Solutions to problem set 12

12.1 Photon emission

a) We know that $\boldsymbol{\epsilon}_{\mathbf{k}a}$ for $a \in \{1, 2\}$ are orthonormal vectors in the plane perpendicular to $\mathbf{k}$, and we can write $\mathbf{k} = k\mathbf{e}_k$. Thus, a general (real) vector can be decomposed as follows:

$$\mathbf{a} = (\mathbf{a} \cdot \boldsymbol{\epsilon}_{\mathbf{k}1})\, \boldsymbol{\epsilon}_{\mathbf{k}1} + (\mathbf{a} \cdot \boldsymbol{\epsilon}_{\mathbf{k}2})\, \boldsymbol{\epsilon}_{\mathbf{k}2} + (\mathbf{a} \cdot \mathbf{e}_k)\, \mathbf{e}_k$$

Then:

$$\begin{aligned}
\sum_a (\mathbf{a} \cdot \boldsymbol{\epsilon}_{\mathbf{k}a})^2 &= (\mathbf{a} \cdot \boldsymbol{\epsilon}_{\mathbf{k}1})^2 + (\mathbf{a} \cdot \boldsymbol{\epsilon}_{\mathbf{k}2})^2 \\
&= \underbrace{(\mathbf{a} \cdot \boldsymbol{\epsilon}_{\mathbf{k}1})^2 + (\mathbf{a} \cdot \boldsymbol{\epsilon}_{\mathbf{k}2})^2 + (\mathbf{a} \cdot \mathbf{e}_k)^2}_{=\mathbf{a}^2} - (\mathbf{a} \cdot \mathbf{e}_k)^2 \\
&= \mathbf{a}^2 - \left(\mathbf{a} \cdot \frac{\mathbf{k}}{k}\right)^2
\end{aligned}$$

b) For the 1D harmonic oscillator, we have the standard ladder operators:

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega\hat{x} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega\hat{x} - i\hat{p})$$

This gives the momentum operator:

$$\begin{aligned}
\hat{p} &= \frac{1}{2i} \left(\sqrt{2m\hbar\omega}\hat{a} - m\omega\hat{x} + m\omega\hat{x} - \sqrt{2m\hbar\omega}\hat{a}^\dagger \right) \\
&= -i\sqrt{\frac{m\hbar\omega}{2}} \left(\hat{a} - \hat{a}^\dagger \right)
\end{aligned}$$

These ladder operators act on the $|n\rangle$ states.

$$\begin{aligned}
\langle n-1|\hat{p}|n\rangle &= -i\sqrt{\frac{m\hbar\omega}{2}} \langle n-1| \left(\hat{a} - \hat{a}^\dagger \right) |n\rangle \\
&= -i\sqrt{\frac{m\hbar\omega}{2}} \langle n-1|\hat{a}|n\rangle - \langle n-1|\hat{a}^\dagger|n\rangle \\
&= -i\sqrt{\frac{m\hbar\omega}{2}} \langle n-1|\sqrt{n}|n-1\rangle - \underbrace{\langle n-1|\sqrt{n+1}|n+1\rangle}_{=0} \\
&= -i\sqrt{\frac{m\hbar\omega n}{2}}
\end{aligned}$$

Where we note that this is in the z-direction as noted in the exercise text.

c) From the text:

$$
\begin{aligned}
p(\theta,\phi) &= \kappa\sum_a |\ \langle n-1, 1_{\mathbf{k}a}|\hat{H}_{emis}|n,0\rangle\ |^2 \\
&= \kappa\sum_a |\ \langle n-1, 1_{\mathbf{k}a}| -\frac{e}{m}\sum_{\mathbf{k}'a'}\sqrt{\frac{\hbar}{2V\epsilon_0\omega}}\hat{\mathbf{p}}\cdot\boldsymbol{\epsilon}_{\mathbf{k}'a'}\hat{a}^\dagger_{\mathbf{k}'a'}|n,0\rangle\ |^2 \\
&= \kappa\ |\ \frac{e}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega}}\ |^2\sum_a\sum_{\mathbf{k}'a'}|\ \langle n-1, 1_{\mathbf{k}a}|\hat{\mathbf{p}}\cdot\boldsymbol{\epsilon}_{\mathbf{k}'a'}\hat{a}^\dagger_{\mathbf{k}'a'}|n,0\rangle\ |^2
\end{aligned}
$$

