omega_n$

Using this we can now find the time evolution of a general state in the $\{|+,n\rangle, |-,n+1\rangle\}$ space:

$$|\psi(0)\rangle = c_n^+(0)|+,n\rangle + c_n^-(0)|-,n+1\rangle$$

$$= d_n^+|\psi_n^+\rangle + d_n^-|\psi_n^-\rangle$$

$$\xrightarrow{\text{time}} d_n^+ e^{-\frac{i}{\hbar}E_n^+ t}|\psi_n^+\rangle + d_n^- e^{-\frac{i}{\hbar}E_n^- t}|\psi_n^-\rangle$$

$$= c_n^+(t)|+,n\rangle + c_n^-(t)|-,n+1\rangle$$

With the initial state $|-,n+1\rangle$ we have $c_n^+(0)=0$, $c_n^-(0)=1$

and get $c_n^+(t) = -e^{-\frac{i}{\hbar}\epsilon_n t}\sin\theta_n\sin\frac{\Omega_n t}{2}$

$$c_n^-(t) = -e^{-\frac{i}{\hbar}\epsilon_n t}\left(\cos\frac{\Omega_n t}{2} + i\cos\theta_n\sin\frac{\Omega_n t}{2}\right)$$

Probability for the excited state is

$$P_+(t) = |c_n^+(t)|^2 = \sin^2\theta_n\sin^2\frac{\Omega_n t}{2}$$

Comparing to the Rabi problem, this is the same provided we identify $\omega_1 \leftrightarrow g_n$

g) We have $|\psi(t)\rangle = c_n^+(t)|+,n\rangle + c_n^-(t)|-,n+1\rangle$ with $c_n^\pm(t)$ given in b).

Density matrix: $\rho = |\psi(t)\rangle\langle\psi(t)|$

$$= |c_n^+(t)|^2|+,n\rangle\langle +,n| + c_n^+(t)\, c_n^-(t)^*\, |+,n\rangle\langle -,n+1|$$

$$+\ c_n^+(t)^*\, c_n^-(t)|-,n+1\rangle\langle +,n| + |c_n^-(t)|^2|-,n+1\rangle\langle -,n+1|$$

Tracing over the photon mode:

$$\rho_{TLS} = \mathrm{Tr}_{\text{photon}}\, \rho = \sum_m \langle m|\rho|m\rangle = |c_n^+(t)|^2|+\rangle\langle+| + |c_n^-(t)|^2|-\rangle\langle -|$$

We have $|c_n^+(t)|^2 = \sin^2\theta_n \sin^2\frac{\Omega_n t}{2} = p^+$

$|c_n^-(t)|^2 = 1 - \sin^2\theta_n \sin^2\frac{\Omega_n t}{2} = p^-$

Entanglement entropy:

$$S = -\mathrm{Tr}\, \rho_{TLS} \ln \rho_{TLS} = -p^+ \ln p^+ - p^- \ln p^-$$

$$= -\sin^2\theta_n \sin^2\frac{\Omega_n t}{2} \ln\left(\sin^2\theta_n \sin^2\frac{\Omega_n t}{2}\right)$$

$$-\left(1-\sin^2\theta_n \sin^2\frac{\Omega_n t}{2}\right) \ln\left(1-\sin^2\theta_n \sin^2\frac{\Omega_n t}{2}\right)$$

Maximal entropy when p^+ and p^- are as equal as possible.

If $\sin^2\theta_n > \frac{1}{2}$, $\theta_n > \pi/4$ we can get $p^+ = p^- = \frac{1}{2}$

with $S_{max} = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \ln 2$

This happens when $\sin^2\theta_n \sin^2\frac{\Omega_n t}{2} = \frac{1}{2}$

$$\Rightarrow t = \frac{2}{\Omega_n}\arcsin\left[\frac{1}{\sqrt{2}\sin\theta_n}\right] = \frac{2}{\Omega_n}\arcsin\left[\frac{\Omega_n}{\sqrt{2}\,g_n}\right]$$

If $\sin^2\theta_n < \frac{1}{2}$ we have $p^+ < \frac{1}{2}$ and maximal when $\frac{\Omega_n t}{2} = \frac{\pi}{2} + m\pi \quad (m \in \mathbb{Z})$

$$p^+_{max} = \sin^2\theta_n \quad , \quad S_{max} = -\sin^2\theta_n \ln\sin^2\theta_n - \cos^2\theta_n \ln\cos^2\theta_n$$

d) For the Rabi model (in rotating frame):

$$|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$$

$$c_0(t) = \cos\frac{\Omega t}{2} + i\sin\frac{\Omega t}{2}\cos\theta \qquad c_1(t) = i\sin\frac{\Omega t}{2}\sin\theta$$

This is a pure state and the Bloch vector has components

$$m_x^R = 2\,\mathrm{Re}(c_0^* c_1) = \sin 2\theta \sin^2\frac{\Omega t}{2}$$

$$m_y^R = 2\,\mathrm{Im}(c_0^* c_1) = \sin\theta \sin\Omega t$$

$$m_z^R = |c_0|^2 - |c_1|^2 = \cos^2\frac{\Omega t}{2} + \sin^2\frac{\Omega t}{2}\cos^2\theta - \sin^2\frac{\Omega t}{2}\sin^2\theta$$

$$= 1 - 2\sin^2\theta \sin^2\frac{\Omega t}{2}$$

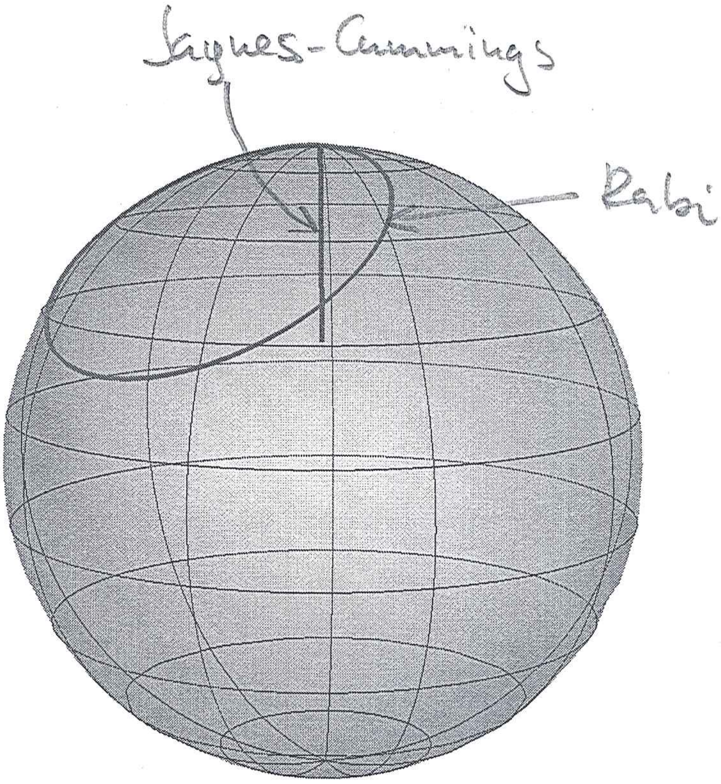
For the JC model we use $\rho_{TLS} = \frac{1}{2}(\mathbb{1} + \vec{m}^{JC}\cdot\vec{\sigma})$

$$\rho_{TLS} = p^-|-\rangle\langle-| + p^+|+\rangle\langle+| = \frac{1}{2}(\mathbb{1} + (p^- - p^+)\sigma_z)$$

$$\Rightarrow \quad m_x^{JC} = m_y^{JC} = 0$$

$$m_z^{JC} = p^- - p^+ = 1 - 2\sin^2\theta_n \sin^2\frac{\Omega_n t}{2}$$

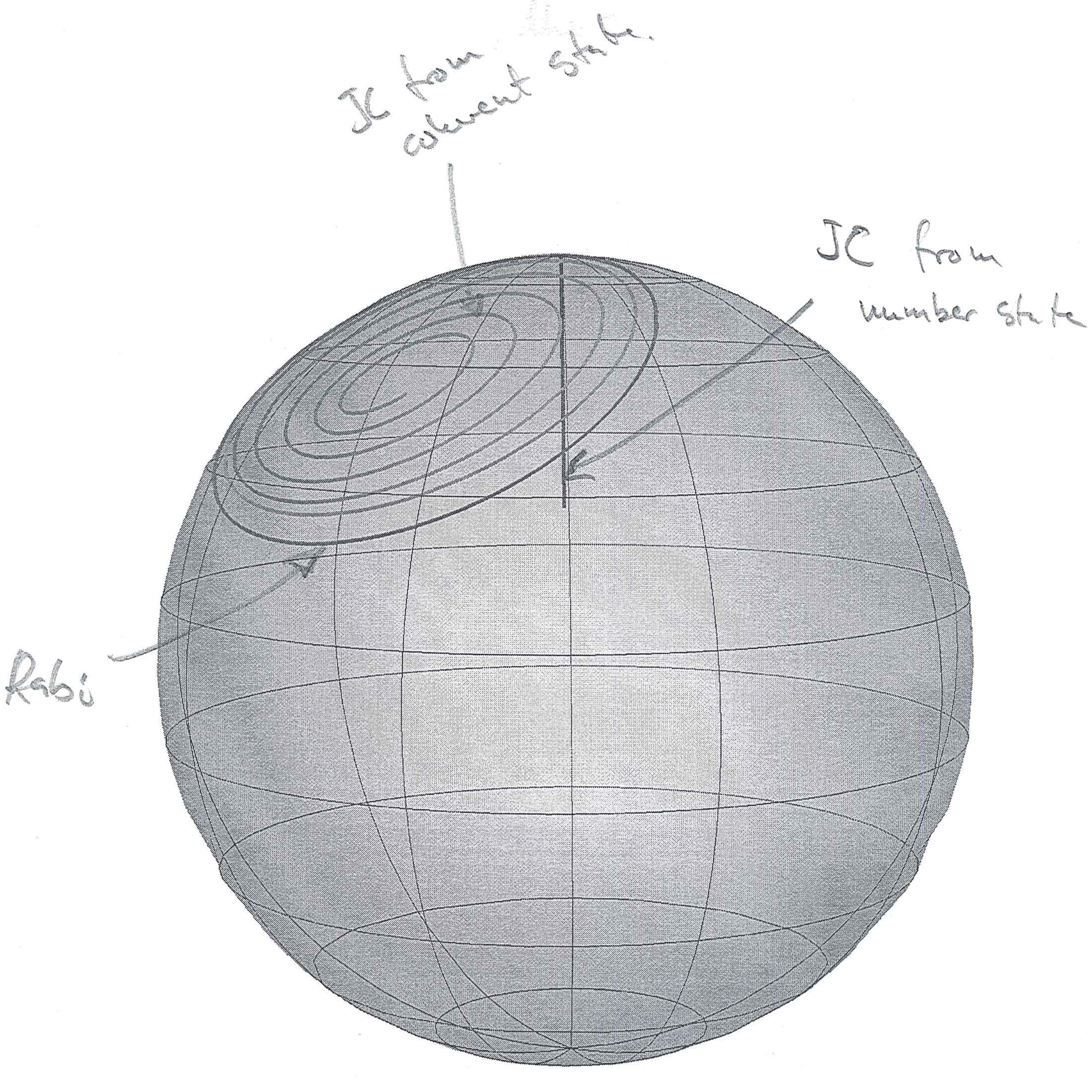
⑤b



In the Rabi model, the state is always pure, and the Bloch vector presesses in a circle on the surface of the Bloch sphere. In the JC model, the qubit is entangled with the photon mode, and the reduced density matrix describes a mixed state. The Bloch vector oscillates along the axis of the Bloch sphere with $m_z^{JC} = m_z^{R}$.

e) $n \to \infty$:
$$\Omega_n = \sqrt{\Delta^2 + g_n^2} = \sqrt{\Delta^2 + 4\lambda^2(n+1)} \to g_n$$
$$\sin\theta_n = \frac{g_n}{\Omega_n} \to 1$$

The amplitude and frequency of the oscillations increase as $n \to \infty$, but the Bloch vector is always on the axis of the Bloch sphere and entanglement is not reduced. An idea for a classical limit is to assume that the photon mode starts in a coherent state instead of an eigenstate. We know that coherent states are the link to classical mechanics for the harmonic oscillator, and we can hope that it will extend to the JC model as well.



It works to some extent, but it becomes a spiral instead of circle. Here I used an average photon number of 9, maybe it should be bigger for the limit, but numerics gets slow. More work is needed...

Problem 2

a) $H = \hbar\omega_r(a^\dagger a + \frac{1}{2}) + \frac{\hbar\Omega}{2}\sigma^z + \hbar g(a^\dagger\sigma^- + a\sigma^+)$

$|\downarrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$|\uparrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

Non-interacting eigenstates: $\{|\uparrow,n\rangle, |\downarrow,n\rangle\}$

We know that the interaction only mixes the states $|\downarrow,n\rangle$ and $|\uparrow,n+1\rangle$.

$$H|\downarrow,n\rangle = \underbrace{\left(\hbar\omega_r(n+\tfrac{1}{2}) + \tfrac{\hbar\Omega}{2}\right)}_{E_{\downarrow,n}}|\downarrow,n\rangle + \hbar g\sqrt{n+1}\,|\uparrow,n+1\rangle$$

$$H|\uparrow,n+1\rangle = \underbrace{\left(\hbar\omega_r(n+\tfrac{3}{2}) - \tfrac{\hbar\Omega}{2}\right)}_{E_{\uparrow,n+1}}|\uparrow,n+1\rangle + \hbar g\sqrt{n+1}\,|\downarrow,n\rangle$$

$\Delta = \Omega - \omega_r$

$$H_n = \frac{\hbar}{2}\begin{pmatrix}\Delta & 2g\sqrt{n+1}\\ 2g\sqrt{n+1} & -\Delta\end{pmatrix} + \hbar\omega_r(n+1)\,\mathbb{1}$$

$$= \frac{\hbar\Omega_n}{2}\begin{pmatrix}\cos\theta_n & \sin\theta_n\\ \sin\theta_n & -\cos\theta_n\end{pmatrix} + \hbar\omega_r(n+1)\,\mathbb{1}$$

$$= \frac{\hbar\Omega_n}{2}\underbrace{\left(\cos\theta_n\,\sigma_z + \sin\theta_n\,\sigma_x\right)}_{\vec{n}\cdot\vec{\sigma}} + \hbar\omega_r(n+1)\,\mathbb{1}$$

$\vec{n} = (\sin\theta_n, 0, \cos\theta_n)$

$\Omega_n = \sqrt{\Delta^2 + 4g^2(n+1)}$ $\cos\theta_n = \frac{\Delta}{\Omega_n}$ $\sin\theta_n = \frac{2g\sqrt{n+1}}{\Omega_n}$

$\vec{n}\cdot\vec{\sigma}$ has eigenvalues ± 1 and eigenstates

$$|+,n\rangle = \cos\theta_n|\downarrow,n\rangle + \sin\theta_n|\uparrow,n+1\rangle$$

$$|-,n\rangle = -\sin\theta_n|\downarrow,n\rangle + \cos\theta_n|\uparrow,n+1\rangle$$

These are also eigenstates of H_n and the eigenvalues are

$$E_{\pm n} = \pm\frac{\hbar\Omega_n}{2} + \hbar\omega_r(n+1)$$

b) For $\Delta \gg g$ the energies are

$$E_{\pm n} = \pm \frac{\hbar\Delta}{2}\sqrt{1+\frac{4g^2(n+1)}{\Delta^2}} + \hbar\omega_r(n+1)$$

$$\approx \pm\frac{\hbar\Delta}{2}\left(1+\frac{2g^2(n+1)}{\Delta^2}+\dots\right)+\hbar\omega_r(n+1)$$

$$= (n+1)\left(\hbar\omega_r \pm \frac{\hbar g^2}{\Delta}\right) \pm \frac{\hbar\Delta}{2}$$

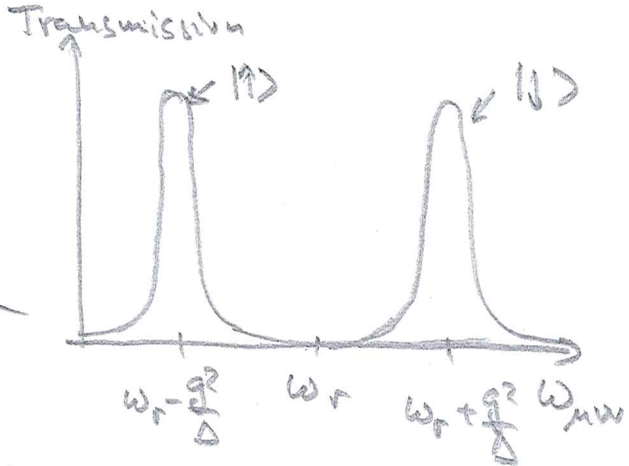
Level spacing: $E_{\pm,n+1}-E_{\pm n} = \hbar\omega_r \pm \frac{\hbar g^2}{\Delta}$ independent of n.

When $\Delta \gg g$ $\cos\theta_n \approx 1$ $\sin\theta_n \approx \frac{2g\sqrt{n+1}}{\Delta} \ll 1$

$\Rightarrow$ $|+,n\rangle \approx |\downarrow,n\rangle$ $|-,n\rangle \approx |\uparrow,n\rangle$

$\Rightarrow$ Level spacing depends on qubit state.

c) The transmission is large when the microwave frequency ω_{MW} is resonant with transitions in the system. Since the level spacing depends on the qubit state we can determine it from the position of the resonance line. The frequency should be choosen as one of the resonance frequencies, e.g. $\omega_r - \frac{g^2}{\Delta}$



If we get large transmission amplitude $\Rightarrow$ $|\uparrow\rangle$

— " — small — " — $\Rightarrow$ $|\downarrow\rangle$

d) We use $e^{\lambda A} B e^{-\lambda A} = B + \lambda [A,B] + \frac{\lambda^2}{2}[A,[A,B]] + \dots$
with $A = a\sigma^+ - a^\dagger\sigma^-$ and $B = H$.

Basic relations: $[a, a^\dagger] = 1 \quad [\sigma^+, \sigma^-] = \sigma^z$

$$[\sigma^\pm, \sigma^z] = \mp 2\sigma^\pm$$

$$\sigma^+\sigma^- = \tfrac{1}{2}(1+\sigma^z)$$

$$[AB, C] = A[B,C] + [A,C]B \qquad \sigma^-\sigma^+ = \tfrac{1}{2}(1-\sigma^z)$$

$$\sigma^+\sigma^- + \sigma^-\sigma^+ = 1$$

$$[a\sigma^+ - a^\dagger\sigma^-, a^\dagger a] = \underbrace{[a, a^\dagger a]}_{a^\dagger \underbrace{[a,a]}_{0} + \underbrace{[a,a^\dagger]}_{1} a}\sigma^+ - \underbrace{[a^\dagger, a^\dagger a]}_{a^\dagger \underbrace{[a^\dagger,a]}_{-1} + \underbrace{[a^\dagger,a^\dagger]}_{0} a}\sigma^- = a\sigma^+ + a^\dagger\sigma^-$$

$$[a\sigma^+ - a^\dagger\sigma^-, \sigma^z] = a\underbrace{[\sigma^+, \sigma^z]}_{-2\sigma^+} - a^\dagger\underbrace{[\sigma^-, \sigma^z]}_{2\sigma^-} = -2(a\sigma^+ + a^\dagger\sigma^-)$$

$$[a\sigma^+ - a^\dagger\sigma^-, a^\dagger\sigma^- + a\sigma^+] = \underbrace{[a\sigma^+, a^\dagger\sigma^-]}_{a\underbrace{[\sigma^+, a^\dagger\sigma^-]}_{a^\dagger\underbrace{[\sigma^+,\sigma^-]}_{\sigma^z}} + \underbrace{[a, a^\dagger\sigma^-]}_{\underbrace{[a,a^\dagger]}_{1}\sigma^-}\sigma^+} - \underbrace{[a^\dagger\sigma^-, a\sigma^+]}_{a^\dagger\underbrace{[\sigma^-, a\sigma^+]}_{-a\sigma^z} + \underbrace{[a^\dagger, a\sigma^+]}_{-\sigma^+}\sigma^-}$$

$$= \underbrace{aa^\dagger}_{a^\dagger a+1}\sigma^z + \sigma^-\sigma^+ + a^\dagger a\sigma^z + \sigma^+\sigma^-$$

$$= (2a^\dagger a + 1)\sigma^z + 1$$

$$[A, B] = -\hbar\Delta(a\sigma^+ + a^\dagger\sigma^-) + \hbar g[(2a^\dagger a+1)\sigma^z + 1]$$

$$[A,[A,B]] = -\hbar\Delta[(2a^\dagger a+1)\sigma^z + 1] + \underbrace{(\ )g}_{\text{Only contributes to } g^3}$$

$$UHU^\dagger \approx \hbar\omega_r\left(a^\dagger a+\tfrac{1}{2}\right)+\frac{\hbar\Omega}{2}\sigma^z+\hbar g\left(a^\dagger\sigma^-+a\sigma^+\right)$$
$$+\frac{g}{\Delta}\left[-\hbar\Delta\left(a\sigma^++a^\dagger\sigma^-\right)+\hbar g\left((2a^\dagger a+1)\sigma^z+1\right)\right]$$
$$+\frac{1}{2}\left(\frac{g}{\Delta}\right)^2(-\hbar\Delta)\left[(2a^\dagger a+1)\sigma^z+1\right]$$

$$=\underbrace{\hbar\left(\omega_r+\frac{g^2}{\Delta}\sigma^z\right)a^\dagger a}_{\text{Level spacing.}}+\frac{\hbar}{2}\left(\Omega+\frac{g^2}{\Delta}\right)\sigma^z+\underbrace{\frac{1}{2}\hbar\omega_r+\frac{\hbar g^2}{2\Delta}}_{\text{constant}}$$