We note that the ladder operator in this case acts on the photon states, while $\hat{\mathbf{p}}$ acts on the particle states. Further:

$$
\begin{aligned}
\langle n-1, 1_{\mathbf{k}a}|\hat{\mathbf{p}}\cdot\boldsymbol{\epsilon}_{\mathbf{k}'a'}\hat{a}^\dagger_{\mathbf{k}'a'}|n,0\rangle &= \langle n-1|\otimes\langle 1_{\mathbf{k}a}|\left(\hat{\mathbf{p}}\cdot\boldsymbol{\epsilon}_{\mathbf{k}'a'}\otimes\mathbb{1}\right)\left(\mathbb{1}\otimes\hat{a}^\dagger_{\mathbf{k}'a'}\right)|n\rangle\otimes|0\rangle \\
&= \left[\left(\langle n-1|\hat{\mathbf{p}}\cdot\boldsymbol{\epsilon}_{\mathbf{k}'a'}\right)\otimes\langle 1_{\mathbf{k}a}|\right]\left[|n\rangle\otimes\left(\hat{a}^\dagger_{\mathbf{k}'a'}|0\rangle\right)\right] \\
&= \langle n-1|\hat{\mathbf{p}}\cdot\boldsymbol{\epsilon}_{\mathbf{k}'a'}|n\rangle\underbrace{\langle 1_{\mathbf{k}a}|\hat{a}^\dagger_{\mathbf{k}'a'}|0\rangle}_{=\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'}} \\
&= \langle n-1|\hat{\mathbf{p}}|n\rangle\cdot\boldsymbol{\epsilon}_{\mathbf{k}'a'}\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'}
\end{aligned}
$$

Where I marked the quantum numbers in order to have the possibility for the photon state to differ from the one in the emmision hamiltonian. So far, we have:

$$
\begin{aligned}
p(\theta,\phi) &= \kappa\ |\ \frac{e}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega}}\ |^2\sum_a\sum_{\mathbf{k}'a'}|\ \langle n-1|\hat{\mathbf{p}}|n\rangle\cdot\boldsymbol{\epsilon}_{\mathbf{k}'a'}\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'}\ |^2 \\
&= \kappa\ |\ \frac{e}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega}}\ |^2\sum_a|\ \langle n-1|\hat{\mathbf{p}}|n\rangle\cdot\boldsymbol{\epsilon}_{\mathbf{k}a}\ |^2
\end{aligned}
$$

Using the result from b) yields:

$$
\begin{aligned}
p(\theta,\phi) &= \kappa\ |\ \frac{e}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega}}\ |^2\sum_a|-i\sqrt{\frac{m\hbar\omega n}{2}}\mathbf{e}_z\cdot\boldsymbol{\epsilon}_{\mathbf{k}a}\ |^2 \\
&= \kappa\ |\ \frac{e}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega}}\sqrt{\frac{m\hbar\omega n}{2}}\ |^2\sum_a|\ \mathbf{e}_z\cdot\boldsymbol{\epsilon}_{\mathbf{k}a}\ |^2
\end{aligned}
$$

Noting that both $\mathbf{e}_z$ and $\boldsymbol{\epsilon}_{\mathbf{k}a}$ are real, we see that it is on the same form as the identity in exercise a).

$$
\begin{aligned}
p(\theta,\phi) &= \frac{\kappa e^2\hbar^2 n}{4mV\epsilon_0}\left(\mathbf{e}_z^2-\left(\mathbf{e}_z\cdot\frac{\mathbf{k}}{k}\right)^2\right) \\
&= \frac{\kappa e^2\hbar^2 n}{4mV\epsilon_0}\left(1-\left(\mathbf{e}_z\cdot\frac{\mathbf{k}}{k}\right)^2\right)
\end{aligned}
$$

The probability is given in terms of angular coordinates, and $\mathbf{k}/k$ has unit length, which allows us to write:

$$
\frac{\mathbf{k}}{k} = \cos\phi\sin\theta\mathbf{e}_x + \sin\phi\sin\theta\mathbf{e}_y + \cos\theta\mathbf{e}_z
$$

Inserting this yields:

$$p(\theta,\phi) = \frac{\kappa e^2\hbar^2 n}{4mV\epsilon_0}\left(1-\cos^2\theta\right)$$

This expression may or may not be normalized, we check:

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, p(\theta,\phi) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, \frac{\kappa e^2\hbar^2 n}{4mV\epsilon_0}\left(1-\cos^2\theta\right)$$

Using the substitution:

$$\sin\theta d\theta = -\frac{d\cos\theta}{d\theta}d\theta = -d\cos\theta$$

the boundries change to $\cos 0 = 1$ to $\cos\pi = -1$, and the minus interchanges the boundaries:

$$\begin{aligned}
\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, p(\theta,\phi) &= \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta\, \frac{\kappa e^2\hbar^2 n}{4mV\epsilon_0}\left(1-\cos^2\theta\right) \\
&= 2\pi\frac{\kappa e^2\hbar^2 n}{4mV\epsilon_0}\int_{-1}^{1} d\cos\theta\left(1-\cos^2\theta\right) \\
&= \frac{\pi\kappa e^2\hbar^2 n}{2mV\epsilon_0}\left[\cos\theta - \frac{1}{3}\cos^3\theta\right]_{\cos\theta=-1}^{\cos\theta=1} \\
&= \frac{\pi\kappa e^2\hbar^2 n}{2mV\epsilon_0}\left[1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right] \\
&= \frac{\pi\kappa e^2\hbar^2 n}{2mV\epsilon_0}\left[2-\frac{2}{3}\right] \\
&= \frac{2}{3}\frac{\pi\kappa e^2\hbar^2 n}{mV\epsilon_0}
\end{aligned}$$