$\sigma^z|\downarrow\rangle=+|\downarrow\rangle \Rightarrow \hbar\omega_r+\frac{\hbar g^2}{\Delta}$

$\sigma^z|\uparrow\rangle=-|\uparrow\rangle \Rightarrow \hbar\omega_r-\frac{\hbar g^2}{\Delta}$

as we found in b).

e) $H_{\mu w}=\hbar\mathcal{E}\left(a^\dagger e^{-i\omega_{\mu w}t}+a\,e^{i\omega_{\mu w}t}\right)$

$$\left[a\sigma^+-a^\dagger\sigma^-,\ a^\dagger e^{-i\omega_{\mu w}t}+a\,e^{i\omega_{\mu w}t}\right]=\sigma^+e^{-i\omega_{\mu w}t}+\sigma^-e^{i\omega_{\mu w}t}$$

$$UH_{\mu w}U^\dagger\approx\hbar\mathcal{E}\left(a^\dagger e^{-i\omega_{\mu w}t}+a\,e^{i\omega_{\mu w}t}\right)+\frac{\hbar g\mathcal{E}}{\Delta}\left(\sigma^+e^{-i\omega_{\mu w}t}+\sigma^-e^{i\omega_{\mu w}t}\right)$$

f) Transformation of Hamiltonian: $H'=THT^\dagger+i\hbar\frac{dT}{dt}T^\dagger$

$T=e^{i\frac{\omega_{\mu w}t}{2}\sigma_z+i\omega_{\mu w}t\,a^\dagger a}$

$\frac{dT}{dt}T^\dagger=i\omega_{\mu w}\left(\frac{1}{2}\sigma_z+a^\dagger a\right)$

T commutes with $UHU^\dagger$

$$e^{\lambda a^\dagger a}\, a\, e^{-\lambda a^\dagger a} = a + \lambda \underbrace{[a^\dagger a, a]}_{-a} + \frac{\lambda^2}{2!}\underbrace{[a^\dagger a,[a^\dagger a, a]]}_{a} + \cdots$$

$$= a - \lambda a + \frac{\lambda^2}{2!} a - \cdots = e^{-\lambda} a$$

$$\Rightarrow e^{i\omega_{\mu w} t\, a^\dagger a}\, a\, e^{-i\omega_{\mu w} t\, a^\dagger a} = a\, e^{-i\omega_{\mu w} t}$$

$$e^{i\omega_{\mu w} t\, a^\dagger a}\, a^\dagger\, e^{-i\omega_{\mu w} t\, a^\dagger a} = a^\dagger e^{i\omega_{\mu w} t}$$

$$e^{\lambda\sigma^z}\sigma^\pm e^{-\lambda\sigma^z} = \sigma^\pm + \lambda\underbrace{[\sigma^z,\sigma^\pm]}_{\pm 2\sigma^\pm} + \frac{\lambda^2}{2!}\underbrace{[\sigma^z,[\sigma^z,\sigma^\pm]]}_{4\sigma^\pm} + \cdots$$

$$= \sigma^\pm\left(1 \pm 2\lambda + \frac{(2\lambda)^2}{2!} \pm \frac{(2\lambda)^3}{3!} \cdots\right) = \sigma^\pm e^{\pm 2\lambda}$$

$$\Rightarrow e^{i\frac{\omega_{\mu w}}{2} t\,\sigma^z}\,\sigma^\pm\, e^{-i\frac{\omega_{\mu w}}{2} t\,\sigma^z} = \sigma^\pm e^{\pm i\omega_{\mu w} t}$$

$$H_{iq} = TU(H + H_{\mu w})U^\dagger T^\dagger$$

$$= \hbar\left(\omega_r + \frac{g^2}{\Delta}\sigma^z\right) a^\dagger a + \frac{\hbar}{2}\left(\Omega + \frac{g^2}{\Delta}\right)\sigma^z$$

$$+ \hbar\varepsilon(a^\dagger + a) + \frac{\hbar\varepsilon g}{\Delta}\underbrace{(\sigma^+ + \sigma^-)}_{\sigma_x} - \hbar\omega_{\mu w}\left(\frac{1}{2}\sigma^z + a^\dagger a\right)$$

$$= \frac{\hbar}{2}\left[\Omega + 2\frac{g^2}{\Delta}\left(a^\dagger a + \frac{1}{2}\right) - \omega_{\mu w}\right]\sigma^z + \frac{\hbar\varepsilon g}{\Delta}\sigma_x$$

$$+ \hbar(\omega_r - \omega_{\mu w})a^\dagger a + \hbar\varepsilon(a^\dagger + a)$$

g) With $\omega_{mw} = \Omega + (2n+1)\frac{g^2}{\Delta} - 2\frac{g\varepsilon}{\Delta}$ we have

$$\omega_r - \omega_{mw} = \underbrace{\omega_r - \Omega}_{-\Delta} + (\,)\frac{g}{\Delta} \approx -\Delta \quad \text{when } \Delta \gg g.$$

If we also assume $\Delta \gg \varepsilon$ the term $\hbar\varepsilon(a^\dagger a)$ will only induce small variations in the photon number n. We will ignore this term and replace $a^\dagger a \to n$.

$$H_{1q} = \frac{\hbar\varepsilon g}{\Delta}(\sigma^x + \sigma^z) + \text{constant.}$$

$$U(t) = e^{-\frac{i}{\hbar}H_{1q}t}$$

$$U\left(\frac{\pi\Delta}{2\sqrt{2}g\varepsilon}\right) = e^{-\frac{i\pi}{2}\frac{1}{\sqrt{2}}(\sigma^x+\sigma^z)} = \underbrace{\cos\frac{\pi}{2}}_{0}\cdot\mathbb{1} - i\underbrace{\sin\frac{\pi}{2}}_{1}\frac{1}{\sqrt{2}}(\sigma^x+\sigma^z)$$

$$= -\frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -i\cdot H$$

h) Let $\omega_{mw} = \Omega + \frac{g^2}{\Delta}(2n+1) \Rightarrow H_{1q} = \frac{\hbar\varepsilon g}{\Delta}\sigma_x + \text{constant}$

Rotation around x-axis with angle θ: $e^{-i\frac{\theta}{2}\sigma_x}$

$$\Rightarrow \frac{\varepsilon g}{\Delta}t = \frac{\theta}{2} \quad \Rightarrow t = \frac{\Delta\theta}{2g\varepsilon}$$

i) $H = \hbar\omega_r(a^\dagger a + \frac{1}{2}) + \frac{\hbar\Omega}{2}(\sigma_1^z + \sigma_2^z) + \hbar g[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)]$

$$u = e^{\frac{g}{\Delta}\underbrace{[a(\sigma_1^+ + \sigma_2^+) - a^\dagger(\sigma_1^- + \sigma_2^-)]}_{A}}$$

$$[A, a^\dagger a] = a(\sigma_1^+ + \sigma_2^+) + a^\dagger(\sigma_1^- + \sigma_2^-)$$

$$[A, \sigma_1^z + \sigma_2^z] = -2[a(\sigma_1^+ + \sigma_2^+) + a^\dagger(\sigma_1^- + \sigma_2^-)]$$

$$\begin{aligned}
&[A, a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)] \\
&= [a(\sigma_1^+ + \sigma_2^+), a^\dagger(\sigma_1^- + \sigma_2^-)] - [a^\dagger(\sigma_1^- + \sigma_2^-), a(\sigma_1^+ + \sigma_2^+)] \\
&= 2[\underbrace{a a^\dagger}_{a^\dagger a + 1}(\sigma_1^z + \sigma_2^z) + \underbrace{(\sigma_1^- + \sigma_2^-)(\sigma_1^+ + \sigma_2^+)}_{1 - \frac{1}{2}(\sigma_1^z + \sigma_2^z) + \sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+}] \\
&= 2[(a^\dagger a + \tfrac{1}{2})(\sigma_1^z + \sigma_2^z) + \underline{1} + \sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+]
\end{aligned}$$

$$[A, H] = -\hbar\Delta[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)] + 2\hbar g[(a^\dagger a + \tfrac{1}{2})(\sigma_1^z + \sigma_2^z) + 1 + \sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+]$$

$$[A,[A,H]] = -\hbar\Delta\, 2[(a^\dagger a + \tfrac{1}{2})(\sigma_1^z + \sigma_2^z) + 1 + \sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+] + (\ )g$$

$$\begin{aligned}
uHu^\dagger &= \hbar\omega_r(a^\dagger a + \tfrac{1}{2}) + \frac{\hbar\Omega}{2}(\sigma_1^z + \sigma_2^z) + \hbar g[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)] \\
&\quad + \frac{g}{\Delta}\{-\hbar\Delta[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)] + 2\hbar g[(a^\dagger a + \tfrac{1}{2})(\sigma_1^z + \sigma_2^z) + 1 + \sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+]\} \\
&\quad + \frac{1}{2}\left(\frac{g}{\Delta}\right)^2(-2\hbar\Delta)[(a^\dagger a + \tfrac{1}{2})(\sigma_1^z + \sigma_2^z) + 1 + \sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+] \\
&= \hbar\left[\omega_r + \frac{g^2}{\Delta}(\sigma_1^z + \sigma_2^z)\right]a^\dagger a + \frac{\hbar}{2}\left(\Omega + \frac{g^2}{\Delta}\right)(\sigma_1^z + \sigma_2^z) \\
&\quad + \frac{\hbar g^2}{\Delta}(\sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+) + \text{constant}
\end{aligned}$$

j) Transformation to rotating frame

$T(t) = e^{i\frac{\Omega t}{2}(\sigma_1^z+\sigma_2^z) + i\omega_r t a^\dagger a}$ $\quad\quad \frac{dT}{dt}T^\dagger = i\frac{\Omega}{2}(\sigma_1^z+\sigma_2^z) + i\omega_r a^\dagger a$

$T(t)$ commutes trivially with the twos first terms in $UHU^\dagger$, and in fact also

$$[\sigma_1^z+\sigma_2^z, \sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+] = [\sigma_1^z,\sigma_1^-]\sigma_2^+ + [\sigma_1^z,\sigma_1^+]\sigma_2^-$$
$$+ [\sigma_2^z,\sigma_2^-]\sigma_1^+ + [\sigma_2^z,\sigma_2^+]\sigma_1^-$$

$$= -2\sigma_1^-\sigma_2^+ + 2\sigma_1^+\sigma_2^- - 2\sigma_2^-\sigma_1^+ + 2\sigma_2^+\sigma_1^- = 0$$

So $H_{2q} = TUHU^\dagger T^\dagger + i\hbar\frac{dT}{dt}T^\dagger$

$$= \frac{\hbar g^2}{\Delta}(\sigma_1^z+\sigma_2^z)(a^\dagger a+\tfrac{1}{2}) + \frac{\hbar g^2}{\Delta}(\sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+)$$

k) We have shown that the to terms in H_{2q} commute

$$\Rightarrow U_{2q} = e^{-\frac{i}{\hbar}H_{2q}t} = e^{-\frac{ig^2}{\Delta}t(\sigma_1^z+\sigma_2^z)(a^\dagger a+\frac{1}{2})}\, e^{-\frac{ig^2}{\Delta}t\underbrace{(\sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+)}_{A}}$$

$$A = \sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+ = \begin{pmatrix}0&0\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\0&0\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$$

$$= \begin{pmatrix}0&0&0&0\\0&\multicolumn{2}{c}{\multirow{2}{*}{(σ_x)}}&0\\0&&&0\\0&0&0&0\end{pmatrix}$$

$$e^{-i\frac{g^2 t}{\Delta}A} = \begin{pmatrix}1&0&0&0\\0&\multicolumn{2}{c}{\multirow{2}{*}{$\left(e^{-\frac{ig^2}{\Delta}t\sigma_x}\right)$}}&0\\0&&&0\\0&0&0&1\end{pmatrix}$$

$$e^{-\frac{ig^2}{\Delta}t\sigma_x} = \cos\frac{g^2t}{\Delta}\cdot\mathbb{1} - i\sin\frac{g^2t}{\Delta}\cdot\sigma_x = \begin{pmatrix}\cos\frac{g^2t}{\Delta} & -i\sin\frac{g^2t}{\Delta}\\ -i\sin\frac{g^2t}{\Delta} & \cos\frac{g^2t}{\Delta}\end{pmatrix}$$

1.) $t = \frac{3\pi\Delta}{2g^2} \Rightarrow \frac{g^2 t}{\Delta} = \frac{3\pi}{2} \Rightarrow M\left(\frac{3\pi\Delta}{2g^2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

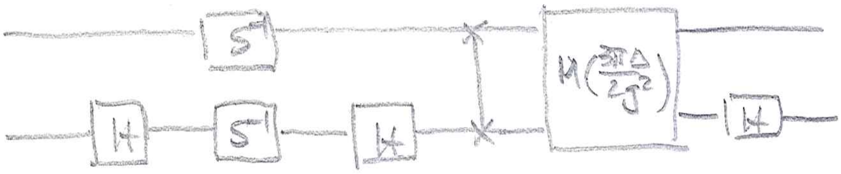
A Hadamard gate on the second qubit is

$$\mathbb{1} \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

S^{-1} on both qubits is

$$S^{-1} \otimes S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The circuit



is then given by

$$\underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{\mathbb{1} \otimes H} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{M\left(\frac{3\pi\Delta}{2g^2}\right)} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{SWAP} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{S^{-1} \otimes S^{-1}} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{\mathbb{1} \otimes H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = CNOT$$

Fys 4110 Midterm exam 2018. Solutions

a) $S(\zeta) = e^{\frac{1}{2}(\zeta^* a^2 - \zeta a^{\dagger 2})}$ $B = -\frac{1}{2}(\zeta^* a^2 - \zeta a^{\dagger 2})$ $B^\dagger = -B$

$$S^\dagger a S = e^{B} a e^{-B} = a + [B,a] + \frac{1}{2}[B,[B,a]] + \dots$$

$$[B,a] = \frac{1}{2}\zeta[a^{\dagger 2},a] = \frac{1}{2}\zeta\Big(a^\dagger \underbrace{[a^\dagger,a]}_{-1} + \underbrace{[a^\dagger,a]}_{-1} a^\dagger\Big) = -\zeta a^\dagger$$

$$[B,a^\dagger] = -\frac{1}{2}\zeta^*[a^2,a^\dagger] = -\frac{1}{2}\zeta^*\Big(a \underbrace{[a,a^\dagger]}_{1} + \underbrace{[a,a^\dagger]}_{1} a\Big) = -\zeta^* a$$

$$S^\dagger a S = a - \zeta a^\dagger + \frac{1}{2}\zeta\zeta^* a - \frac{1}{3!}\zeta^2\zeta^* a^\dagger + \frac{1}{4!}\zeta^2\zeta^{*2} a + \dots$$

$$= \Big[1 + \frac{1}{2!}|\zeta|^2 + \frac{1}{4!}|\zeta|^4 + \dots\Big]a - \Big[\zeta + \frac{1}{3!}\zeta^2\zeta^* + \frac{1}{5!}\zeta^3\zeta^{*2} + \dots\Big]a^\dagger$$

$$= \Big[1 + \frac{1}{2!}r^2 + \frac{1}{4!}r^4 + \dots\Big]a - e^{i\theta}\Big[r + \frac{1}{3!}r^3 + \frac{1}{5!}r^5 + \dots\Big]a^\dagger$$

$$= \cosh r\, a - e^{i\theta}\sinh r\, a^\dagger$$

$$S^\dagger a^\dagger S = \cosh r\, a^\dagger - e^{-i\theta}\sinh r\, a$$

b) $\langle sq_\zeta | x | sq_\zeta \rangle = \langle 0 | S^\dagger x S | 0 \rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0 | S^\dagger (a^\dagger + a) S | 0 \rangle$

$$= \sqrt{\frac{\hbar}{2m\omega}}\langle 0 | (\cosh r - e^{i\theta}\sinh r)a^\dagger + (\cosh r - e^{-i\theta}\sinh r)a | 0 \rangle = 0$$

$\langle sq_\zeta | p | sq_\zeta \rangle = \langle 0 | S^\dagger p S | 0 \rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle 0 | S^\dagger (a^\dagger - a) S | 0 \rangle$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\langle 0 | (\cosh r + e^{i\theta}\sinh r)a^\dagger - (\cosh r + e^{-i\theta}\sinh r)a | 0 \rangle = 0$$

$$\Delta x^2 = \langle Sq_\xi | x^2 | Sq_\xi \rangle = \langle 0 | S^\dagger x S S^\dagger x S | 0 \rangle$$

$$= \frac{\hbar}{2m\omega}\left(\cosh r - e^{i\theta}\sinh r\right)\left(\cosh r - e^{-i\theta}\sinh r\right)$$

$$= \frac{\hbar}{2m\omega}\Big[\underbrace{\cosh^2 r + \sinh^2 r}_{\cosh 2r} - \underbrace{\cosh r \sinh r}_{\frac{1}{2}\sinh 2r}\underbrace{(e^{i\theta}+e^{-i\theta})}_{2\cos\theta}\Big]$$

$$= \frac{\hbar}{2m\omega}\left(\cosh 2r - \sinh 2r \cos\theta\right)$$

$$\Delta p^2 = \langle Sq_\xi | p^2 | Sq_\xi \rangle = \langle 0 | S^\dagger p S S^\dagger p S | 0 \rangle$$

$$= \frac{\hbar m\omega}{2}\left(\cosh r + e^{i\theta}\sinh r\right)\left(\cosh r + e^{-i\theta}\sinh r\right)$$

$$= \frac{\hbar m\omega}{2}\left[\cosh^2 r + \sinh^2 r + \cosh r \sinh r \left(e^{i\theta}+e^{-i\theta}\right)\right]$$

$$= \frac{\hbar m\omega}{2}\left(\cosh 2r + \sinh 2r \cos\theta\right)$$

c)

$$\Delta x \Delta p = \frac{\hbar}{2}\sqrt{\cosh^2 2r - \sinh^2 2r \underbrace{\cos^2\theta}_{1-\sin^2\theta}}$$

$$= \frac{\hbar}{2}\sqrt{1 + \sinh^2 2r \sin^2\theta}$$

Minimal uncertainty when $\sin\theta = 0$ $\qquad \theta = n\pi$

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\cosh 2r - \sinh 2r\,(-1)^n} = \sqrt{\frac{\hbar}{2m\omega}}\, e^{(-1)^{n+1} r}$$

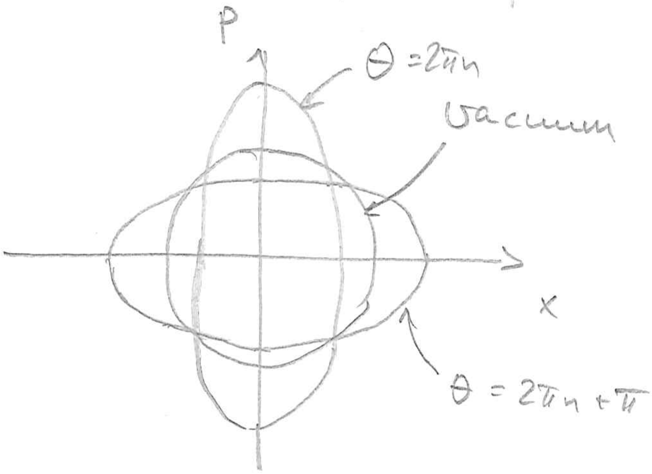
$$\Delta p = \sqrt{\frac{\hbar m\omega}{2}}\sqrt{\cosh 2r + \sinh 2r\,(-1)^n} = \sqrt{\frac{\hbar m\omega}{2}}\, e^{(-1)^n r}$$

When n is even: Δx decreases by a factor e^{-r}

Δp increases by a factor e^{r}

When n is odd: Δx increases, Δp decreases.

We can illustrate this by ~~the~~ spread ~~of~~ the wavefunction in phase space (Wigner function)



d) $\langle sq_\xi | a^\dagger a | sq_\xi \rangle = \langle 0 | S^\dagger a^\dagger S S^\dagger a S | 0 \rangle$

$= \langle 0 | (\cosh r\, a^\dagger - e^{-i\theta} \sinh r\, a)(\cosh r\, a - e^{i\theta} \sinh r\, a^\dagger) | 0 \rangle$

$= \sinh^2 r$

e) From the lecture notes (1.196): $D(\alpha)^\dagger x D(\alpha) = x + x_c$

$D(\alpha)^\dagger p D(\alpha) = p + p_c$

with $\alpha = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x_c + i p_c)$ (1.187)

From this it follows that

$\langle \alpha, sq_\xi | x | \alpha, sq_\xi \rangle = x_c = \sqrt{\frac{2\hbar}{m\omega}}\, \mathrm{Re}\, \alpha$

$\langle \alpha, sq_\xi | p | \alpha, sq_\xi \rangle = p_c = \sqrt{2m\hbar\omega}\, \mathrm{Im}\, \alpha$

and the uncertainties are unchanged

f) $D(\alpha)S(\zeta) = S S^\dagger D S \qquad D = e^{\alpha a^\dagger - \alpha^* a}$

$$S^\dagger D S = S^\dagger\left(1 + (\alpha a^\dagger - \alpha^* a) + \frac{1}{2!}(\alpha a^\dagger - \alpha^* a)^2 + \dots\right) S$$

$$S^\dagger(\alpha a^\dagger - \alpha^* a)^n S = S^\dagger(\alpha a^\dagger - \alpha^* a)S S^\dagger(\ ) \cdots (\ ) S$$

$$S^\dagger(\alpha a^\dagger - \alpha^* a) S = \alpha(\cosh r\, a^\dagger - e^{-i\theta} \sinh r\, a) - \alpha^*(\cosh r\, a - e^{i\theta} \sinh r\, a^\dagger)$$

$$= (\underbrace{\alpha \cosh r + e^{i\theta}\alpha^* \sinh r}_{\beta^*}) a^\dagger - (\underbrace{\alpha^* \cosh r + e^{-i\theta} a \sinh r}_{\beta}) a$$

$\Rightarrow \quad S^\dagger D(\alpha) S = D(\beta)$

$\Rightarrow \quad D(\alpha) S(\zeta) = S(\zeta) D(\beta)$ with $\beta = \alpha^* \cosh r + e^{-i\theta} a \sinh r$

Thus, we get the same states, just with different displacement parameter.

g) If a length l is changed by the gravitational wave by the length Δl, the strain amplitude is $h = \frac{\Delta l}{l}$. That is $\Delta l = l \cdot h = 4 \cdot 10^{3}\,\mathrm{m} \cdot 10^{-21} = 4 \cdot 10^{-18}\,\mathrm{m}$. The diameter of a proton is about $10^{-15}\,\mathrm{m}$, so we need to detect displacements of the order of $\frac{1}{1000}$ times the size of a proton.

h) $|\psi\rangle = S_2(\zeta) D_1(\alpha) |0\rangle$ $\qquad S_2(\zeta) = e^{\frac{1}{2}(\zeta^* a_2^2 - \zeta a_2^{+2})}$

$\qquad D_1(\alpha) = e^{\alpha a_1^+ - \alpha^* a_1}$

$$\langle\psi| b_1^+ b_1 |\psi\rangle = \frac{1}{2}\langle\psi|(a_1^+ - i a_2^+)(a_1 + i a_2)|\psi\rangle$$

$$= \frac{1}{2}\langle\psi| a_1^+ a_1 + a_2^+ a_2 + i(a_1^+ a_2 - a_2^+ a_1)|\psi\rangle$$

$$\langle\psi| b_2^+ b_2 |\psi\rangle = \frac{1}{2}\langle\psi|(a_2^+ - i a_1^+)(a_2 + i a_1)|\psi\rangle$$

$$= \frac{1}{2}\langle\psi| a_1^+ a_1 + a_2^+ a_2 + i(a_2^+ a_1 - a_1^+ a_2)|\psi\rangle$$

$$\langle\psi| b_2^+ b_2 - b_1^+ b_1 |\psi\rangle = i\langle\psi| a_2^+ a_1 - a_1^+ a_2 |\psi\rangle$$

$$\langle\psi| a_2^+ a_1 |\psi\rangle = \langle 0| D_1^+ S_2^+ a_2^+ a_1 S_2 D_1 |0\rangle$$

$$= \langle 0| D_1^+ a_1 D_1 |0\rangle \langle 0| \underbrace{\underbrace{S_2^+ a_2^+ S_2}_{\cosh r\, a^+ - e^{-i\theta} \sinh r\, a}|0\rangle}_{0} = 0$$

$$\langle\psi| a_1^+ a_2 |\psi\rangle = 0$$

$$\Rightarrow \quad \langle\psi| P |\psi\rangle = 0$$

$$p^2 = \left(\frac{2\hbar\omega}{c}\right)^2 \Big[(b_2^\dagger b_2)^2 + (b_1^\dagger b_1)^2 \underbrace{- b_2^\dagger b_2\, b_1^\dagger b_1 - b_1^\dagger b_1\, b_2^\dagger b_2}_{} \Big]$$

These two terms are equal since $[b_1^\dagger, b_2] = \frac{1}{2}[a_1^\dagger - i a_2^\dagger,\ a_2 + i a_1]$

$$= \frac{i}{2}\left([a_1^\dagger, a_1] - [a_2^\dagger, a_2] \right) = 0$$

$$\langle\psi|(b_1^\dagger b_1)^2|\psi\rangle = \frac{1}{4}\langle\psi|\left[a_1^\dagger a_1 + a_2^\dagger a_2 + i(a_1^\dagger a_2 - a_2^\dagger a_1)\right]^2|\psi\rangle$$

$$= \frac{1}{4}\langle\psi|(a_1^\dagger a_1)^2 + (a_2^\dagger a_2)^2 - (a_1^\dagger a_2 - a_2^\dagger a_1)^2 + 2a_1^\dagger a_1 a_2^\dagger a_2$$

$$+ i\left[a_1^\dagger a_1(a_1^\dagger a_2 - a_2^\dagger a_1) + (a_1^\dagger a_2 - a_2^\dagger a_1)a_1^\dagger a_1\right.$$

$$\left. + a_2^\dagger a_2(a_1^\dagger a_2 - a_2^\dagger a_1) + (a_1^\dagger a_2 - a_2^\dagger a_1)a_2^\dagger a_2\right]|\psi\rangle$$

$$\langle\psi|(a_1^\dagger a_1)^2|\psi\rangle = \langle\psi|a_1^\dagger \underbrace{a_1 a_1^\dagger}_{a_1^\dagger a_1 + 1} a_1|\psi\rangle$$

$$= \langle 0|S_2^\dagger D_1^\dagger (a_1^{\dagger 2} a_1^2 + a_1^\dagger a_1) D_1 S_2|0\rangle = |\alpha|^4 + |\alpha|^2$$

$$S_2\, S_2^\dagger$$

$$\langle\psi|(a_2^\dagger a_2)^2|\psi\rangle = \langle 0|S_2^\dagger a_2^\dagger a_2 a_2^\dagger a_2 S_2|0\rangle$$

$\theta = 0$

$$= \langle 0|(a_2^\dagger \cosh r - a_2 \sinh r)(a_2 \cosh r - a_2^\dagger \sinh r)$$

$$(a_2^\dagger \cosh r - a_2 \sinh r)(a_2 \cosh r - a_2^\dagger \sinh r)|0\rangle$$

$$= \sinh^2 r \langle 0|a_2(a_2\cosh r - a_2^\dagger \sinh r)(a_2^\dagger \cosh r - a_2 \sinh r)a_2^\dagger|0\rangle$$

$$= \sinh^2 r \langle 0|\underbrace{a_2 a_2 a_2^\dagger a_2^\dagger}_{\Downarrow \atop 2} \cosh^2 r + \underbrace{a_2 a_2^\dagger a_2 a_2^\dagger}_{\Downarrow \atop 1} \sinh^2 r|0\rangle$$

$$= \sinh^4 r + 2\sinh^2 r \cosh^2 r$$

$$\langle\psi|(a_1^\dagger a_2 - a_2^\dagger a_1)^2|\psi\rangle = \langle\psi|a_1^\dagger a_2 a_1^\dagger a_2 - a_1^\dagger a_2 a_2^\dagger a_1 - a_2^\dagger a_1 a_1^\dagger a_2$$

$$+ a_2^\dagger a_1 a_2^\dagger a_1|\psi\rangle$$

$$\langle\psi|a_1^\dagger a_2 a_1^\dagger a_2|\psi\rangle = \underbrace{\langle 0|D_1^\dagger a_1^\dagger a_1^\dagger D_1|0\rangle}_{\alpha^{*2}} \underbrace{\langle 0|S_2^\dagger a_2 \overset{\swarrow S_2 S_2^\dagger}{} a_2 S_2|0\rangle}_{\underbrace{\langle 0|(a\cosh r - a^\dagger\sinh r)(a\cosh r - a^\dagger\sinh r)|0\rangle}_{-\cosh r\sinh r}}$$

α real

$$= -\alpha^2\cosh r\sinh r$$

$$\langle\psi|a_2^\dagger a_1 a_2^\dagger a_1|\psi\rangle = -\alpha^2\cosh r\sinh r$$

$$\langle\psi|\underbrace{a_1^\dagger a_1}_{\alpha^2}\underbrace{a_2 a_2^\dagger}_{1+\underbrace{a_2^\dagger a_2}_{d)\hookrightarrow\sinh^2 r}}|\psi\rangle = \alpha^2(1+\sinh^2 r) = \alpha^2\cosh^2 r$$

$$\langle\psi|\underbrace{a_1 a_1^\dagger}_{1+a_1^\dagger a_1} a_2^\dagger a_2|\psi\rangle = (1+\alpha^2)\sinh^2 r$$

$$\langle\psi|a_1^\dagger a_1 a_1^\dagger a_2|\psi\rangle = (\ )_1\langle 0|\underbrace{S_2^\dagger a_2 S_2}_{a_2\cosh r - a_2^\dagger\sinh r}|0\rangle = 0$$

The same will happen with all expectations of products containing an odd number of a_2 and $a_2^\dagger$.

$$\Rightarrow \langle\psi|(b_1^\dagger b_1)^2|\psi\rangle = \frac{1}{4}\Big[\alpha^4 + \alpha^2 + \sinh^4 r + 2\sinh^2\cosh^2 r$$

$$-(-2\alpha^2\cosh r\sinh r - \alpha^2\cosh^2 r - (1+\alpha^2)\sinh^2 r)$$

$$+ 2\alpha^2\sinh^2 r\Big]$$

$$\langle\psi|(b_2^\dagger b_2)^2|\psi\rangle = \frac{1}{4}\langle\psi|\left[a_1^\dagger a_1 + a_2^\dagger a_2 - i(a_1^\dagger a_2 - a_2^\dagger a_1)\right]^2|\psi\rangle$$

$$= \langle\psi|(b_1^\dagger b_1)^2|\psi\rangle$$

$$\langle\psi| b_2^\dagger b_2\, b_1^\dagger b_1|\psi\rangle = \frac{1}{4}\langle\psi|\left[a_1^\dagger a_1 + a_2^\dagger a_2 - i(a_1^\dagger a_2 - a_2^\dagger a_1)\right]$$

$$\left[a_1^\dagger a_1 + a_2^\dagger a_2 + i(a_1^\dagger a_2 - a_2^\dagger a_1)\right]|\psi\rangle$$

$$= \frac{1}{4}\langle\psi|(a_1^\dagger a_1)^2 + (a_2^\dagger a_2)^2 + (a_1^\dagger a_2 - a_2^\dagger a_1)^2 + 2a_1^\dagger a_1\, a_2^\dagger a_2\,|\psi\rangle$$

$$\langle\psi|p^2|\psi\rangle = -\left(\frac{2\hbar\omega}{c}\right)^2\langle\psi|(a_1^\dagger a_2 - a_2^\dagger a_1)^2|\psi\rangle$$

$$= \left(\frac{2\hbar\omega}{c}\right)^2\left[2\alpha^2\cosh r\sinh r + \alpha^2\cosh^2 r + (1+\alpha^2)\sinh^2 r\right]$$

$$= \left(\frac{2\hbar\omega}{c}\right)^2\left[\alpha^2(\underbrace{\cosh^2 r + 2\cosh r\sinh r + \sinh^2 r}_{(\underbrace{\cosh r + \sinh r}_{e^r})^2}) + \sinh^2 r\right]$$

$$= \left(\frac{2\hbar\omega}{c}\right)^2\left[\alpha^2 e^{2r} + \sinh^2 r\right]$$

i) After the time τ a momentum p is given to the end mirror. It will have a velocity $v = \frac{p}{m}$. The average velocity is $\bar{v} = \frac{1}{2}v = \frac{p}{2m}$, which gives a displacement $z = \bar{v}\tau = \frac{p}{2m}\tau$. Therefore $\Delta z_{pp} = \frac{\tau}{2m}\Delta p$ and from h) we get

$$\Delta z_{rp} = \frac{\hbar\omega\tau}{mc}\sqrt{\alpha^2 e^{2r} + \sinh^2 r}$$

The mean number of photons in input 1 is α^2, so the input power must be proportional to α^2.

Thus, Δz_{rp} increases with increasing laser power and decreases with increasing mass m.

To reduce it we can either reduce the power or increase the mass of the mirrors. We see that if $\alpha >> 1$ which usually is the case, we can also reduce the noise by squeezing with a negative r.

j) For the beamsplitter we had the relations

$$b_1 = \frac{1}{\sqrt{2}}(a_1 + i a_2) \qquad b_2 = \frac{1}{\sqrt{2}}(a_2 + i a_1)$$

Assuming a symmetrical beamsplitter we apply the same relations when the light passes the beamsplitter the second time, adding phasefactors to describe the phase change in traversing the arms of the interferometer.

$$c_1 = \frac{1}{\sqrt{2}}\left(b_1 e^{i\theta_1} + i b_2 e^{i\theta_2}\right)$$

$$c_2 = \frac{1}{\sqrt{2}}\left(b_2 e^{i\theta_2} + i b_1 e^{i\theta_1}\right)$$

Expressed in terms of a_1, a_2 :

$$c_1 = \frac{1}{2}\left[(a_1+ia_2)e^{i\theta_1} + i(a_2+ia_1)e^{i\theta_2}\right]$$

$$= \frac{1}{2}\left[a_1\left(e^{i\theta_1}-e^{i\theta_2}\right) + ia_2\left(e^{i\theta_1}+e^{i\theta_2}\right)\right]$$

$$= e^{i\Phi}\left[-ia_1\sin\phi + ia_2\cos\phi\right]$$

$$c_2 = \frac{1}{2}\left[(a_2+ia_1)e^{i\theta_2} + i(a_1+ia_2)e^{i\theta_1}\right]$$

$$= \frac{1}{2}\left[ia_1\left(e^{i\theta_2}+e^{i\theta_1}\right) + a_2\left(e^{i\theta_2}-e^{i\theta_1}\right)\right]$$

$$= e^{i\Phi}\left[ia_1\cos\phi + ia_2\sin\phi\right]$$

Here $\Phi = \frac{1}{2}(\theta_1+\theta_2)$ is the average phase change and $\phi = \frac{1}{2}(\theta_2-\theta_1)$ is half the phase difference.

k) $\langle\psi|c_2^\dagger c_2|\psi\rangle = \langle\psi|(a_1^\dagger\cos\phi + a_2^\dagger\sin\phi)(a_1\cos\phi + a_2\sin\phi)|\psi\rangle$

$$= \langle\psi|a_1^\dagger a_1|\psi\rangle\cos^2\phi + \langle\psi|a_2^\dagger a_2|\psi\rangle\sin^2\phi$$

$$+ \underbrace{\langle\psi|a_1^\dagger a_2 + a_2^\dagger a_1|\psi\rangle}_{0}\cos\phi\sin\phi$$

$$= \alpha^2\cos^2\phi + \sinh^2 r\sin^2\phi$$

$$(c_2^\dagger c_2)^2 = (a_1^\dagger a_1)^2\cos^4\phi + (a_2^\dagger a_2)^2\sin^4\phi + (a_1^\dagger a_2 + a_2^\dagger a_1)^2\cos^2\phi\sin^2\phi$$

$$+\, 2 a_1^\dagger a_1 a_2^\dagger a_2 \cos^2\phi\sin^2\phi \;+$$ Terms with 1 or 3 $a_2, a_2^\dagger$ which average to 0 as on p ②

$$\langle\psi|(c_2^\dagger c_2)^2|\psi\rangle = (\alpha^4+\alpha^2)\cos^4\phi + (\sinh^4 r + 2\sinh^2 r\cosh^2 r)\sin^4\phi$$

$$+\left[2\alpha^2\sinh^2 r - 2\alpha^2\cosh r\sinh r + \alpha^2\cosh^2 r + (1+\alpha^2)\sinh^2 r\right]\cos^2\phi\sin^2\phi$$

$$(\Delta N_2)^2 = \langle\psi|N_2^2|\psi\rangle - \langle\psi|N_2|\psi\rangle^2$$

$$= \alpha^2\cos^4\phi + 2\sinh^2 r\cosh^2 r\sin^4\phi$$

$$+\underbrace{\left[\cancel{2\alpha^2\sinh^2 r} - 2\alpha^2\cosh r\sinh r + \alpha^2\underset{1+\sinh^2 r}{\cosh^2 r} + (1+\alpha^2)\sinh^2 r - \cancel{2\alpha^2\sinh^2 r}\right]}\cos^2\phi\sin^2\phi$$

$$= \alpha^2\cos^4\phi + 2\sinh^2 r\cosh^2 r\sin^4\phi$$

$$+\left[\alpha^2\Big(1 + 2\underbrace{\sinh^2 r}_{\frac{1}{4}\underbrace{(e^r-e^{-r})^2}_{e^{2r}+e^{-2r}-2}} - 2\underbrace{\cosh r\sinh r}_{\frac{1}{4}(e^{2r}-e^{-2r})}\Big) + \sinh^2 r\right]\cos^2\phi\sin^2\phi$$

$$= \alpha^2\cos^4\phi + 2\sinh^2 r\cosh^2 r\sin^4\phi$$

$$+\left(\alpha^2 e^{-2r} + \sinh^2 r\right)\cos^2\phi\sin^2\phi$$

1) To relate uncertainty in phase with uncertainty in position we use the fact that $\theta_1 = \frac{2(l+z_1)}{\lambda} 2\pi = \frac{\omega}{c} 2(l+z_1)$

$$\theta_2 = \frac{\omega}{c} 2(l+z_2)$$

$$\Rightarrow \phi = \frac{1}{2}(\theta_2 - \theta_1) = \frac{\omega}{c} z \qquad (z = z_2 - z_1)$$

A change Δz then gives a phase change

$$\Delta\phi = \frac{\omega}{c}\Delta z$$

A change $\Delta\phi$ in phase gives a change in average photon count

$$\Delta N_2 = \frac{dN_2}{d\phi}\cdot\Delta\phi = -2\alpha^2\cos\phi\sin\phi\cdot\Delta\phi$$

$$= -2\alpha^2\cos\phi\sin\phi\cdot\frac{\omega}{c}\Delta z$$

The error in Δz due to photon count error is

$$\Delta z_{pc} = \frac{c}{2\omega\alpha^2\cos\phi\sin\phi}\Delta N_2$$

$$= \frac{c}{2\omega\alpha^2\cos\phi\sin\phi}\sqrt{\alpha^2\cos^4\phi + 2\sinh^2 r\cosh^2 r\sin^4\phi + (\alpha^2 e^{-2r} + \sinh^2 r)\underset{\cos^2\phi\sin^2\phi}{\wedge}}$$

$$= \frac{c}{2\omega}\sqrt{\frac{\cot^2\phi}{\alpha^2} + \frac{2\tan^2\phi\sinh^2 r\cosh^2 r}{\alpha^4} + \frac{e^{-2r}}{\alpha^2} + \frac{\sinh^2 r}{\alpha^4}}$$

4) $\Delta z_{pc} = \frac{c}{2\omega}\frac{e^{-r}}{\alpha}$ $\Delta z_{rp} = \frac{\hbar\omega\tau}{mc}\alpha e^{r}$

$$\Delta z = \sqrt{\Delta z_{pc}^2 + \Delta z_{rp}^2} = \sqrt{\left(\frac{c}{2\omega}\right)^2 \frac{e^{-2r}}{\alpha^2} + \left(\frac{\hbar\omega\tau}{mc}\right)^2 e^{2r}\alpha^2}$$

Δz_{pc} decreases with increasing power (increasing α)

Δz_{rp} increases ———

$\Rightarrow$ There must exist an optimal power. $P_{opt} \sim \alpha_{opt}^2$

$$\frac{d\Delta z}{d\alpha^2} = \frac{-\left(\frac{c}{2\omega}\right)^2 e^{-2r}\frac{1}{\alpha^4} + \left(\frac{\hbar\omega\tau}{mc}\right)^2 e^{2r}}{2\Delta z} = 0$$

$$\alpha_{opt}^2 = \frac{c\,mc}{2\omega\,\hbar\omega\tau}e^{-2r} = \underbrace{\frac{mc^2}{2\hbar\omega^2\tau}}_{\alpha_0^2}e^{-2r} = \alpha_0^2 e^{-2r}$$

$\Rightarrow$ $P_{opt} = P_0 e^{-2r}$ where P_0 is the optimal power without squeezing.

For the optimal noise we get

$$\Delta z_{opt} = \sqrt{\left(\frac{c}{2\omega}\right)^2 \frac{1}{\alpha_0^2} + \left(\frac{\hbar\omega\tau}{mc}\right)^2 \alpha_0^2}$$ independent of r.

$\Rightarrow$ Squeezing does not improve the sensitivity but we can obtain the same sensitivity with a smaller power.

h) We have shown that $\langle N_2 \rangle = \alpha^2 \cos^2\phi + \sinh^2 r \sin^2\phi \approx \alpha^2 \cos^2\phi$ and for sufficiently large α: $\Delta N_2 \approx \alpha^2 e^{-2r} \cos^2\phi \sin^2\phi$

To produce a plot similar to the one given, we need in principle the probability distribution $P(N_2)$. It is tempting (and not too unreasonable) to approximate this with a Gaussian with mean $\langle N_2 \rangle$ and standard deviation ΔN_2. One problem with this is that without squeezing $(r=0)$ we get $\Delta N_2 = \langle N_2 \rangle \sin^2\phi$ and we have assumed that $\cot\phi < e^{-r}$ when we dropped the first term in the expression for ΔZ_{pc}. This means $\sin\phi$ is not small, and the standard deviation is of the same order as the mean. This gives a non-negligible pobability for a negative N_2 which is not physical. Another reasonable choice of distribution could be the Poisson distribution, as this is what we expect for independent, uncorrelated events. However, it has always the same value for the mean and the variance. We need the unsqueezed and squeezed to have the same mean but different variances, so both can not be Poisson.

Can we calculate the true distribution?

The joint probability for n_1 photons in detector 1 and n_2 photons in detector 2 is $P(n_1, n_2) = |\langle n_1 n_2 | \psi \rangle|^2$

We have $|n_1\rangle = \frac{1}{\sqrt{n_1!}} c_1^{+n_1} |0\rangle \qquad |n_2\rangle = \frac{1}{\sqrt{n_2!}} c_2^{+n_2} |0\rangle$

$$|\psi\rangle = S_2 D_1 |0\rangle$$

$$c_1 = -a_1 \sin\phi + a_2 \cos\phi$$

$$c_2 = a_1 \cos\phi + a_2 \sin\phi$$

(ignoring irrelevant phases)

$$\langle n_1 n_2 | \psi \rangle = \frac{1}{\sqrt{n_1! n_2!}} \langle 0 | c_2^{n_2} c_1^{n_1} S_2 D_1 |0\rangle$$

$$= \frac{1}{\sqrt{n_1! n_2!}} \langle 0 | \underbrace{(a_1 \cos\phi + a_2 \sin\phi)}_{\sum_k \binom{n_2}{k} \cos^{n_2-k}\phi \sin^k\phi \, a_1^{n_2-k} a_2^k}{}^{n_2} (\underbrace{-a_1 \sin\phi + a_2 \cos\phi}_{\sum_l \binom{n_1}{l} (-\sin\phi)^{n_1-l} \cos\phi^l a_1^{n_2-l} a_2^l})^{n_1} S_2 D_1 |0\rangle$$

We get terms of the form

$$\langle 0 | a_1^{n_1+n_2-k-l} a_2^{k+l} S_2 D_1 |0\rangle =$$

$$= \underbrace{\langle 0 | a_1^{n_1+n_2-k-l} D_1 |0\rangle}_{\alpha^{n_1+n_2-k-l} e^{-\frac{1}{2}\alpha^2}} \underbrace{\langle 0 | a_2^{k+l} S_2 |0\rangle}_{\begin{cases} \frac{1}{\sqrt{\cosh r}} (-\tanh r)^{\frac{k+l}{2}} \frac{(k+l)!}{2^{\frac{k+l}{2}} \left(\frac{k+l}{2}\right)!}, & k+l = 0, 2, 4 \ldots \\ 0 & k+l = 1, 3, 5 \ldots \end{cases}}$$

Here we used the representations in number states:

$$|\alpha\rangle = D(\alpha)|0\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{1}{2}\alpha^2} |n\rangle$$

$$|sq\rangle = S(\xi)|0\rangle = \frac{1}{\sqrt{\cosh r}} \sum_n (-\tanh r)^n \frac{\sqrt{(2n)!}}{2^n n!} |2n\rangle$$

This gives

$$\langle n_1 n_2 | \psi \rangle = \sqrt{n_1! n_2!}\ \frac{e^{-\frac{1}{2}\alpha^2}}{\sqrt{\cosh r}} \sum_{k,l} \frac{(-1)^{n_1-l}(k+l)! \cos^{n_2-k+l}\phi \sin^{n_1-l+k}\phi}{(n_2-k)!\, k!\, (n_2-l)!\, l!\, 2^{\frac{k+l}{2}} \left(\frac{k+l}{2}\right)!}\ \alpha^{n_1+n_2-k-l} (-\tanh r)^{\frac{k+l}{2}} R_{k+l}$$

where $R_{k+l} = \begin{cases} 1 & k+l = 0,2,4\ldots \\ 0 & k+l = 1,3,5\ldots \end{cases}$

I can see no hope except numerical calculation of this sum

Then we find $P(n_1, n_2) = |\langle n_1 n_2 | \psi \rangle|^2$

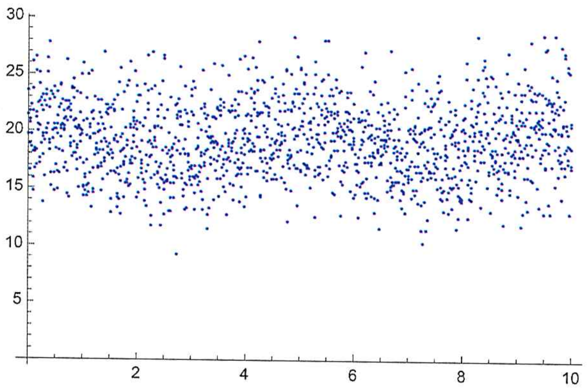
and the marginal distribution $P_2(n_2) = \sum_{n_1} P(n_1, n_2)$

Numerical calculation is complicated by many large numbers because of the factorials, and large numbers of terms. It is easier with small α, so let us try modest squeezing with $r = 1$. In our approximations we have assumed $\cot\phi < e^{-r}$ and $\frac{\tan\phi \sinh r \cosh r}{\alpha} < e^{-r}$

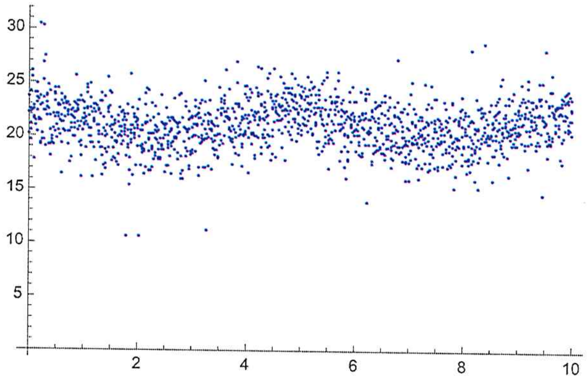
which implies $\alpha > e^{2r} \sinh r \cosh r \overset{r=1}{=} 13$.

We choose $\alpha = 13$ and $\phi = \cot^{-1} e^{-r} \overset{r=1}{=} 1{,}22$

Without squeezing:



With squeezing, $r=1$



The difference is not so impressive as in the paper, we need $r > 1$, but that requires more work to improve the numerical calculation of $\langle n_1 n_2 | \psi \rangle$.

Fys 4110 Midterm exam 2019. Solutions.

Problem 1

a) Commutators:

$$[c,c^\dagger] = \mu^2 \underbrace{[a,a^\dagger]}_{1} + \nu^2 \underbrace{[b,b^\dagger]}_{1} = \mu^2+\nu^2 = 1$$

$$[d,d^\dagger] = \nu^2 [a,a^\dagger] + \mu^2 [b,b^\dagger] = 1$$

$$[c,d] = 0$$

$$[c,d^\dagger] = \mu\nu[a,a^\dagger] - \mu\nu[b,b^\dagger] = 0$$

We solve for a, b to get

$$a = \mu c + \nu d \qquad b = \nu c - \mu d$$

$$H = \hbar\omega_a(\mu c^\dagger + \nu d^\dagger)(\mu c + \nu d) + \hbar\omega_b(\nu c^\dagger - \mu d^\dagger)(\nu c - \mu d)$$

$$+ \frac{\hbar\lambda}{2}\left[(\mu c^\dagger + \nu d^\dagger)(\nu c - \mu d) + (\nu c^\dagger - \mu d^\dagger)(\mu c + \nu d)\right]$$

$$= \hbar\omega_a(\mu^2 c^\dagger c + \mu\nu c^\dagger d + \mu\nu d^\dagger c + \nu^2 d^\dagger d)$$

$$+ \hbar\omega_b(\nu^2 c^\dagger c - \mu\nu c^\dagger d - \mu\nu d^\dagger c + \mu^2 d^\dagger d)$$

$$+ \frac{\hbar\lambda}{2}(\mu\nu c^\dagger c - \mu^2 c^\dagger d + \nu^2 d^\dagger c - \mu\nu d^\dagger d + \mu\nu c^\dagger c + \nu^2 c^\dagger d - \mu^2 d^\dagger c - \mu\nu d^\dagger d)$$

$$= \hbar(\mu^2\omega_a + \nu^2\omega_b + \lambda\mu\nu)c^\dagger c + \hbar(\nu^2\omega_a + \mu^2\omega_b - \lambda\mu\nu)d^\dagger d$$

$$+ \hbar\underbrace{\left(\mu\nu\omega_a - \mu\nu\omega_b - \frac{\lambda\mu^2}{2} + \frac{\lambda\nu^2}{2}\right)}c^\dagger d + \hbar\underbrace{\left(\mu\nu\omega_a - \mu\nu\omega_b + \frac{\lambda\nu^2}{2} - \frac{\lambda\mu^2}{2}\right)}d^\dagger c$$

→ Equal to 0 if ←

$$\mu\nu(\underbrace{\omega_a - \omega_b}_{\Delta}) = \frac{\lambda}{2}(\mu^2 - \nu^2)$$

$$\mu^2 \underset{\substack{\uparrow \\ 1-\mu^2}}{\nu^2} \Delta^2 = \frac{\lambda^2}{4}(\mu^2 - \underset{\substack{\uparrow \\ 1-\mu^2}}{\nu^2})^2$$

$$\Rightarrow \mu^2(1-\mu^2)\Delta^2 = \frac{\lambda^2}{4}(2\mu^2 - 1)^2 = \frac{\lambda^2}{4}(4\mu^4 - 4\mu^2 + 1)$$

$x=\mu^2$: $\Delta^2 x - \Delta^2 x^2 = \lambda^2 x^2 - \lambda^2 x + \frac{\lambda^2}{4}$

$$(\underbrace{\Delta^2+\lambda^2}_{\bar\lambda^2})x^2 - (\underbrace{\Delta^2+\lambda^2}_{\bar\lambda^2})x + \frac{\lambda^2}{4} = 0$$

$$x = \frac{\bar\lambda^2 \pm \sqrt{\bar\lambda^4 - \bar\lambda^2\lambda^2}}{2\bar\lambda^2} = \frac12\left(1\pm\sqrt{1-\frac{\lambda^2}{\bar\lambda^2}}\right)$$

$$= \frac12\left(1\pm\sqrt{\frac{\Delta^2}{\bar\lambda^2}}\right) = \frac12(1\pm\varepsilon) \qquad \varepsilon = \frac{\Delta}{\bar\lambda} = \frac{\Delta}{\sqrt{\Delta^2+\lambda^2}}$$

$$\Rightarrow \mu = \sqrt{\frac12(1+\varepsilon)} \qquad \nu = \sqrt{\frac12(1-\varepsilon)}$$

$$\omega_c = \mu^2\omega_a + \nu^2\omega_b + \lambda\mu\nu = \frac12(1+\varepsilon)\omega_a + \frac12(1-\varepsilon)\omega_b + \frac{\lambda}{2}\sqrt{\underbrace{1-\varepsilon^2}_{\frac{\lambda^2}{\Delta^2+\lambda^2}}}$$

$$= \underbrace{\frac{\omega_a+\omega_b}{2}}_{\bar\omega} + \frac{\omega_a-\omega_b}{2}\varepsilon + \frac{\lambda^2}{2\bar\lambda} = \bar\omega + \frac12\left(\frac{\Delta^2}{\bar\lambda}+\frac{\lambda^2}{\bar\lambda}\right) = \bar\omega+\frac12\bar\lambda$$

Similarly $\omega_d = \bar\omega - \frac12\bar\lambda$

b) $|1_A 0_B\rangle = a^\dagger|0\rangle = (\mu c^\dagger + \nu d^\dagger)|0\rangle = \mu|1_c 0_d\rangle + \nu|0_c 1_d\rangle$

$$\xrightarrow{\text{time}} \mu e^{-i\omega_c t}|1_c0_d\rangle + \nu e^{-i\omega_d t}|0_c1_d\rangle = e^{-i\bar\omega t}\left(\mu e^{-i\frac{\bar\lambda t}{2}}|1_c0_d\rangle + \nu e^{i\frac{\bar\lambda t}{2}}|0_d 1_c\rangle\right)$$

$$= e^{-i\bar\omega t}\left[\mu e^{-i\frac{\bar\lambda t}{2}}c^\dagger + \nu e^{i\frac{\bar\lambda t}{2}}d^\dagger\right]|0\rangle$$

$$= e^{-i\bar\omega t}\left[\mu e^{-i\frac{\bar\lambda t}{2}}(\mu a^\dagger+\nu b^\dagger) + \nu e^{i\frac{\bar\lambda t}{2}}(\nu a^\dagger - \mu b^\dagger)\right]|0\rangle$$

$$= e^{-i\bar\omega t}\left[\underbrace{\left(\mu^2 e^{-i\frac{\bar\lambda t}{2}} + \nu^2 e^{i\frac{\bar\lambda t}{2}}\right)}_{\cos\frac{\bar\lambda t}{2} - i\varepsilon\sin\frac{\bar\lambda t}{2}} a^\dagger + \underbrace{\mu\nu}_{\frac{\lambda}{2\bar\lambda}}\underbrace{\left(e^{-i\frac{\bar\lambda t}{2}} - e^{i\frac{\bar\lambda t}{2}}\right)}_{-2i\sin\frac{\bar\lambda t}{2}} b^\dagger\right]|0\rangle$$

$$= e^{-i\bar\omega t}\left[\left(\cos\frac{\bar\lambda t}{2} - i\varepsilon\sin\frac{\bar\lambda t}{2}\right)|1_A0_B\rangle - i\delta\sin\frac{\bar\lambda t}{2}|0_A1_B\rangle\right]$$

$$\delta = \frac{\lambda}{\bar\lambda} = \frac{\lambda}{\sqrt{\Delta^2+\lambda^2}}$$

c) $|\psi\rangle = e^{-i\bar{\omega}t}\left[\left(\cos\frac{\bar{\lambda}t}{2} - i\epsilon\sin\frac{\bar{\lambda}t}{2}\right)|10\rangle - i\delta\sin\frac{\bar{\lambda}t}{2}|01\rangle\right]$

$$\langle N_A\rangle = \langle\psi|a^\dagger a|\psi\rangle = \cos^2\frac{\bar{\lambda}t}{2} + \epsilon^2\sin^2\frac{\bar{\lambda}t}{2}$$

$$\langle N_B\rangle = \langle\psi|b^\dagger b|\psi\rangle = \delta^2\sin^2\frac{\bar{\lambda}t}{2}$$

If $\omega_a = \omega_b$: $\Delta = 0$, $\bar{\lambda} = \lambda$, $\epsilon = 0$, $\delta = 1$

$$\langle N_A\rangle = \cos^2\frac{\lambda t}{2} \qquad \langle N_B\rangle = \sin^2\frac{\lambda t}{2}$$

The energy is oscillating between the two oscillators. The transfer of energy is complete, and at certain times oscillator A is with certainty in the ground state, while all the energy is in oscillator B.

If $\omega_a \gg \omega_b$ so that $\Delta \gg \lambda$

To lowest order in $\frac{\lambda}{\Delta}$: $\delta = \frac{\lambda}{\bar{\lambda}} \approx \frac{\lambda}{\Delta}$

$$\epsilon^2 = \left(\frac{\Delta}{\bar{\lambda}}\right)^2 = \frac{\Delta^2}{\lambda^2+\Delta^2} = \frac{1}{1+\left(\frac{\lambda}{\Delta}\right)^2} \approx 1 - \frac{\lambda^2}{\Delta^2}$$

$$\langle N_A\rangle \approx \cos^2\frac{\bar{\lambda}t}{2} + \left(1 - \frac{\lambda^2}{\Delta^2}\right)\sin^2\frac{\bar{\lambda}t}{2} = 1 - \frac{\lambda^2}{\Delta^2}\sin^2\frac{\bar{\lambda}t}{2}$$

$$\langle N_B\rangle \approx \frac{\lambda^2}{\Delta^2}\sin^2\frac{\bar{\lambda}t}{2}$$

Oscillator A retains almost all the energy, and only a small fraction is transferred to B and back.

d) $$|\psi\rangle = e^{-i\bar{\omega}t}\Big[\underbrace{\Big(\cos\frac{\bar{\lambda}t}{2} - i\,\varepsilon\sin\frac{\bar{\lambda}t}{2}\Big)}_{A}|10\rangle \underbrace{-i\,\delta\sin\frac{\bar{\lambda}t}{2}}_{B}|01\rangle\Big]$$

$$\rho = |\psi\rangle\langle\psi| = \big(A\,|10\rangle + B\,|01\rangle\big)\big(A^*\langle 10| + B^*\langle 01|\big)$$

$$= |A|^2|10\rangle\langle 10| + AB^*|10\rangle\langle 01| + BA^*|01\rangle\langle 10| + |B|^2|01\rangle\langle 01|$$

$$\rho_A = \mathrm{Tr}_B\,\rho = |A|^2|1\rangle\langle 1| + |B|^2|0\rangle\langle 0|$$

$$= \Big(\cos^2\frac{\bar{\lambda}t}{2} + \varepsilon^2\sin^2\frac{\bar{\lambda}t}{2}\Big)|1\rangle\langle 1| + \delta^2\sin^2\frac{\bar{\lambda}t}{2}\,|0\rangle\langle 0|$$

Entanglement entropy:

$$S = -\mathrm{Tr}\,\rho\ln\rho = -|A|^2\ln|A|^2 - |B|^2\ln|B|^2$$

Maximal S if $|A|^2$ and $|B|^2$ are as equal as possible

$\Delta < \lambda$: $\quad \delta^2 = \frac{\lambda^2}{\Delta^2+\lambda^2} > \frac{1}{2} \qquad S_{max} = \ln 2$

$\Delta > \lambda$: $\quad S_{max}$ when $\sin\frac{\bar{\lambda}t}{2} = 1$

$$S_{max} = -\varepsilon^2\ln\varepsilon^2 - \delta^2\ln\delta^2 = -\frac{\Delta^2}{\Delta^2+\lambda^2}\ln\frac{\Delta^2}{\Delta^2+\lambda^2} - \frac{\lambda^2}{\Delta^2+\lambda^2}\ln\frac{\lambda^2}{\Delta^2+\lambda^2}$$

e) $$|n,0\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle = \frac{1}{\sqrt{n!}}(\mu c^\dagger + \nu d^\dagger)^n|0\rangle \xrightarrow{\text{time}} \frac{e^{-in\bar{\omega}t}}{\sqrt{n!}}\Big(\mu e^{-i\frac{\bar{\lambda}t}{2}}c^\dagger + \nu e^{i\frac{\bar{\lambda}t}{2}}d^\dagger\Big)^n|0\rangle$$

$$= \frac{e^{-in\bar{\omega}t}}{\sqrt{n!}}\big[Aa^\dagger + Bb^\dagger\big]^n|0\rangle = \frac{e^{-in\bar{\omega}t}}{\sqrt{n!}}\sum_{k=0}^{n}\binom{n}{k}A^{n-k}a^{\dagger\,n-k}B^k b^{\dagger k}|0\rangle$$

$$= e^{-in\bar{\omega}t}\sum_{k=0}^{n}\sqrt{\binom{n}{k}}\,A^{n-k}B^k\,|n-k,k\rangle$$

Probability $$P_{|n-k,k\rangle} = \binom{n}{k}|A|^{2(n-k)}|B|^{2k}$$

$$= \binom{n}{k}\Big(\cos^2\frac{\bar{\lambda}t}{2} + \varepsilon^2\sin^2\frac{\bar{\lambda}t}{2}\Big)^{n-k}\Big(\delta^2\sin^2\frac{\bar{\lambda}t}{2}\Big)^k$$

$$\langle N_A\rangle = \sum_{k=0}^{n}(n-k)\,P_{|n-k,k\rangle} = n - \sum_k k\,P_{|n-k,k\rangle}$$

$$\langle N_B\rangle = \sum_k k\,P_{|n-k,k\rangle} = \sum_{k=0}^{n} k\binom{n}{k}\left(|A|^2\right)^{n-k}\left(|B|^2\right)^k$$

$$= n\sum_{k=1}^{n}\binom{n-1}{k-1}\left(|A|^2\right)^{n-k}\left(|B|^2\right)^k = n\sum_{k=0}^{n-1}\binom{n-1}{k}\left(|A|^2\right)^{n-k-1}\left(|B|^2\right)^{k+1}$$

$$= n|B|^2\underbrace{\sum_{k=0}^{n-1}\binom{n-1}{k}\left(|A|^2\right)^{n-1-k}\left(|B|^2\right)^k}_{\left(|A|^2+|B|^2\right)^{n-1}=1} = n|B|^2 = n\,\frac{\lambda^2}{\Delta^2+\lambda^2}\sin^2\frac{\bar\lambda t}{2}$$

$$\langle N_A\rangle = n\left(1-\frac{\lambda^2}{\Delta^2+\lambda^2}\sin^2\frac{\bar\lambda t}{2}\right)$$

Problem 2.

a) $c|\psi(0)\rangle = z_{c0}|\psi(0)\rangle$

$$c|\psi(t)\rangle = c\,e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle = c\,e^{-i\omega_c c^\dagger c - i\omega_d d^\dagger d}|\psi(0)\rangle$$

$$= e^{-i\omega_d d^\dagger d}e^{-i\omega_c c^\dagger c}\underbrace{e^{i\omega_c c^\dagger c}\,c\,e^{-i\omega_c c^\dagger c}}_{e^{-i\omega_c t}c}|\psi(0)\rangle$$

$$= \underbrace{z_{c0}e^{-i\omega_c t}}_{z_c(t)}|\psi(0)\rangle$$

b) $$a|\psi(t)\rangle = (\mu c+\nu d)|\psi(t)\rangle = \underbrace{\left(\mu z_{c0}e^{-i\omega_c t}+\nu z_{d0}e^{-i\omega_d t}\right)}_{z_a(t)}|\psi(t)\rangle$$

$$b|\psi(t)\rangle = (\nu c-\mu d)|\psi(t)\rangle = \underbrace{\left(\nu z_{c0}e^{-i\omega_c t}-\mu z_{d0}e^{-i\omega_d t}\right)}_{z_b(t)}|\psi(t)\rangle$$

$$\left.\begin{array}{l} z_{a0} = \mu z_{c0} + \nu z_{d0} \\ z_{b0} = \nu z_{c0} - \mu z_{d0} \end{array}\right\} \Rightarrow \begin{array}{l} z_{c0} = \mu z_{a0} + \nu z_{b0} \\ z_{d0} = \nu z_{a0} - \mu z_{d0} \end{array}$$

$$z_a(t) = \left(\mu^2 e^{-i\omega_c t} + \nu^2 e^{-i\omega_d t}\right) z_{a0} + \mu\nu\left(e^{-i\omega_c t} - e^{-i\omega_d t}\right) z_{b0}$$

$$= e^{-i\bar{\omega} t}\left[\left(\cos\frac{\Delta t}{2} - i\varepsilon \sin\frac{\Delta t}{2}\right) z_{a0} - i\delta \sin\frac{\Delta t}{2} z_{b0}\right]$$

$$z_b(t) = \mu\nu\left(e^{-i\omega_c t} - e^{-i\omega_d t}\right) z_{a0} + \left(\nu^2 e^{-i\omega_c t} + \mu^2 e^{-i\omega_d t}\right) z_{b0}$$

$$= e^{-i\bar{\omega} t}\left[-i\delta \sin\frac{\Delta t}{2} z_{a0} + \left(\cos\frac{\Delta t}{2} + i\varepsilon \sin\frac{\Delta t}{2}\right) z_{b0}\right]$$

9) We know that $a|\psi\rangle = z_a|\psi\rangle$ $\quad b|\psi\rangle = z_b|\psi\rangle$

Generally $|\psi\rangle = \sum_{mn} C_{mn} |m\rangle_A \otimes |n\rangle_B$

$$a|\psi\rangle = \sum_{mn} C_{mn} \sqrt{m}\, |m-1\rangle \otimes |n\rangle = z_a |\psi\rangle = \sum_{mn} z_a C_{mn} |m\rangle \otimes |n\rangle$$

$$\Rightarrow \sqrt{m}\, C_{mn} = z_a C_{m-1,n}$$

$$b|\psi\rangle = \sum_{mn} C_{mn} \sqrt{n}\, |m\rangle \otimes |n-1\rangle = z_b|\psi\rangle = \sum_{mn} z_b C_{mn} |m\rangle \otimes |n\rangle$$

$$\Rightarrow \sqrt{n}\, C_{mn} = z_b C_{m,n-1}$$

$$C_{m+1,n} = \frac{z_a}{\sqrt{m+1}} C_{mn} \qquad C_{m,n+1} = \frac{z_b}{\sqrt{n+1}} C_{mn}$$

Start from C_{00} and get $C_{10} = \frac{z_a}{\sqrt{1}} C_{00}$, $C_{20} = \frac{z_a^2}{\sqrt{2!}} C_{00}$

$$\ldots\ C_{m0} = \frac{z_a^m}{\sqrt{m!}} C_{00}$$

Start from C_{m0} and get $C_{m1} = \frac{z_b}{\sqrt{1}} C_{m0}$, $C_{m2} = \frac{z_b^2}{\sqrt{2!}} C_{m0}$

$$\ldots\ C_{mn} = \frac{z_b^n}{\sqrt{n!}} C_{m0} = \frac{z_a^m z_b^n}{\sqrt{m!\,n!}} C_{00}$$

$$\Rightarrow |\psi\rangle = \sum_{mn} C_{mn} |m\rangle \otimes |n\rangle = \sum_{mn} \frac{z_a^m z_b^n}{\sqrt{m!}\sqrt{n!}} |m\rangle \otimes |n\rangle$$

$$= \sum_m \frac{z_a^m}{\sqrt{m!}} |m\rangle \otimes \sum_n \frac{z_b^n}{\sqrt{n!}} |n\rangle = |z_a\rangle \otimes |z_b\rangle$$

The same argument shows that since the initial state $|\psi(0)\rangle = |z_{a0}\rangle_A \otimes |z_{b0}\rangle_B$ is coherent with respect to c and d it must be $|z_{c0}\rangle_C \otimes |z_{d0}\rangle_D$.

We then have

$$|z_{a0}\rangle_A \otimes |z_{b0}\rangle_B = |z_{c0}\rangle_C \otimes |z_{d0}\rangle_D \xrightarrow{\text{time}} |z_c\rangle_C \otimes |z_d\rangle_D = |z_a\rangle_A \otimes |z_b\rangle_B$$

No entanglement is generated between A and B.

d) If A and B have degenerate eigenstates:

$A|z_a^0\rangle_A = z_a |z_a^0\rangle_A$ $\qquad$ $B|z_b^0\rangle_B = z_b |z_b^0\rangle_B$

$A|z_a^1\rangle_A = z_a |z_a^1\rangle_A$ $\qquad$ $B|z_b^1\rangle_B = z_b |z_b^1\rangle_B$

We can choose them to be orthogonal: $\langle z_a^0|z_a^1\rangle = \langle z_b^0|z_b^1\rangle = 0$

The state $|\psi\rangle = C_0 |z_a^0\rangle_A \otimes |z_b^0\rangle_B + C_1 |z_a^1\rangle_A \otimes |z_b^1\rangle_B$, $|C_0|^2 + |C_1|^2 = 1$

is then entangled and satisfies

$A \otimes \mathbb{1} |\psi\rangle = z_a |\psi\rangle$ $\qquad$ $\mathbb{1} \otimes B |\psi\rangle = z_b |\psi\rangle$

e) With $z_{b0} = 0$ we get

$$\langle N_A\rangle = \langle z_a|a^\dagger a|z_a\rangle = |z_a|^2 = \left(\cos^2\frac{\Delta t}{2} + \varepsilon^2 \sin^2\frac{\Delta t}{2}\right)|z_{a0}|^2$$

$$\langle N_B\rangle = \langle z_b|b^\dagger b|z_b\rangle = |z_b|^2 = \delta^2 \sin^2\frac{\Delta t}{2}\, |z_{a0}|^2$$

These are the same as those found in 1c) from the state $|10\rangle$ scaled by $|z_{a0}|^2$. Also the same as in 1e) from $|n0\rangle$ if we replace $n \to |z_{a0}|^2$

Problem 3.

a) In writing the Lindblad eq. in the form

$$\dot{\rho} = -i[H,\rho] - \frac{\gamma}{2}(a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger)$$
$$- \frac{\gamma}{2}(b^\dagger b\rho + \rho b^\dagger b - 2b\rho b^\dagger)$$

we have used the two jump operators a and b. This means that we only can reduce the excitation of the oscillators, emitting energy to the environment. But we can not absorb any energy from the environment. This means that there are no exitations of the environment, which means that the temperature of the environment is zero. Since the energies of the oscillators are $\hbar\omega_a$ and $\hbar\omega_b$ zero temperature means $k_B T \ll \hbar\omega_a, \hbar\omega_b$.

b) $H = \omega_a a^\dagger a + \omega_b b^\dagger b + \frac{\lambda}{2}(a^\dagger b + b^\dagger a)$

$$\langle mn|[H,\rho]|m'n'\rangle = \langle mn|\, \omega_a a^\dagger a \rho + \omega_b b^\dagger b \rho + \frac{\lambda}{2}(a^\dagger b + b^\dagger a)\rho - \omega_a \rho a^\dagger a - \omega_b \rho b^\dagger b - \frac{\lambda}{2}\rho(a^\dagger b + b^\dagger a)\, |m'n'\rangle$$

$$= [\omega_a(m-m') + \omega_b(n-n')]\,\rho_{mn,m'n'}$$

$$+ \frac{\lambda}{2}\Big[\langle m-1,n+1|\sqrt{m(n+1)}\,\rho\,|m'n'\rangle + \langle m+1,n-1|\sqrt{(m+1)n}\,\rho\,|m'n'\rangle - \langle mn|\rho\sqrt{(m'+1)n'}\,|m'+1,n'-1\rangle - \langle mn|\rho\sqrt{m'(n'+1)}\,|m'-1,n'+1\rangle\Big]$$

$$= [\omega_a(m-m') + \omega_b(n-n')]\,\rho_{mn,m'n'}$$

$$+ \frac{\lambda}{2}\Big[\sqrt{m(n+1)}\,\rho_{m-1,n+1;m'n'} + \sqrt{(m+1)n}\,\rho_{m+1,n-1;m'n'} - \sqrt{(m'+1)n'}\,\rho_{mn;m'+1,n'-1} - \sqrt{m'(n+1)}\,\rho_{mn;m'-1,n'+1}\Big]$$

$$\langle mn|\underbrace{a^\dagger a}_{m}\rho + \rho\underbrace{a^\dagger a}_{m'} - 2\underbrace{a}_{\sqrt{m+1}\langle m+1|}\rho\underbrace{a^\dagger}_{\sqrt{m'+1}|m'+1\rangle}|m'n'\rangle = (m+m')\rho_{mn,m'n'} - 2\sqrt{(m+1)(m'+1)}\,\rho_{m+1,n;m'+1,n'}$$

$$\langle mn|b^\dagger b\rho + \rho b^\dagger b - 2b\rho b^\dagger|m'n'\rangle = (n+n')\rho_{mn,m'n'} - 2\sqrt{(n+1)(n'+1)}\,\rho_{m,n+1;m',n'+1}$$

This gives the eq. for the elements of the density matrix:

$$\dot\rho_{mn,m'n'} = -i[\omega_a(m-m') + \omega_b(n-n')]\,\rho_{mn,m'n'}$$

$$-i\frac{\lambda}{2}\Big[\sqrt{m(n+1)}\,\rho_{m-1,n+1;m'n'} + \sqrt{(m+1)n}\,\rho_{m+1,n-1;m'n'} - \sqrt{(m'+1)n'}\,\rho_{mn;m'+1,n'-1} - \sqrt{m'(n'+1)}\,\rho_{mn;m'-1,n'+1}\Big]$$

$$-\frac{\gamma}{2}\Big[(m+m'+n+n')\rho_{mn,m'n'} - 2\sqrt{(m+1)(m'+1)}\,\rho_{m+1,n;m'+1,n'} - 2\sqrt{(n+1)(n'+1)}\,\rho_{m,n+1;m',n'+1}\Big]$$

If we start from $|1_A 0_B\rangle$ we have $\rho_{10,10}(0) = 1$ and all other $\rho_{mn,m'n'}(0) = 0$.
At $t=0$ the only matrix elements that will start to grow are those where $\rho_{10,10}$ appear on the RHS of the eq. These are $\rho_{01,10}$, $\rho_{10,01}$ and $\rho_{00,00}$.
Looking for those with one of these on the RHS we find in addition only $\rho_{01,01}$. These will be the only nonzero elements. Physically, it means that if we start with only one excitation on one oscillator we can transfer it to the other oscillator or emit it to the environment. Since the environment is at zero temperature, we will never absorb any energy and no higher excitation of the oscillators will be populated.
The eq. for these matrix elements are:

$$\dot{\rho}_{10,10} = -\gamma \rho_{10,10} - \frac{i\lambda}{2}(\rho_{01,10} - \rho_{10,01})$$

$$\dot{\rho}_{01,10} = (i\Delta - \gamma)\rho_{01,10} - \frac{i\lambda}{2}(\rho_{10,10} - \rho_{01,01})$$

$$\dot{\rho}_{10,01} = (-i\Delta - \gamma)\rho_{10,01} + i\frac{\lambda}{2}(\rho_{10,10} - \rho_{01,01})$$

$$\dot{\rho}_{01,01} = -\gamma\rho_{01,01} + i\frac{\lambda}{2}(\rho_{01,10} - \rho_{10,01})$$

$$\dot{\rho}_{00,00} = \gamma(\rho_{10,10} + \rho_{01,01})$$

c) We calculate the time derivatives and compare to the RHS. ⑪

$$c = \cos\frac{\bar\lambda t}{2} \quad , \quad s = \sin\frac{\bar\lambda t}{2}$$

$$\rho_{01,10} - \rho_{10,01} = -2i\delta c s e^{-\gamma t}$$

$$\rho_{10,10} - \rho_{01,01} = \underbrace{\Big(c^2 + \underbrace{(\varepsilon^2-\delta^2)}_{1-2\delta^2}s^2\Big)}_{1-2\delta^2 s^2} e^{-\gamma t} = (1-2\delta^2 s^2)e^{-\gamma t}$$

$$\dot\rho_{10,10} = -\gamma\rho_{10,10} + \underbrace{[-2cs + 2\varepsilon^2 cs]}_{2(\underbrace{\varepsilon^2-1}_{-\delta^2})cs}\frac{\overset{\swarrow\frac{\lambda}{\delta}}{\bar\lambda}}{2}e^{-\gamma t} = -2\delta cs\frac{\lambda}{2}e^{-\gamma t} - \gamma\rho_{10,01}$$

$$= -\gamma\rho_{10,01} - \frac{i\lambda}{2}(-2i\delta cs e^{-\gamma t}) = -\gamma\rho_{10,01} - \frac{i\lambda}{2}(\rho_{01,10} - \rho_{10,01})$$

$$\dot\rho_{01,10} = -\gamma\rho_{01,10} + \left[+i\delta s^2 - i\delta c^2 + \varepsilon\delta\cdot 2sc\right]\frac{\bar\lambda}{2}e^{-\gamma t}$$

$$= -\gamma\rho_{01,10} - \frac{i\lambda}{2}\left[-s^2 + c^2 + i2\varepsilon sc\right]e^{-\gamma t}$$

$$= -\gamma\rho_{01,10} - \frac{i\lambda}{2}\underbrace{(1-2\delta^2 s^2)e^{-\gamma t}}_{\rho_{10,10}-\rho_{01,01}} - \frac{i\lambda}{2}\Big[\underbrace{-s^2 + c^2 + 2i\varepsilon cs - 1 + 2\delta^2 s^2}_{A}\Big]e^{-\gamma t}$$

$$A = -2(\underbrace{1-\delta^2}_{\varepsilon^2})s^2 + 2i\varepsilon cs = -2\varepsilon^2 s^2 + 2i\varepsilon cs$$

$$-\frac{\lambda}{2}A = -i\lambda\varepsilon cs + \lambda\varepsilon^2 s^2 \qquad\qquad \lambda\varepsilon = \lambda\frac{\Delta}{\bar\lambda} = \delta\Delta$$

$$= -i\delta\Delta cs + \Delta\delta\varepsilon s^2 = \Delta(-i\delta cs + \varepsilon\delta s^2)$$

$$\Rightarrow \dot\rho_{01,10} = (i\Delta - \gamma)\rho_{01,10} - \frac{i\lambda}{2}(\rho_{10,10} - \rho_{01,01})$$

$\dot\rho_{10,01}$ is complex conjugate of $\dot\rho_{01,10}$

$$\dot\rho_{01,01} = -\gamma\rho_{01,01} + \frac{\overset{\swarrow\frac{\lambda}{\delta}}{\bar\lambda}}{2}2\delta^2 cs e^{-\gamma t} = -\gamma\rho_{01,01} + i\frac{\lambda}{2}(\rho_{01,10} - \rho_{10,01})$$

$$\dot\rho_{00,00} = \gamma e^{-\gamma t} = \gamma(\rho_{10,10} + \rho_{01,01})$$

d) From 1b): $|\psi\rangle = e^{-i\bar{\omega}t}\left[(c - i\varepsilon s)|1_A 0_B\rangle - i\delta s|0_A 1_B\rangle\right]$

$$\rho = |\psi\rangle\langle\psi| = (c^2+\varepsilon^2 s^2)|10\rangle\langle 10| + i(c-i\varepsilon s)\delta s|10\rangle\langle 01|$$

$$+ i(c+i\varepsilon s)\delta s|01\rangle\langle 10| + \delta^2 s^2|01\rangle\langle 01|$$

This gives the same matrix elements as Eq (5) if $\gamma = 0$.

e) $$\rho = \begin{pmatrix} \rho_{00,00} & 0 & 0 & 0 \\ 0 & \rho_{01,01} & \rho_{01,10} & 0 \\ 0 & \rho_{10,01} & \rho_{10,10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & B & C & 0 \\ 0 & D & A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues: $$\begin{vmatrix} \lambda - E & 0 & 0 & 0 \\ 0 & \lambda - B & -C & 0 \\ 0 & -D & \lambda - A & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda\underbrace{(\lambda - \overset{1-e^{-\gamma t}}{E})}\left[(\lambda - A)(\lambda - B) - CD\right]$$

If $\gamma = 0$ we know that $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle$ given in 3d) so ρ is pure

$\to$ If $\gamma > 0$ this eigenvalue is > 0 and at least one more must be nonzero to keep $\mathrm{Tr}\,\rho = 1$

$\Rightarrow$ ρ is mixed

Only if $t = 0$ or $t \to \infty$ it is pure.

f) Let A and B be TLS with $\rho_A = \rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\rho = \rho_A \otimes \rho_B$ is then separable while the von Neumann entropy of each reduced density matrix is $\ln 2$, maximal for TLS. In fact, the concept of entanglement entropy is ill-defined for mixed ρ. If $\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\rho_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\rho = \rho_A \otimes \rho_B$ the reduced density matrices for A and B do not have the same von Neumann entropy.

g) With basis $\{|m_A\rangle\}$ for A and $\{|n_B\rangle\}$ for B a general density matrix is $\rho = \sum_{\substack{mn\\m'n'}} \rho_{mn,m'n'} |m_A n_B\rangle\langle m'_A n'_B|$

The partial transpose with respect to B is

$$\rho^{T_B} = \sum_{\substack{mn\\m'n'}} \rho_{mn';m'n} |mn\rangle\langle m'n'|$$

The positive partial transpose criterion (or Peres-Horodecki criterion) states that if ρ is separable, ρ^{T_B} has non-negative eigenvalues. In general, the test is not sufficient to show that a state is separable. That is, there exist non-separable states with positive partial transpose.

For A and B both TLS the state is entangled if and only if ρ^{T_B} has a negative eigenvalue.

Source: Wikipedia page: Peres-Horodecki criterion.

h) In the state (5) only the lowest two levels of each oscillator are excited, and each system is effectively a TLS.

$$\rho = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & B & C & 0 \\ 0 & D & A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \rho^{T_B} = \begin{pmatrix} E & 0 & 0 & C \\ 0 & B & 0 & 0 \\ 0 & 0 & A & 0 \\ D & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues
$$\begin{vmatrix} \lambda-E & 0 & 0 & -C \\ 0 & \lambda-B & 0 & 0 \\ 0 & 0 & \lambda-A & 0 \\ -D & 0 & 0 & \lambda \end{vmatrix} = (\lambda-E)(\lambda-B)(\lambda-A)\lambda + D(-C)(\lambda-B)(\lambda-A)$$

$$= (\lambda-A)(\lambda-B)\left[\lambda(\lambda-E) - DC\right] = 0$$

$$\Rightarrow \lambda = A \text{ or } \lambda = B \text{ or } \lambda^2 - E\lambda - DC = 0$$

$$\Rightarrow \lambda_\pm = \frac{E \pm \sqrt{E^2+4DC}}{2}$$

A and B > 0 $\quad E \geq 0 \quad D = C^* \Rightarrow DC = |C|^2 \geq 0$

$\Rightarrow \lambda_+ \geq 0$ and $\lambda_- \leq 0 \quad \lambda_- < 0$ if $DC > 0$

$$DC = e^{-2\gamma t}\left[\delta^2 \varepsilon^2 \sin^4\frac{\bar{\lambda}t}{2} + \delta^2 \cos^2\frac{\bar{\lambda}t}{2}\sin\frac{\bar{\lambda}t}{2}\right] = e^{-2\gamma t}\delta^2 \sin^2\frac{\bar{\lambda}t}{2}\left(\varepsilon^2 \sin^2\frac{\bar{\lambda}t}{2} + \cos^2\frac{\bar{\lambda}t}{2}\right)$$

$DC = 0$ if $\sin\frac{\bar{\lambda}t}{2} = 0 \Rightarrow \frac{\bar{\lambda}t}{2} = n\pi$, $n = 0, 1, 2 \ldots$

At these times ρ is seperable, otherwise it is entangled.
Note that these are the same times that we found the isolated system to be non-entangled in 1d)

i) The concurrence is for two coupled TLS defined as

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

where λ_i are the square roots in descending order of the matrix $M = \rho\, \sigma_y \otimes \sigma_y\, \rho^*\, \sigma_y \otimes \sigma_y$ $\quad \rho^*$ is elementwise complex conjugate.

Source: M. Plenio and S. Virmani: An introduction to entanglement measures: Quant. Inf. Comput. 7, 1 (2007)
(arxiv: quant-ph/0504163)

j) $$\rho = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & B & C & 0 \\ 0 & D & A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \rho^* = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & B & D & 0 \\ 0 & C & A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ since $D^* = C$ and A, B, E are real

$$\sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & -i\sigma_y \\ i\sigma_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & B & C & 0 \\ 0 & D & A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} E & 0 & 0 & 0 \\ 0 & B & D & 0 \\ 0 & C & A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & DC+AB & 2BC & 0 \\ 0 & 2AD & DC+AB & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Define } X = AB + CD = (c^2 + \varepsilon^2 s^2)\delta^2 s^2 e^{-2\gamma t} + \overbrace{(\varepsilon^2\delta^2 s^4 + \delta^2 c^2 s^2)}^{\delta^2 s^2 (c^2 + \varepsilon^2 s^2)} e^{-2\gamma t}$$
$$= 2(c^2 + \varepsilon^2 s^2)e^{-2\gamma t}$$

Eigenvalues $\begin{vmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu - x & -2BC & 0 \\ 0 & -2AD & \mu - x & 0 \\ 0 & 0 & 0 & \mu \end{vmatrix} = \mu^2\left[(\mu-x)^2 - 4ABCD\right] = 0$

$\Rightarrow \mu_{1,2} = 0 \qquad \mu_{3,4} = x \pm 2\sqrt{ABCD} = \begin{cases} 0 \\ 4(c^2+\varepsilon^2 s^2)\delta^2 s^2 e^{-2\gamma t} \end{cases}$

$\Rightarrow \lambda_1 = 2\delta|s|\sqrt{c^2+\varepsilon^2 s^2}\, e^{-\gamma t} \qquad \lambda_2 = \lambda_3 = \lambda_4 = 0$

$\Rightarrow C = 2\delta|s|\sqrt{c^2+\varepsilon^2 s^2}\, e^{-\gamma t}$

k) $x = \dfrac{1+\sqrt{1-C^2}}{2} = \frac{1}{2}\left(1 + \sqrt{1 - 4\delta^2 s^2(c^2+\varepsilon^2 s^2)e^{-2\gamma t}}\right)$

$E_F = -x \ln x - (1-x)\ln(1-x)$

For $\gamma = 0$: $c^2 + \varepsilon^2 s^2 = c^2 + (1-\delta^2)s^2 = 1 - \delta^2 s^2$

$\sqrt{1 - 4\delta^2 s^2(1-\delta^2 s^2)} = 2\delta^2 s^2 - 1 \qquad \Rightarrow x = \delta^2 s^2$

$E_F = -\delta^2 s^2 \ln \delta^2 s^2 - (1-\delta^2 s^2)\ln(1-\delta^2 s^2)$

which agrees with what we found in 1d)

FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2020. Solution

Problem 1: Bloch sphere for three-level system

a) Hermitian matrices have real diagonal elements and the lower triangular elements are determined by the upper triangular ones, and are in general complex. This means that we need n^2 real parameters to specify a Hermitian $n \times n$ matrix. The traceless condition reduces this by one, so that the number of λ_i-matrices is $n^2 - 1$.

b) We use the fact that for pure states is $\mathrm{Tr}\,\rho^2 = 1$. We have

$$\mathrm{Tr}\,\rho^2 = \frac{1}{n^2}\,\mathrm{Tr}(\mathbb{1} + 2\alpha m_i \lambda_i + \alpha^2 m_i m_j \lambda_i \lambda_j) = \frac{1}{n^2}(n + 2\alpha^2 |\mathbf{m}|) = 1$$

If we set $|\mathbf{m}| = 1$, we get that

$$\alpha = \sqrt{\frac{n(n-1)}{2}}.$$

c) For n-level systems, the general pure state is $|\psi\rangle = \sum_{i=1}^{n} c_i |i\rangle$, which means that we have n complex coefficients, or $2n$ real coefficients. Normalization reduces the number by one, and the global phase by one, so we have that the space of pure states is $2(n-1)$ dimensional.

d) We have shown that with proper choice of α the pure states have $|\mathbf{m}| = 1$, so they are on the surface of the Bloch sphere. The surface of the Bloch sphere in a space of $n^2 - 1$ dimensions is $n^2 - 2$ dimensional. But the space of pure states is $2(n-1)$ dimensional, and for $n > 2$ is $n^2 - 2 > 2(n-1)$, so the pure states do not cover the whole surface. This means that there are many points on the surface of the Bloch sphere that does not represent any physical quantum state.

e)

$$\rho = \frac{1}{3}\left[\mathbb{1} + \sqrt{3}(m_1\lambda_1 + m_8\lambda_8)\right] = \begin{pmatrix} \frac{1}{3}(1+m_8) & \frac{1}{\sqrt{3}}m_1 & 0 \\ \frac{1}{\sqrt{3}}m_1 & \frac{1}{3}(1+m_8) & 0 \\ 0 & 0 & \frac{1}{3}(1-2m_8) \end{pmatrix}.$$

The eigenvalues are found from

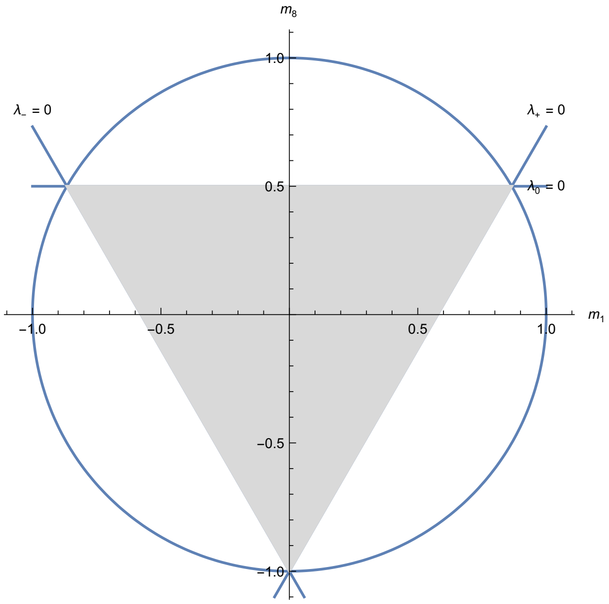
$$\begin{vmatrix} \frac{1}{3}(1+m_8) - \lambda & \frac{1}{\sqrt{3}}m_1 & 0 \\ \frac{1}{\sqrt{3}}m_1 & \frac{1}{3}(1+m_8) - \lambda & 0 \\ 0 & 0 & \frac{1}{3}(1-2m_8) - \lambda \end{vmatrix} = \left[\frac{1}{3}(1-2m_8) - \lambda\right]\left\{\left[\frac{1}{3}(1+m_8) - \lambda\right]^2 - \frac{1}{3}m_1^2\right\} = 0.$$

This gives three possible solutions

$$\lambda_0 = \frac{1}{3}(1 - 2m_8)$$

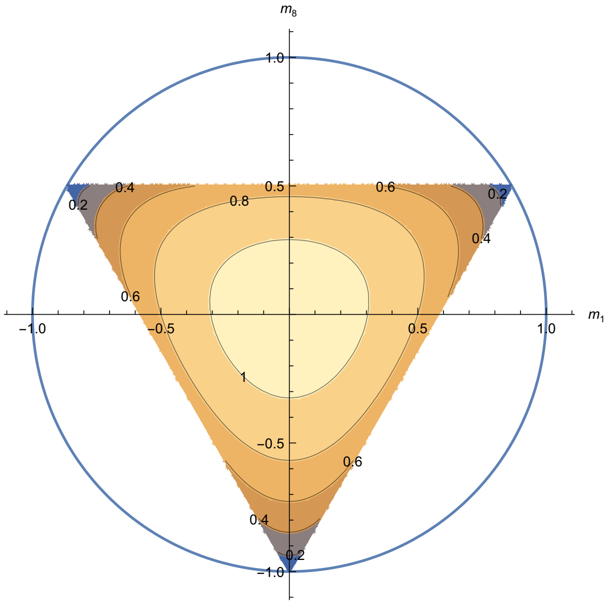
$$\lambda_\pm = \frac{1}{3}(1 + m_8) \pm \frac{1}{\sqrt{3}}m_1$$

f)



g) The entropy is given by

$$S = -\lambda_0 \log \lambda_0 - \lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-$$



We can see from the plot that the entropy depends on the direction as well as the length of the Bloch vector.

Problem 2: Entanglement transformations using local operations and classical communication

a) If both A and B are 2-level systems, the vectors α and β have both only two elements. We also know that for the state to be normalized we have $\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 1$. This means that there is only one nontrivial inequality to be considered in the majorization condition [Eq. (1) of the problem set], namely the one for $k = 1$. If $\alpha_1 \leq \beta_1$ we have that $\alpha \prec \beta$ and then $|\psi\rangle \to |\phi\rangle$. If $\beta_1 \leq \alpha_1$ we have that $\beta \prec \alpha$ and or $|\phi\rangle \to |\psi\rangle$. If $\alpha_1 = \beta_1$ both transformations are possible.

b) The state that is majorized by all other states must have the smallest possible α_1. Since the Schmidt coefficients are assumed to be in decreasing order, this means that $\alpha_1 = \frac{1}{2}$. Any state with this property is majorized by all other states, and consequently can be converted to all other states by LOCC. As an example, we can use one of our familiar Bell states

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

c) We apply the unitary transformation σ_z to system A. It has the action

$$\sigma_z|0\rangle = |0\rangle, \qquad \sigma_z|1\rangle = -|1\rangle$$

on the basis states. This gives exactly the specified transformation on the given total state for A and B. No measurements are required, and no classical information has to be transferred.

d) From the given matrix for U_θ we can read the action of the operator on the basis states

$$\begin{aligned}
U_\theta|00\rangle &= \cos\theta|00\rangle - \sin\theta|10\rangle \\
U_\theta|01\rangle &= \cos\theta|01\rangle + \sin\theta|10\rangle \\
U_\theta|10\rangle &= \sin\theta|00\rangle + \cos\theta|10\rangle \\
U_\theta|11\rangle &= -\sin\theta|01\rangle + \cos\theta|11\rangle
\end{aligned}$$

Then we get

$$\begin{aligned}
U_\theta|\psi\rangle_1 \otimes |\chi\rangle_2 = \frac{1}{\sqrt{2}} & [(\cos\phi\cos\theta + \sin\phi\sin\theta)|00\rangle + (\cos\phi\cos\theta - \sin\phi\sin\theta)|01\rangle \\
& +(-\cos\phi\sin\theta + \sin\phi\cos\theta)|10\rangle + (\cos\phi\sin\theta + \sin\phi\cos\theta)|11\rangle] \\
= & \frac{1}{\sqrt{2}}[\cos(\phi-\theta)|00\rangle + \cos(\phi+\theta)|01\rangle + \sin(\phi-\theta)|10\rangle + \sin(\phi+\theta)|11\rangle]
\end{aligned}$$

If we measure the second particle, the state of the first particle would be (if we normalize the states)

$$\begin{aligned}
&\text{Measurement outcome 0:} \quad |\psi\rangle_1 = \cos(\phi-\theta)|0\rangle + \sin(\phi-\theta)|1\rangle \\
&\text{Measurement outcome 1:} \quad |\psi\rangle_1 = \cos(\phi+\theta)|0\rangle + \sin(\phi+\theta)|1\rangle
\end{aligned}$$

e) An interaction Hamiltonian of the form $H = -\hbar\omega\sigma_y \otimes \sigma_z$ gives the time evolution $e^{-\frac{i}{\hbar}Ht} = U_{\omega t}$. This mans that the Bloch vector of the first particle will rotate around the y-axis with a direction depedent on the state of the second particle. Since the second particle is in a superposition of the two states, both rotations take place at the same time. When measuring the state of the second particle, the wavefunction collapses, and the corresponding rotation is the only one that is realized.

f) We write the qubits in the order from top to bottom (as indicated by the numbers on the left). Note that we have to be careful when applying the U_θ as it is stated that the lower line should correspond to the first qubit, which is opposite to what we write here. We know that

$$e^{i\frac{\pi}{2}\sigma_y} = \cos\frac{\pi}{2}\mathbb{1} + i\sin\frac{\pi}{2}\sigma_y = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

That is, it exchanges the $|0\rangle$ and $|1\rangle$ states with a change of sign in one case. We then get

$$\begin{aligned}
|000\rangle &\overset{H_1\otimes H_3}{\rightarrow} \frac{1}{2}(|0\rangle + |1\rangle)|0\rangle(|0\rangle + |1\rangle) \\
&\overset{CNOT}{\rightarrow} \frac{1}{2}(|00\rangle + |11\rangle)(|0\rangle + |1\rangle) \\
&\overset{e^{i\frac{\pi}{2}\sigma_y}}{\rightarrow} \frac{1}{2}(|01\rangle - |10\rangle)(|0\rangle + |1\rangle) \\
&\overset{U_\theta}{\rightarrow} \frac{1}{2}\left[|0\rangle\left(\cos\theta|10\rangle + \sin\theta|11\rangle - \sin\theta|10\rangle + \cos\theta|11\rangle\right) - |1\rangle\left(\cos\theta|00\rangle - \sin\theta|01\rangle + \sin\theta|00\rangle + \cos\theta|01\rangle\right))\right] \\
&= \frac{1}{2}\left[(\cos\theta - \sin\theta)|01\rangle - (\cos\theta + \sin\theta)|10\rangle\right]|0\rangle + \frac{1}{2}\left[(\cos\theta + \sin\theta)|01\rangle - (\cos\theta - \sin\theta)|10\rangle\right]|1\rangle
\end{aligned}$$

We now measure the third qubit and get the state of the first two qubits

$$\begin{aligned}
&\text{Measurement outcome 0:} \frac{1}{\sqrt{2}}\left[(\cos\theta - \sin\theta)|01\rangle - (\cos\theta + \sin\theta)|10\rangle\right] \\
&\text{Measurement outcome 1:} \frac{1}{\sqrt{2}}\left[(\cos\theta + \sin\theta)|01\rangle - (\cos\theta - \sin\theta)|10\rangle\right]
\end{aligned}$$

where we have normalized the states. We are told that $V_0 = W_0 = \mathbb{1}$ which means that if the measurement gives 0 we do nothing to any of the first two qubits. To get the same state also if the measurement gives 1, we have to swich the two terms, which we gan achieve by applying $V_1 = W_1 = \sigma_x$.

g) To determine the probabilities of the two outcomes, we need the reduced density matrix of qubit 3. We can rewrite the final state as

$$|\psi\rangle = \frac{1}{2}|01\rangle\left[(\cos\theta - \sin\theta)|0\rangle + (\cos\theta + \sin\theta)|1\rangle\right] - \frac{1}{2}|10\rangle\left[(\cos\theta + \sin\theta)|0\rangle + (\cos\theta - \sin\theta)|1\rangle\right]$$

Then we find that

$$
\begin{aligned}
\rho_3 &= Tr_{12}\rho = Tr_{12}|\psi\rangle\langle\psi| \\
&= \frac{1}{4}\left[(\cos\theta - \sin\theta)^2|0\rangle\langle 0| + (\cos^2\theta - \sin^2\theta)(|0\rangle\langle 1| + |1\rangle\langle 0|) + (\cos\theta + \sin\theta)^2|1\rangle\langle 1|\right] \\
&= \frac{1}{4}\left[(\cos\theta + \sin\theta)^2|0\rangle\langle 0| + (\cos^2\theta - \sin^2\theta)(|0\rangle\langle 1| + |1\rangle\langle 0|) + (\cos\theta - \sin\theta)^2|1\rangle\langle 1|\right] \\
&= \frac{1}{2}(|0\rangle\langle 0| + \cos 2\theta(|0\rangle\langle 1| + |1\rangle\langle 0|) + |1\rangle\langle 1|)
\end{aligned}
$$

The probabilities for the outcames are

$$
\begin{aligned}
P_0 &= \mathrm{Tr}(\rho_3|0\rangle\langle 0|) = \frac{1}{2} \\
P_1 &= \mathrm{Tr}(\rho_3|1\rangle\langle 1|) = \frac{1}{2}.
\end{aligned}
$$

h) The Hadamard gate on the first qubit prepares a superposition of the basis states. The CNOT entangles this with the second qubit. The $e^{i\frac{\pi}{2}\sigma_y}$ flips the second qubit. Together, these three gates prepares the initial state $\frac{1}{\sqrt{2}}(ket01 - |10\rangle)$ that is to be transformed. CNOT is the only gate that is nonlocal in qubits 1 and 2, and they would have to be close enough to intract at that point. Later they are separated, so that qubit 1 is with observer A and qubit 2 with observer B. To execute the transformation, we will measure qubit 2, but only non-projectively. This we do by entangling it with qubit 3 (which we consider to be close to qubit 2, with observer B) in the U_θ-gate and then measuring qubit 3. The outcome of this measurement is used to determine the action V_i on qubit 2 and is sent via classical communication to A to inform about which local unitary W_i should be applied.

i) A proof can be found in M. Nielsen and G. Vidal, Quantum Information and Computation, **1**, 76 (2001). All proofs that I have seen use, like that one, some more general theorem that requires some non-trivial mathematical tools. I have never seen a simple direct proof, but it probably can be found in the literature. The following argument is direct and should make it clear that the entropy can never increase using LOCC.

First, we know that any LOCC process leads to a state with a vector β of squared Schmidt coefficients that majorizes the vector α corresponding to the original state. We also know that the entanglement entropy is given in terms of the vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ by

$$
S(\alpha) = -\sum_i \alpha_i \ln \alpha_i.
$$

We need therefore to prove that if $\alpha \prec \beta$ then $S(\alpha) \geq S(\beta)$. If we consider the function $-x\ln x$ it has a derivative that is monotonously decreasing. This means that if we increase one of the α_i while decreasing a smaller α by the same amount (remember that $\sum_i \alpha_i = 1$), keeping the rest fixed, the entropy decreases. we need a way to change from α to β so that we always increase a larger α_i and decrease a smaller. Start by incresing α_1 and decreasing α_n until one of the partial sums $\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \beta_i$. If $k = 1$ or $k = n - 1$, we know that $\alpha_1 = \beta_1$ or $\alpha_n = \beta_n$, and we repeat the procedure for the remaining α_i. If the partial sums agree at some intermediate k, we split the vectors at that point, and repeat the procedure for each part independently. We can continue this until $\alpha_i = \beta_i$ for all i.

j) Both the states are already in Schmidt decomposed form, so we read directly that

$$\alpha_1 = \frac{1}{2}, \qquad \alpha_2 = \frac{2}{5}, \qquad \alpha_3 = \frac{1}{10}$$
$$\beta_1 = \frac{3}{5}, \qquad \beta_2 = \frac{1}{5}, \qquad \beta_3 = \frac{1}{5}$$

From this we find the sums

n	1	2	3
$\sum_{i=1}^{n} \alpha_i$	0.5	0.9	1
$\sum_{i=1}^{n} \beta_i$	0.6	0.8	1

From this we see that we do not have $\alpha \prec \beta$ or $\beta \prec \alpha$ and therefore neither $|\psi\rangle \to |\phi\rangle$ nor $|\phi\rangle \to |\psi\rangle$.

k) One problem with classifying different types of entanglement by whether they are convertible using LOCC or not is the fact that states that are very close to each other may be classified as having completely different type of entanglement. One example is given in Martin B. Plenio and S. Virmani, An introduction to entanglement measures, Quant.Inf.Comput. **7**, 1 (2007). The initial state $(|00\rangle + |11\rangle)/sqrt2$ can be transformed by LOCC to $0.8|00\rangle + 0.6|11\rangle$ but not to $(0.8|00\rangle + 0.6|11\rangle + \epsilon|22\rangle)/\sqrt{1+\epsilon^2}$ even if the two final states are arbitrary close for small ϵ. Classification of states according to LOCC transformation does not capture the fact that these states are close. Different modifications have been proposed, where one studies the number of states of one type are needed to get one state of another type, or allows the process to succeed only with a certain probability, see the paper cited above or R. Horodecki et al., Rev. Mod. Phys. **81**, 865 (2009).

l) Since the states already are in Schmidt form, we read directly the vectors α (corresponding to $|\psi_1\rangle$) and β (corresponding to $|\psi_2\rangle$).

$$\alpha_1 = 0.4, \qquad \alpha_2 = 0.4, \qquad \alpha_3 = 0.1, \qquad \alpha_4 = 0.1$$
$$\beta_1 = 0.5, \qquad \beta_2 = 0.25, \qquad \beta_3 = 0.25, \qquad \beta_4 = 0$$

From this we find the sums

n	1	2	3	4
$\sum_{i=1}^{n} \alpha_i$	0.4	0.8	0.9	1
$\sum_{i=1}^{n} \beta_i$	0.5	0.75	1	1

From this we see that we do not have $\alpha \prec \beta$ or $\beta \prec \alpha$ and therefore neither $|\psi_1\rangle \to |\psi_2\rangle$ nor $|\psi_2\rangle \to |\psi_1\rangle$.

m) We have in total 4 systems, two at A and two at B. A basis for the states of the two systems at A is

$$|ij\rangle_A = |i\rangle_A \otimes |j\rangle_A$$

where $i = 1 \ldots 4$ and $j = 5, 6$. The systems at B has a similar basis, and we can then write

$$\begin{aligned} |\psi_1\rangle|\phi\rangle = {} & \sqrt{0.24}|15\rangle_A \otimes |15\rangle_B + \sqrt{0.24}|25\rangle_A \otimes |25\rangle_B + \sqrt{0.06}|35\rangle_A \otimes |35\rangle_B + \sqrt{0.06}|45\rangle_A \otimes |45\rangle_B \\ & + \sqrt{0.16}|16\rangle_A \otimes |16\rangle_B + \sqrt{0.16}|26\rangle_A \otimes |26\rangle_B + \sqrt{0.04}|36\rangle_A \otimes |36\rangle_B + \sqrt{0.04}|46\rangle_A \otimes |46\rangle_B . \end{aligned}$$

This is in Schmidt form, and sorting we get the coefficients

$$\begin{aligned}
&\alpha_1 = 0.24, \qquad \alpha_2 = 0.24, \qquad \alpha_3 = 0.16, \qquad \alpha_4 = 0.16,\\
&\alpha_5 = 0.06, \qquad \alpha_6 = 0.06, \qquad \alpha_7 = 0.04, \qquad \alpha_8 = 0.04.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
|\psi_2\rangle|\phi\rangle = {} & \sqrt{0.3}|15\rangle_A \otimes |15\rangle_B + \sqrt{0.15}|25\rangle_A \otimes |25\rangle_B + \sqrt{0.15}|35\rangle_A \otimes |35\rangle_B \\
& + \sqrt{0.2}|16\rangle_A \otimes |16\rangle_B + \sqrt{0.1}|26\rangle_A \otimes |26\rangle_B + \sqrt{0.1}|36\rangle_A \otimes |36\rangle_B
\end{aligned}$$

$$\begin{aligned}
&\beta_1 = 0.3, \qquad \beta_2 = 0.2, \qquad \beta_3 = 0.15, \qquad \beta_4 = 0.15,\\
&\beta_5 = 0.1, \qquad \beta_6 = 0.1, \qquad \beta_7 = 0, \qquad \beta_8 = 0.
\end{aligned}$$

From this we find the sums

n	1	2	3	4	5	6	7	8
$\sum_{i=1}^n \alpha_i$	0.24	0.48	0.64	0.8	0.86	0.92	0.96	1
$\sum_{i=1}^n \beta_i$	0.3	0.5	0.65	0.8	0.9	1	1	1

We see that $\alpha \prec \beta$ which means that $|\psi_1\rangle|\phi\rangle \to |\psi_2\rangle|\phi\rangle$.

FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2021. Solution

Problem 1: Superradiance

a) From the lecture notes we have

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k}a} \sqrt{\frac{\hbar}{2V\omega_0\epsilon_0}} \left[\hat{a}_{\mathbf{k}a} e^{i\mathbf{k}\mathbf{r}} + \hat{a}^\dagger_{\mathbf{k}a} e^{-i\mathbf{k}\mathbf{r}}\right] \epsilon_{\mathbf{k}a}.$$

Restricted to the $\{|0\rangle, |1\rangle\}$ subspace we can write

$$\mathbf{p} = \langle 0|\mathbf{p}|1\rangle |0\rangle\langle 1| + \langle 1|\mathbf{p}|0\rangle |1\rangle\langle 0| = \langle 0|\mathbf{p}|1\rangle \sigma^- + \langle 1|\mathbf{p}|0\rangle \sigma^+$$

When calculating transition rates, there will appear a δ-function ensuring energy conservation. This means that terms of the form $\hat{a}\sigma^-$ or $\hat{a}^\dagger\sigma^+$ never will contribute. We choose the position of the atom to be $\mathbf{r} = 0$ and in the dipole approximation it means that $e^{-i\mathbf{k}\mathbf{r}} \approx 1$ and we get

$$H_{int} = -\frac{e}{m} \sum_{\mathbf{k}a} \sqrt{\frac{\hbar}{2V\omega_0\epsilon_0}} \left[\hat{a}_{\mathbf{k}a}\sigma^+ + \hat{a}^\dagger_{\mathbf{k}a}\sigma^-\right] \langle 0|\mathbf{p}|1\rangle \cdot \epsilon_{\mathbf{k}a} = \sum_{\mathbf{k}a} g_{\mathbf{k}a} (\hat{a}_{\mathbf{k}a}\sigma^+ + \hat{a}^\dagger_{\mathbf{k}a}\sigma^-)$$

with

$$g_{\mathbf{k}a} = -\frac{e}{m}\sqrt{\frac{\hbar}{2V\omega_0\epsilon_0}} \langle 0|\mathbf{p}|1\rangle \cdot \epsilon_{\mathbf{k}a}$$

The relative phase of $|0\rangle$ and $|1\rangle$ can always be choosen so that $\langle 1|\mathbf{p}|0\rangle = \langle 0|\mathbf{p}|1\rangle$ is real.

b) The rate of spontaneous emission is

$$w_1 = \sum_{\mathbf{k}a} \frac{2\pi}{\hbar} |\langle 0, 1_{\mathbf{k}a}|H_{int}|1, 0\rangle|^2 \delta(E_0 + \hbar\omega_k - E_1)$$

where E_0 and E_1 are the energies of $|0\rangle$ and $|1\rangle$ and $|1, 0\rangle$ refers to the atom in state $|1\rangle$ and field in vacuum state. As in the lecture notes, eq (4.101) we get

$$w_1 = \frac{e^2\omega}{3\pi c^3\hbar m^2\epsilon_0} |\langle 0|\mathbf{p}|1\rangle|^2$$

where $\hbar\omega = E_1 - E_0$. To compare with (4.101) recall (4.80): $\langle 0|\mathbf{p}|1\rangle = im\omega\langle 0|\mathbf{r}|1\rangle$.

c) As indicated in the problem, we write $|10\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle + |\psi^-\rangle)$ with $|\psi^\pm\rangle\frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle)$. The state $|\psi^-\rangle$ is an eigenstate of the Hamiltonian (both the Hamiltonian of the atom, and the interaction) and this part of the initial state will not decay. The remaining $|\psi^+\rangle$ has a nonzero matrix element $\langle 00|D^-|\psi^+\rangle$ and will decay to the ground state $|00\rangle$.

d) There is a probability $\frac{1}{2}$ to be in the state $|\psi^-\rangle$ and therefore not decay. Otherwise, one photon is emitted. On average, $\frac{1}{2}$ photon is emitted for each repetition of the experiment.

e) We have

$$(D^-)^{J-M}|11\cdots 1\rangle \sim |\underbrace{00\cdots 0}_{J-M}\underbrace{11\cdots 1}_{J+M}\rangle + \text{All permutations with J+M atoms in } |1\rangle \text{ and J-M atoms in } |0\rangle$$

Therefore $\langle JM|JM'\rangle = 0$ if $M \neq M'$ since the number of excited atoms are different. To check normalization we note that there are $\binom{N}{J-M} = \frac{N!}{(J-M)!(J+M)!}$ different terms in $(D^-)^{J-M}|11\cdots 1\rangle$. But the operator generates each term seveal times. For a given set of $J-M$ atoms to be de-excited, the order in which they are de-excited does not matter, which means that

$$|JM\rangle = A(D^-)^{J-M}|11\cdots 1\rangle = A(J-M)!(|\underbrace{00\cdots 0}_{J-M}\underbrace{11\cdots 1}_{J+M}\rangle + \text{permutations})$$

where A is the normalization to be determined. We then have

$$\langle JM|JM\rangle = |A|^2[(J-M)!]^2(\langle 00\cdots 011\cdots 1| + \text{permutations})(|00\cdots 011\cdots 1\rangle + \text{permutations}).$$

Each permutation has inner product 1 with itself and 0 with all other permutations, so

$$\langle JM|JM\rangle = |A|^2[(J-M)!]^2\frac{N!}{(J-M)!(J+M)!}.$$

Requiring $\langle JM|JM\rangle = 1$ gives

$$A = \sqrt{\frac{(J+M)!}{N!(J-M)!}}.$$

f) The decay rate from the state $|JM\rangle$ is

$$w_{JM} = \sum_{\mathbf{k}a}\frac{2\pi}{\hbar}|\langle J, M-1, 1_{\mathbf{k}a}|H_{int}|JM, 0\rangle|^2\delta(E_{J,M-1} + \hbar\omega_k - E_{JM}).$$

The difference from the one atom case is that $\langle 0|\sigma^-|1\rangle$ is replaced by

$$\langle J, M-1|D^-|JM\rangle = \sqrt{\frac{(J+M)!}{N!(J-M)!}}\langle J, M-1|(D^-)^{J-M+1}|11\cdots 1\rangle = \sqrt{(J+M)(J-M+1)}$$

where we used that

$$(D^-)^{J-M+1}|11\cdots 1\rangle = \sqrt{\frac{N!(J-M+1)!}{(J+M-1)!}}|J, M-1\rangle.$$

This gives

$$w_{JM} = (J+M)(J-M+1)w_1.$$

g) The decy rate is maximal for $M = 0$ and $M = 1$.

$$w_{J0} = W_{J1} = J(J+1)w_1 = \frac{N}{2}(\frac{N}{2}+1)w_1 \approx \frac{N^2}{4}w_1.$$

One atom emits a photon at the rate w_1, so N independent atoms will emit at the rate $N\omega_1$. For $N \gg 1$ we see that $w_{J0} \gg Nw_1$ so the emission rate is much larger than for N independent atoms.

h)

$$|\langle J, M-1|D^-|JM\rangle|^2 = \langle JM|D^+|J, M-1\rangle\langle J, M-1|D^-|JM\rangle = \langle JM|D^+ \sum_{M'} |JM'\rangle\langle JM'|D^-|JM\rangle$$

since $\langle JM'|D^-|JM\rangle = 0$ for all $M' \neq M-1$. Since the states $|JM\rangle$ constitute a complete set, the sum of projectors is the identity and we get

$$|\langle J, M-1|D^-|JM\rangle|^2 = \langle JM|D^+D^-|JM\rangle.$$

i) If $|a_1 \cdots a_N\rangle$ with $a_k = 0$ or 1 is some state, we have

$$\sigma_i^+\sigma_i^-|a_1 \cdots a_N\rangle = a_i|a_1 \cdots a_N\rangle.$$

This means that if $a_k = 0$ for $J - M$ atoms and $a_k = 1$ for $J + M$ atoms

$$\sum_i \sigma_i^+\sigma_i^-|a_1 \cdots a_N\rangle = (J+M)|a_1 \cdots a_N\rangle.$$

This applies to all permutations and depends only on the number of excited atoms, so $\sum_i \sigma_i^+\sigma_i^-|JM\rangle = (J+M)|JM\rangle$, which menas that

$$\langle JM|\sum_i \sigma_i^+\sigma_i^-|JM\rangle = J + M.$$

j) We have

$$\langle JM|D^+D^-|JM\rangle = \langle JM|\sum_{ij}\sigma_i^+\sigma_j^-|JM\rangle = \langle JM|\sum_i \sigma_i^+\sigma_i^-|JM\rangle + \langle JM|\sum_{i\neq j}\sigma_i^+\sigma_j^-|JM\rangle.$$

Due to the permutation symmetry of the state, the last sum consists of $N(N-1)$ identical terms. From f) and h) we have that

$$\langle JM|D^+D^-|JM\rangle = |\langle J, M-1|D^-|JM\rangle|^2 = (J+M)(J-M+1)$$

which gives

$$\langle JM|\sigma_i^+\sigma_j^-|JM\rangle = \frac{J^2 - M^2}{N(N-1)}.$$

k) We have

$$\sigma_i^+\sigma_j^- = \frac{1}{4}(\sigma_x^i + i\sigma_y^i)(\sigma_x^j - i\sigma_y^j) = \frac{1}{4}(\sigma_x^i\sigma_x^j + \sigma_y^i\sigma_y^j - i\sigma_x^i\sigma_y^j + i\sigma_y^i\sigma_x^j).$$

From the permutation symmetry of $|JM\rangle$ we get

$$\langle JM|\sigma_x^i\sigma_y^j|JM\rangle = \langle JM|\sigma_y^i\sigma_x^j|JM\rangle.$$

There is also symmetry with respect to x and y, so

$$\langle JM|\sigma_x^i\sigma_x^j|JM\rangle = \langle JM|\sigma_y^i\sigma_y^j|JM\rangle$$

which means that

$$\langle JM|\sigma_x^i\sigma_x^j|JM\rangle = 2\langle JM|\sigma_i^+\sigma_j^-|JM\rangle = 2\frac{J^2 - M^2}{N(N-1)}.$$

We denote the probability that the measurements of σ_x^i and σ_y^j gives the same result as P_+ and the probability to get opposite results as $P_- = 1 - P_+$. Then $\langle JM|\sigma_x^i\sigma_x^j|JM\rangle = P_+ - P_- = 2P_+ - 1$ which gives that

$$P_+ = \frac{1}{2} + \frac{J^2 - M^2}{N(N-1)}.$$

For $N = 2$ and $M = 0$ we get $P_+ = 1$. For large N and $M = 0$ we get $P_+ \approx \frac{3}{4}$.

l) We have $N = 4$, $J = 2$, $M = -2, -1, 0, 1, 2$.

$$\begin{aligned}
&M = 2 |22\rangle = |1111\rangle \\
&\qquad \rho_1 = |1\rangle\langle 1| \\
&\qquad S = 0 \text{ (no entanglement)} \\
&M = 1 |21\rangle = \frac{1}{2}(|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle) \\
&\qquad \rho_1 = \frac{1}{4}(|0\rangle\langle 0| + 3|1\rangle\langle 1|) \\
&\qquad S = -\frac{1}{4}\ln\frac{1}{4} - \frac{3}{4}\ln\frac{3}{4} \\
&M = 0 |20\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle) \\
&\qquad \rho_1 = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \\
&\qquad S = -2\frac{1}{2}\ln\frac{1}{2} = \ln 2.
\end{aligned}$$

Negative M gives the same with 0 and 1 interchanged.

m)

$$\rho_{12} = \frac{1}{6}(|00\rangle\langle 00| + 2|01\rangle\langle 01| + 2|10\rangle\langle 10| + |11\rangle\langle 11| + 2|01\rangle\langle 10| + 2|10\rangle\langle 01|) = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where we use the matrix representation $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Two eigenvalues are $p_1 = p_4 = 1/6$. We find the other two eigenvalues

$$\begin{vmatrix} \frac{1}{3} - p & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} - p \end{vmatrix} = p^2 - \frac{2}{3}p + \frac{1}{12} = 0,$$

which gives

$$p_2 = \frac{2}{3} \qquad p_3 = 0.$$

The entropy is

$$S = -\sum_n p_n \ln p_n = \frac{1}{3}\ln 6 - \frac{2}{3}\ln\frac{2}{3} = \ln 3 - \frac{1}{3}\ln 2.$$

n) We have

$$D^- = \sigma^- \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and

$$H = -\frac{\omega_0}{2}(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) = -\omega_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We parametrize the density matrix

$$\rho = \begin{pmatrix} p & a & b & c \\ a^* & q & d & e \\ b^* & d^* & r & f \\ c^* & e^* & f^* & s \end{pmatrix}. \tag{1}$$

with $p, q, r, s \in \mathbb{R}$ and $p + q + r + s = 1$. Using the Lindblad equation we find (after some calculations)

$$\frac{d\rho}{dt} = -i\omega_0 \begin{pmatrix} 0 & -a & -b & -2c \\ a^* & 0 & 0 & -e \\ b^* & 0 & 0 & -f \\ 2c^* & e^* & f^* & 0 \end{pmatrix}$$

$$- \frac{\gamma}{2} \begin{pmatrix} 4p & 3a+b & a+3b & 2c \\ 3a^*+b^* & 2q+d+d^*-2p & q+r+2d-2p & e+f-2a-2b \\ a^*+3b^* & q+r+2d^*-2p & 2r+d+d^*-2p & e+f-2a-2b \\ 2c^* & e+f^*-2a^*-2b^* & e+f^*-2a^*-2b^* & -2(q+r+d+d^*) \end{pmatrix}.$$

A stationary state is a state with $\frac{d\rho}{dt} = 0$, which means that all matrix elements of $\frac{d\rho}{dt}$ are 0. The 11 element gives that $p = 0$. The 23 and 32 elements give that $d = d^*$ and then the 22 ad 23 elements give that $q = r = -d$. The condition $p + q + r + s = 1$ then implies $q = \frac{1}{2}(1 - s)$. The 12 and 13 elements together imply that $a = b = 0$ and if we know that, the elements 42 and 43 give $e = f = 0$. The 14 element gives $c = 0$. The only remaining free parameter is s, and the density matrix has the form

$$\rho = s|00\rangle\langle 00| + (1-s)|\psi^-\rangle\langle\psi^-|.$$

o) IF the initial state is $|10\rangle$, the initial density matrix has $q = 1$ and all other elements are =0. From the expression for $\frac{d\rho}{dt}$ we see that only the elements q, r, s and d will ever be nonzero. They satisfy the equations

$$\begin{aligned} \dot{q} &= -\gamma(q+d) \\ \dot{r} &= -\gamma(r+d) \\ \dot{d} &= -\frac{\gamma}{2}(q+r+2d) \\ \dot{s} &= \gamma(q+r+2d) \end{aligned}$$

Summing the first two equations and subtracting twice the third we get

$$\frac{d}{dt}(q+r-2d) = 0$$

which implies that $q + r - 2d = 1$ since this is the value at $t = 0$. In the final stationary state we have $q + r + 2d = 0$, so we have $d = -\frac{1}{4}$. Then $q + r = \frac{1}{2}$ and $s = \frac{1}{2}$. The final stationary state is then

$$\rho = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|\psi^-\rangle\langle\psi^-|$$

in accordance with what we found in c).

p) With independent environments for each atom (e.g. distiguishable photon modes) we have one Lindblad operator for each process (atom 1 emits and atom 2 emits).

$$\frac{d\rho}{dt} = -i[H,\rho] - \frac{\gamma_1}{2}(\sigma_1^+\sigma_1^-\rho + \rho\sigma_1^+\sigma_1^- - 2\sigma_1^-\rho\sigma_1^+) - \frac{\gamma_2}{2}(\sigma_2^+\sigma_2^-\rho + \rho\sigma_2^+\sigma_2^- - 2\sigma_2^-\rho\sigma_2^+)$$

where $\sigma_1^\pm = \sigma^\pm \otimes \mathbb{1}$ and $\sigma_2^\pm = \mathbb{1} \otimes \sigma^\pm$. With the density matrix as in Eq. (1) we get

$$\begin{aligned}\frac{d\rho}{dt} = -i\omega_0 &\begin{pmatrix} 0 & -a & -b & -2c \\ a^* & 0 & 0 & -e \\ b^* & 0 & 0 & -f \\ 2c^* & e^* & f^* & 0 \end{pmatrix} \\ &- \frac{\gamma_1}{2}\begin{pmatrix} 2p & 2a & b & c \\ 2a^* & 2q & d & e \\ b^* & d^* & -2p & -2a \\ c^* & e^* & -2a^* & -2q \end{pmatrix} - \frac{\gamma_2}{2}\begin{pmatrix} 2p & a & 2b & c \\ a^* & -2p & d & -2b \\ 2b^* & d^* & 2r & f \\ c^* & -2b^* & f^* & -2r \end{pmatrix}.\end{aligned}$$

In a stationary state we have $\frac{d\rho}{dt} = 0$ which gives $p = a = b = c = d = e = f = r = q = 0$ and $s = 1$, so the only stationary state is $|00\rangle\langle 00|$ which means that any initial state will decay to the ground state.

FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2022. Solution

Problem 1: Supersymmetric quantum mechanics

a) We have

$$A^\dagger A = \left(-\frac{ip}{\sqrt{2m}} + W(x)\right)\left(\frac{ip}{\sqrt{2m}} + W(x)\right) = \frac{p^2}{2m} - \frac{i}{\sqrt{2m}}[p, W(x)] + W^2.$$

Using (remember that the derivative should act on all functions to the right)

$$\frac{\partial}{\partial x}W = \frac{\partial W}{\partial x} + W\frac{\partial}{\partial x}$$

we get

$$[p, W(x)] = -i\frac{\partial}{\partial x}W + iW\frac{\partial}{\partial x} = -i\frac{\partial W}{\partial x} -$$

Thus we have

$$W^2 - \frac{1}{\sqrt{2m}}\frac{dW}{dx} = V_-.$$

b) This is just multiplying matrices.

c) For a system of two particles, the total Hilbert space would be the tensor product of the individual Hilbert spaces. In this case, the Hamiltonian is the direct sum of the individual Hamiltonians, and corresponds to a single particle that is confined in one of the two potentials with no amplitude for tunneling between them.

d) The ground state energy is

$$E_0 = \langle\Psi_0|H|\Psi_0\rangle = \langle\Psi_0|\left\{Q, Q^\dagger\right\}|\Psi_0\rangle = \langle\Psi_0|QQ^\dagger|\Psi_0\rangle + \langle\Psi_0|Q^\dagger Q|\Psi_0\rangle = |Q^\dagger|\Psi_0\rangle|^2 + |Q|\Psi_0\rangle|^2 \geq 0$$

e)

$$\begin{aligned} AH_- &= AA^\dagger A = H_+ A, \\ A^\dagger H_+ &= A^\dagger A A^\dagger = H_- A^\dagger. \end{aligned}$$

f)

$$H_+ A|\psi_n^-\rangle = AH_-|\psi_n^-\rangle = E_n^- A|\psi_n^-\rangle.$$

g) For unbroken SUSY we have

$$H|\Psi_0\rangle = H_-|\psi_0^-\rangle = 0.$$

This implies that

$$\langle\psi_0^-|H_-|\psi_0^-\rangle = \langle\psi_0^-|A^\dagger A|\psi_0^-\rangle = |A|\psi_0^-\rangle|^2 = 0$$

which means that $A|\psi_0^-\rangle = 0$.

h) We order the eigenstates $|\psi_0^\pm\rangle$ according to increasing energy, and we know that the ground state $|\psi_0^-\rangle$ of H_- does not have a corresponding eigenstate of H_+ while all the other states do. So $E_{n-1}^+ = E_n^-$ and

$$|\psi_n^+\rangle = NA|\psi_{n+1}^-\rangle$$

with some normalization N. To determine this we calculate

$$1 = \langle\psi_n^+|\psi_n^+\rangle = N^2\langle\psi_{n+1}^-|A^\dagger A|\psi_{n+1}^-\rangle = N^2E_{n+1}^-$$

which gives $N^2 = 1/E_{n+1}^- = 1/E_n^+$ and we get

$$|\psi_n^+\rangle = \frac{A}{\sqrt{E_n^+}}|\psi_{n+1}^-\rangle. \tag{1}$$

i) We know that $A|\psi_0^-\rangle = 0$, which in the position basis takes the form

$$\left[\frac{1}{\sqrt{2m}}\frac{\partial}{\partial x} + W\right]\psi_0^-(x) = 0.$$

Solving this differential equation gives

$$\psi_0^-(x) = Ne^{-\sqrt{2m}\int_0^x W(x')dx'}.$$

j)

$$V_\pm = W^2 \pm \frac{1}{\sqrt{2m}}\frac{\partial W}{\partial x} = b^2\frac{\cos^2 x}{\sin^2 x} \pm \frac{b}{\sqrt{2m}}\frac{1}{\sin^2 x} = -b^2 + b\left(b \pm \frac{1}{\sqrt{2m}}\right)\frac{1}{\sin^2 x}.$$

If we choose $b = \frac{1}{\sqrt{2m}}$ we get

$$\begin{aligned} V_- &= -\frac{1}{2m} \\ V_+ &= -\frac{1}{2m} + \frac{1}{m\sin^2 x} \end{aligned}$$

on the interval $0 \leq x \leq \pi$ with both potentials being ∞ outside this interval.

k) Normally the potential is 0 at the bottom of the well and the eigenstates are written as $\sqrt{2/\pi}\sin nx$ with $n = 1, 2, \ldots$ with the eigenvalues $n^2/2m$. Since we start numbering from $n = 0$ we write

$$\psi_n^-(x) = \sqrt{\frac{2}{\pi}}\sin(n+1)x$$

with

$$E_n^- = \frac{(n+1)^2 - 1}{2m}.$$

where we have subtracted the $1/2m$ energy at the bottom of the potential.

l) To simplify the expresions, we define $n' = n + 1$. We have

$$A\psi_{n'}^{-}(x) = \frac{1}{\sqrt{2m}}\left[\frac{\partial}{\partial x} - \frac{1}{\tan x}\right]\psi_{n'}^{-}(x) = \frac{1}{\sqrt{\pi m}}\left[n'\cos n'x - \frac{\sin n'x}{\tan x}\right].$$

With

$$E_n^{+} = E_{n'}^{-} = \frac{(n'+1)^2 - 1}{2m}$$

we get

$$\psi_n^{+}(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{(n'+1)^2 - 1}}\left[n'\cos n'x - \frac{\sin n'x}{\tan x}\right].$$

m) Since

$$H_{+}(a_0, x) - H_{-}(a_1, x) = g(a_1) - g(a_0)$$

is a number, the two Hamiltonians will have the same eigenfuntions and the difference between the eigenvalues is the same number so that

$$E_n^{+}(a_0) = E_n^{-}(a_1) + g(a_1) - g(a_0).$$

n) If SUSY is unbroken for all n we have $E_0^{-}(a_n) = 0$. From shape invariance(SI) we have

$$E_0^{+}(a_0) = E_0^{-}(a_1) + g(a_1) - g(a_0) = g(a_1) - g(a_0).$$

From SUSY we know that

$$E_1^{-}(a_0) = E_0^{+}(a_0) = g(a_1) - g(a_0).$$

We can repeat the same process

$$\begin{aligned} E_2^{-}(a_0) &\overset{SUSY}{=} E_1^{+}(a_0) \overset{SI}{=} E_1^{-}(a_1) + g(a_1) - g(a_0) \\ &\overset{SUSY}{=} E_0^{+}(a_1) + g(a_1) - g(a_0) \overset{SI}{=} E_0^{-}(a_2) + g(a_2) - g(a_1) + g(a_1) - g(a_0) = g(a_2) - g(a_0). \end{aligned}$$

The same continues for higher levels so that

$$E_n^{-}(a_0) = g(a_n) - g(a_0).$$

o) From Eq. (1) we get

$$A^{\dagger}|\psi_n^{+}\rangle = \frac{A^{\dagger}A}{\sqrt{E_n^{+}}}|\psi_{n+1}^{-}\rangle = \frac{H_{-}}{\sqrt{E_n^{+}}}|\psi_{n+1}^{-}\rangle = \frac{E_{n+1}^{-}}{\sqrt{E_n^{+}}}|\psi_{n+1}^{-}\rangle = \sqrt{E_n^{+}}|\psi_{n+1}^{-}\rangle$$

which means

$$|\psi_n^{-}(a_0)\rangle = \frac{A^{\dagger}(a_0)}{\sqrt{E_{n-1}^{+}(a_0)}}|\psi_{n-1}^{+}(a_0)\rangle.$$

We then have

$$|\psi_1^-(a_0)\rangle = \frac{A^\dagger(a_0)}{\sqrt{E_0^+(a_0)}}|\psi_0^+(a_0)\rangle \overset{SI}{=} \frac{A^\dagger(a_0)}{\sqrt{E_0^+(a_0)}}|\psi_0^-(a_1)\rangle$$

$$\begin{aligned}
|\psi_2^-(a_0)\rangle &= \frac{A^\dagger(a_0)}{\sqrt{E_1^+(a_0)}}|\psi_1^+(a_0)\rangle \overset{SI}{=} \frac{A^\dagger(a_0)}{\sqrt{E_1^+(a_0)}}|\psi_1^-(a_1)\rangle \\
&= \frac{A^\dagger(a_0)}{\sqrt{E_1^+(a_0)}}\frac{A^\dagger(a_1)}{\sqrt{E_0^+(a_1)}}|\psi_0^+(a_1)\rangle \overset{SI}{=} \frac{A^\dagger(a_0)}{\sqrt{E_1^+(a_0)}}\frac{A^\dagger(a_1)}{\sqrt{E_0^+(a_1)}}|\psi_0^-(a_2)\rangle.
\end{aligned}$$

Repeating this procedure we get

$$|\psi_n^-(a_0)\rangle = \frac{A^\dagger(a_0)}{\sqrt{E_{n-1}^+(a_0)}}\cdots\frac{A^\dagger(a_{n-2})}{\sqrt{E_1^+(a_{n-2})}}\frac{A^\dagger(a_{n-1})}{\sqrt{E_0^+(a_{n-1})}}|\psi_0^-(a_n)\rangle$$

p) With $\sqrt{2m} = 1$ we have

$$V_\pm(b,x) = -b^2 + b(b\pm 1)\frac{1}{\sin^2 x}.$$

This means that

$$V_+(b,x) = -b^2+b(b+1)\frac{1}{\sin^2 x} = -b^2+(b+1)^2-(b+1)^2+(b+1)(b+1-1)\frac{1}{\sin^2 x} = V_(b+1,x)+(b+1)^2-b^2.$$

If we choose the funtions

$$f(b) = b+1 \qquad g(b) = b^2$$

we satisfy the conditions for shape invariance.

q) Choosing the value $b = 1$ corresponds to the infinite square well for $V_-(1,x)$. We choose $a_0 = 1$ and get $a_n = a_{n-1} + 1 = n + 1$. This means that $g(a_n) = (n+1)^2$. Using (**??**) this gives

$$E_n^-(1) = (n+1)^2 - 1$$

which are the energy eigenvalues if $\sqrt{2m} = 1$. The wavefunctions can b edetermined since we know that

$$\psi_0^-(a_n,x) = Ne^{-\int_0^x W(a_n,x')dx'}$$

We need the integral

$$-\int W(a_n,x')dx' = a_n\int\frac{dx}{\tan x} = a_n \ln|\sin x| + C$$

where C is the integration constant. This gives that up to normalization we have

$$\psi_0^-(1,x) = N\sin x.$$

We can also find

$$\psi_1^-(1,x) = NA^\dagger(a_0)e^{-\int_0^x W(a_1,x')dx'} = NA^\dagger(1)e^{-\int_0^x W(2,x')dx'}\left(\frac{\partial}{\partial x} - \frac{1}{\tan x}\right)e^{2\ln|\sin x|} = N\sin 2x.$$

FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2023. Solution

Problem 1: Quantum error correction

a) If there is one bit flip we have that 000 is received as 100, 010, or 001. By majority vote we will correctly change this to 000, correcting the error. If two bit flips would happen, 000 could be received as 110, which we would erroneously correct to 111. The probability for this to happen is p^2. If p
1 we ahve that $p^2 \ll p$, so the probability that we get the correct result after correction is much larger that without error correction.

b) Cloning the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ would give the state $|\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \neq \alpha|000\rangle + \beta|111\rangle$. That the given encoding operation is in fact unitary is proven by the fact that it is the result of the circuit in the next question.

c) If the initial state $|\psi\rangle_1 = \alpha|0\rangle_1 + \beta|1\rangle_1$ is initially stored in the first qubit we have that the action of the circuit is

$$\begin{aligned} |\psi\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 &= \alpha|000\rangle + \beta|100\rangle \\ &\overset{CNOT_{12}}{\rightarrow} \alpha|000\rangle + \beta|110\rangle \\ &\overset{CNOT_{23}}{\rightarrow} \alpha|000\rangle + \beta|111\rangle. \end{aligned}$$

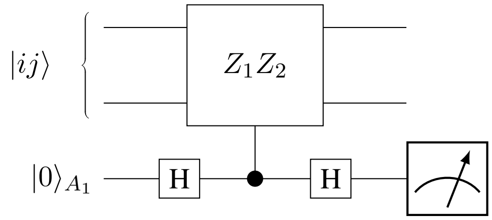
d) We get the following table of eigenvalues of Z_1Z_2 and Z_2Z_3 for the possible cases

Initial state	Error	State	Z_1Z_2	Z_2Z_3
$\|000\rangle$	X_1	$\|100\rangle$	-1	1
	X_2	$\|010\rangle$	-1	-1
	X_3	$\|001\rangle$	1	-1
$\|111\rangle$	X_1	$\|011\rangle$	-1	1
	X_2	$\|101\rangle$	-1	-1
	X_3	$\|110\rangle$	1	-1

As we see, all states are eigenstates of both the stabilizers. The measurement of both stabilizers will then give the corresponding eigenvalues as results. We see that these two eigenvalues uniquely determine which error has happened, independent of the initial state. This also means that the same will be true if we start from a superposition of the two logical states. To correct the error, we then have to apply the corresponding inverse operation (which is the same as the original error since $X_i^2 = I$, the identity). For example, if the measurement of Z_1Z_2 gives 1 and Z_2Z_3 gives 1 we know that the error was X_1 and correct it by applying the operation X_1 to the first qubit. Note that the measurement of the stabilizers only gives information about the error, and no information at all about the initial state. Otherwise we would affect the information stored in the initial state.

e) Measuring the operators Z_1 and Z_2 separately will tell us the state of each qubit individually. This would also reveal information about the initial state and collapse the wavefunction in a way that perturbs the information. Measuring Z_1Z_2 will only tell if the two qubits are in the same state (when the result is 1) or opposite states (when the result is -1). This does not reveal any information about the initial state.

f) It is sufficient to consider only one of the stabilizers. We therefore analyze the simpler circuit



which only involves the first two qubits and the first ancilla. We also only need to consider the action of the circuit on the basis states $|ij\rangle$, and we find that

$$\begin{aligned}
|ij0\rangle &\overset{H}{\to} \frac{1}{\sqrt{2}}(|ij0\rangle + |ij1\rangle) \\
&\overset{CZ_1Z_2}{\to} \frac{1}{\sqrt{2}}(|ij0\rangle + Z_1Z_2|ij1\rangle) \\
&\overset{H}{\to} \frac{1}{2}(|ij0\rangle + |ij1\rangle + Z_1Z_2|ij0\rangle - Z_1Z_2|ij1\rangle) \\
&= \frac{1}{2}(1 + Z_1Z_2)|ij0\rangle + \frac{1}{2}(1 - Z_1Z_2)|ij1\rangle \\
&= \begin{cases} |ij0\rangle & \text{if } i = j \\ |ij1\rangle & \text{if } i \neq j \end{cases}
\end{aligned}$$

So it means that measuring the ancilla A_1 will give 0 if $Z_1Z_2|ij\rangle = 1$ and 1 if $Z_1Z_2|ij\rangle = -1$.

g) Consider for example starting from the state $|000\rangle$ and having such an error act on the first qubit

$$|000\rangle \to U_x^{(1)}(\theta) = \cos\theta|000\rangle + i\sin\theta|100\rangle.$$

Measuring Z_1Z_2 will give $+1$ with the probability $\cos^2\theta$ with the state collapsing to $|000\rangle$ and -1 with the probability $\sin^2\theta$ with the state collapsing to $|100\rangle$. In either case, there is correspondence between the outcome of the measurement and the resulting state (as always). In a sense we can say that the measurement decides if the first bit was unchanged or that the bit flip occured. After the measurement, we can correct the error as before if it happened, or do nothing if no error was detected.

h) Consider for definiteness that the error acts on the first qubit and that $\phi = \pi/2$. If the initial state is $|\phi\rangle = \alpha|000\rangle + \beta|111\rangle$ the state after the error occurs is

$$|\phi'\rangle = U_z^{(1)}(\pi/2)|\phi\rangle = \alpha|000\rangle - \beta|111\rangle.$$

We have that

$$Z_1Z_2|\phi'\rangle = Z_2Z_3|\phi'\rangle = |\phi'\rangle$$

which means that the error is not detected.

i) The stabilizers for the correction of bit flips on the individual qubits are Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8 and Z_8Z_9. For the phase flip we have to use the stabilizers on the qbits that are constructed of bit-flip corrected triples of qubits. These are $X_1^bX_2^b$ and $X_2^bX_3^b$, where $X_1^b = X_1X_2X_3$, $X_2^b = X_4X_5X_6$ and $X_3^b = X_7X_8X_9$, which means that the stabilizers are $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.

j) All the Z_iZ_j commute and also $[X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9] = 0$. We only have to check commutators of the type

$$[Z_1Z_2, X_1X_2X_3X_4X_5X_6] = [Z_1Z_2, X_1X_2]X_3X_4X_5X_6$$

as operators on different qubits commute. We have

$$[Z_1Z_2, X_1X_2] = Z_1[Z_2, X_1X_2] + [Z_1, X_1X_2]Z_2 = Z_1X_1[Z_2, X_2] + [Z_1, X_1]X_2Z_2 = iY_12iY_2 + 2iY_1(-iY_2) = 0.$$

k) Any unitary operation is of the form $U = e^{-iHt}$ for a suitable Hamiltonian H and time t (we use units where $\hbar = 1$). The Hamiltonian, being Hermitian, can be expanded in the Pauli matrices and the identity I as $H = h_0I + \sum_i h_i\sigma_i = h_0I + h\sigma_{\mathbf{n}}$ where the unit vector $\mathbf{n} = \mathbf{h}/|\mathbf{h}|$, $h = |\mathbf{h}|$ and $\sigma_{\mathbf{n}} = \mathbf{h}\cdot\sigma$. We know that these operators have the property $\sigma_{\mathbf{n}}^2 = I$, which means that

$$U = e^{-ih_0t}(\cos htI - i\sin ht\sigma_{\mathbf{n}} = a_0I + a_1X + a_2XZ + a_3Z$$

since we have that $Y = -iXZ$.

Problem 2: Encoding a qbit in an oscillator

a) We use the BCH formula

$$e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+cdots}$$

where the higher order terms in the exponent on the right involve commutators of $[A, B]$ with other operators. Defining $E_\alpha = \alpha\hat{a}^\dagger - \alpha^*\hat{a}$ we have that the commutator

$$[E_\beta, E_\alpha] = \beta\alpha^* - \beta^*\alpha$$

is a number, and therefore commutes with all operators and the series terminates. We then get

$$\hat{D}(\beta)\hat{D}(\alpha) = e^{E_\beta}e^{E_\alpha} = e^{E_\beta+E_\alpha+\frac{1}{2}[E_\beta,E_\alpha]} = e^{E_\beta+E_\alpha+\frac{1}{2}(\beta\alpha^*-\beta^*\alpha)} = \hat{D}(\alpha+\beta)e^{\frac{1}{2}(\beta\alpha^*-\beta^*\alpha)}.$$

Similarly we find that

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)e^{-\frac{1}{2}(\beta\alpha^*-\beta^*\alpha)}.$$

This gives

$$\hat{D}(\beta)\hat{D}(\alpha) = e^{\beta\alpha^*-\beta^*\alpha}.$$

b) We use the expansion

$$e^{B}Ae^{-B} = A + [B, A] + \frac{1}{2}[B, [B, A]] + cdots$$

to get

$$\hat{D}(\alpha)^{\dagger}\hat{x}\hat{D}(\alpha) = e^{-E_{\alpha}}\hat{x}e^{E_{\alpha}} = \hat{x} + [\hat{x}, E_{\alpha}] = \hat{x} + \alpha.$$

Then we check that $\hat{D}(\alpha)|x\rangle$ is an eigenstate of $\hat{x}$

$$\hat{x}\hat{D}(\alpha)|x\rangle = \hat{D}(\alpha)\hat{D}(\alpha)^{\dagger}\hat{x}\hat{D}(\alpha)|x\rangle = \hat{D}(\alpha)(\hat{x} + \alpha)|x\rangle = (x + \alpha)\hat{D}(\alpha)|x\rangle$$

which shows that $\hat{D}(\alpha)|x\rangle = |x + \alpha\rangle$.

c)

$$\hat{D}(\alpha)|0\rangle_L = \sum_{j=-\infty}^{\infty} \hat{D}(\alpha)|2j\alpha\rangle = \sum_{j=-\infty}^{\infty} |2j\alpha + \alpha\rangle = |1\rangle_L,$$

and similarly to show $\hat{D}(\alpha)|1\rangle_L = |0\rangle_L$.

d) We must have $\bar{X}\bar{Z} = -\bar{Z}\bar{X}$ which gives

$$\hat{D}(\alpha)\hat{D}(\beta) = -\hat{D}(\beta)\hat{D}(\alpha) = e^{\beta\alpha^{*}-\beta^{*}\alpha}\hat{D}(\alpha)\hat{D}(\beta).$$

For this to be true we must have $\beta\alpha^{*} - \beta^{*}\alpha = i\pi$. Assuming from now on that α is real and β purely imaginary we can then write this as $\beta = i\frac{\pi}{2\alpha}$ and get

$$\hat{D}(\beta)|0\rangle_L = e^{i\frac{\pi}{2\alpha}(\hat{a}^{\dagger}+\hat{a})}|0\rangle_L = e^{i\frac{\pi}{\alpha}\hat{x}}|0\rangle_L = e^{i\frac{\pi}{\alpha}\hat{x}}\sum_{j=-\infty}^{\infty}|2j\alpha\rangle = \sum_{j=-\infty}^{\infty} e^{i\frac{\pi}{\alpha}2\alpha j}|2j\alpha\rangle = \sum_{j=-\infty}^{\infty} e^{i2\pi j}|2j\alpha\rangle = |0\rangle_L$$

Similarly we get that $\hat{D}(\beta)|1\rangle_L = -|1\rangle_L$, so $\bar{Z} = \hat{D}(\beta)$ acts in the desired way on the logical states. Using this we then easily check the commutation relations.

e) In the position representation we have the wavefuction

$$\psi_0(x) = \langle x|0\rangle_L = \sum_{j=-\infty}^{\infty} \delta(x - 2\alpha j) = \frac{1}{2\alpha}\sum_{j=-\infty}^{\infty} e^{-\frac{i\pi}{\alpha}jx}$$

In the momentum repersentation we have

$$\bar{\psi}_0(p) = \int dx e^{ipx}\phi_0(x) = \frac{1}{2\alpha}\sum_{j=-\infty}^{\infty}\int e^{ipx}e^{-\frac{i\pi}{\alpha}jx} = \frac{\pi}{\alpha}\sum_{j=-\infty}^{\infty}\delta(p - 2|\beta|j).$$

For the state $|1\rangle_L$ we get

$$\bar{\psi}_1(p) = \frac{\pi}{\alpha}\sum_{j=-\infty}^{\infty}(-1)^{j}\delta(p - 2|\beta|j).$$

f) The action of the circuit on an initial state $|\psi\rangle_L$ is

$$
\begin{aligned}
|\psi_L\rangle \otimes |0\rangle &\xrightarrow{H} \frac{1}{\sqrt{2}}|\psi_L\rangle \otimes (|0\rangle + |1\rangle) \\
&\xrightarrow{C\hat{D}(\pm\frac{z}{2})} \frac{1}{\sqrt{2}}\left[\hat{D}(\frac{z}{2})|\psi_L\rangle \otimes |0\rangle + \hat{D}(-\frac{z}{2})|\psi_L\rangle \otimes |1\rangle\right] \\
&= \frac{1}{2}\left\{\left[\hat{D}(\frac{z}{2}) + \hat{D}(-\frac{z}{2})\right]|\psi\rangle_L \otimes |+\rangle + \left[\hat{D}(\frac{z}{2}) - \hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \otimes |-\rangle\right\},
\end{aligned}
$$

which gives

$$
\begin{aligned}
P_\pm &= \frac{1}{4}\langle\psi_L|\left[\hat{D}^\dagger(\frac{z}{2}) \pm \hat{D}^\dagger(-\frac{z}{2})\right]\left[\hat{D}(\frac{z}{2}) \pm \hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \\
&= \frac{1}{4}\left[2 \pm \langle\psi_L|\hat{D}^\dagger(z) + \hat{D}(z)|\psi_L\rangle\right] \\
&= \frac{1}{2}\left[1 \pm \frac{1}{2}\left(\langle\psi_L|\hat{D}(z)|\psi_L\rangle + \langle\psi_L|\hat{D}^\dagger(z)|\psi_L\rangle\right)\right].
\end{aligned}
$$

g) The circuit does the following

$$
\begin{aligned}
|\psi_L\rangle \otimes |0\rangle &\xrightarrow{H} \frac{1}{\sqrt{2}}|\psi_L\rangle \otimes (|0\rangle + |1\rangle) \\
&\xrightarrow{C\hat{D}(\pm\frac{z}{2})} \frac{1}{\sqrt{2}}\left[\hat{D}(\frac{z}{2})|\psi_L\rangle \otimes |0\rangle + \hat{D}(-\frac{z}{2})|\psi_L\rangle \otimes |1\rangle\right] \\
&\xrightarrow{S} \frac{1}{\sqrt{2}}\left[\hat{D}(\frac{z}{2})|\psi_L\rangle \otimes |0\rangle + i\hat{D}(-\frac{z}{2})|\psi_L\rangle \otimes |1\rangle\right] \\
&= \frac{1}{2}\left\{\left[\hat{D}(\frac{z}{2}) + i\hat{D}(-\frac{z}{2})\right]|\psi\rangle_L \otimes |+\rangle + \left[\hat{D}(\frac{z}{2}) - i\hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \otimes |-\rangle\right\}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
P_\pm &= \frac{1}{4}\langle\psi_L|\left[\hat{D}^\dagger(\frac{z}{2}) \mp i\hat{D}^\dagger(-\frac{z}{2})\right]\left[\hat{D}(\frac{z}{2}) \pm i\hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \\
&= \frac{1}{2}\left(1 \pm \mathrm{Im}\left(\langle\psi_L|\hat{D}(z)|\psi_L\rangle\right)\right).
\end{aligned}
$$

If

$$
\hat{D}(z)|\psi_L\rangle = e^{i\theta}|\psi_L\rangle
$$

the state after measurement outcome ± is

$$
|\psi'_L\rangle = \hat{D}(\pm\frac{\epsilon}{2})\left[\hat{D}(\frac{z}{2}) \pm i\hat{D}(-\frac{z}{2})\right]|\psi_L\rangle.
$$

We check that this is and eigenstate of $\hat{D}(z)$

$$\begin{aligned}
\hat{D}(z)|\psi_L'\rangle &= \hat{D}(z)\hat{D}(\pm\frac{\epsilon}{2})\left[\hat{D}(\frac{z}{2}) \pm i\hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \\
&= \hat{D}(\pm\frac{\epsilon}{2})\hat{D}(z)e^{\mp\frac{1}{2}(z\epsilon^*-z^*\epsilon)}\left[\hat{D}(\frac{z}{2}) \pm i\hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \\
&= e^{i\theta\mp\frac{1}{2}(z\epsilon^*-z^*\epsilon)}|\psi_L'\rangle \\
&= e^{i(\theta\mp|z\epsilon|)}|\psi_L'\rangle.
\end{aligned}$$

We also have that

$$P_\pm = \frac{1}{2}(1 \pm \sin\theta)$$

so that if $\theta > 0$ the probability of measuring $+$ is greater thatn $\frac{1}{2}$. In that case, we se that the new exponent $\theta \mp |z\epsilon| < \theta$, so the angle θ is on average reduced. On the other hand, if $\theta < 0$ it is more likely to measure $-$, and the angle will increase. So the state will approach the state with $\theta = 0$, that is, the eigenvalue will be 1.