Then, the normalized probability becomes:

$$\begin{aligned}
p(\theta,\phi) &= \frac{3}{2}\frac{mV\epsilon_0}{\pi\kappa e^2\hbar^2 n}\frac{\kappa e^2\hbar^2 n}{4mV\epsilon_0}\left(1-\cos^2\theta\right) \\
&= \frac{3}{8\pi}\left(1-\cos^2\theta\right)
\end{aligned}$$

This also seems resonable as the probabilities now only depent on angles, and not on things as the charge, potential and what excitation the charge is in.

12.2 Electric dipole transition in hydrogen (Exam 2008)

See solutions to previous exam questions.

12.3 Spinflip radiation

a) The probability distribution for the direction of the emitted photon, $p(\theta, \phi)$ is according to the problem given by

$$p(\theta, \phi) = N \sum_a |(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}a}) \cdot \boldsymbol{\sigma}_{BA}|^2$$

where N is a normalization to be determined later. We have

$$\boldsymbol{\sigma}_{BA} = \langle \downarrow | \boldsymbol{\sigma} | \uparrow \rangle = (1, i, 0)$$

We can choose the polarization vectors so that

$$\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}1} = k\boldsymbol{\epsilon}_{\mathbf{k}2} \qquad \text{and} \qquad \mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}2} = -k\boldsymbol{\epsilon}_{\mathbf{k}1}$$

where $k = |\mathbf{k}|$. This means that we get

$$\sum_a |(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}a}) \cdot \boldsymbol{\sigma}_{BA}|^2 = k^2 \sum_a |\boldsymbol{\epsilon}_{\mathbf{k}a} \cdot \boldsymbol{\sigma}_{BA}|^2 = k^2(|\boldsymbol{\sigma}_{BA}|^2 - |\boldsymbol{\sigma}_{BA} \cdot \frac{\mathbf{k}}{k}|^2$$

Using

$$\frac{\mathbf{k}}{k} = (\sin\theta\cos\phi\sin\theta\sin\phi, \cos\theta)$$

we get

$$|\boldsymbol{\sigma}_{BA}|^2 = 2 \qquad \text{and} \qquad \boldsymbol{\sigma}_{BA} \cdot \frac{\mathbf{k}}{k} = \sin\theta e^{i\phi}.$$

The probability distribution for the direction of the emitted photon is then

$$p(\theta, \phi) = Nk^2(1 + \cos^2\theta).$$

To determine the normalization we calculate

$$\int d\phi \int d\theta \sin\theta p(\theta, \phi) = Nk^2 2\pi \int_0^\pi d\theta \sin\theta(1 + \cos^2\theta) = \frac{16\pi}{3} Nk^2 = 1$$

From which we get $N = 3/16\pi k^2$. The answer is then

$$p(\theta, \phi) = \frac{3}{16\pi}(1 + \cos^2\theta).$$

b) We have $\mathbf{k} = (1, 0, 0)$ and we can chosse the polarization vectors so that $\boldsymbol{\epsilon}_{\mathbf{k}1} = (0, \cos\alpha, \sin\alpha)$. Then

$$p(\alpha) = N|(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}1}) \cdot \boldsymbol{\sigma}_{BA}|^2 = N\sin^2\alpha$$

To determine the normalization, we use the condition $\int_0^\pi d\alpha p(\alpha) = 1$. Here we set the upper limit of the integrtion to π and not 2π since α and $\alpha + \pi$ represents the same polarization state. We then get

$$p(\alpha) = \frac{2}{\pi}\sin^2\alpha$$

c)

$$
\begin{aligned}
w_{BA} &= \frac{V}{(2\pi\hbar)^2}\int d^3k \sum_a |\langle B, 1_{\mathbf{k}a}|H_1|A,0\rangle|^2 \delta(\omega-\omega_B) \\
&= \frac{e^2\hbar}{32\pi^2 m^2\epsilon_0}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} k^2 dk \frac{1}{\omega}\delta(\omega-\omega_B)\sum_a |(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}a})\cdot\boldsymbol{\sigma}_{BA}|^2 \\
&= \frac{e^2\hbar}{32\pi^2 m^2\epsilon_0 c^5} 2\pi \int_0^{\pi} d\theta \sin\theta (1+\cos^2\theta)\int_0^{\infty} d\omega \omega^3 \delta(\omega-\omega_B) \\
&= \frac{e^2\hbar\omega_B^3}{6\pi m^2 \epsilon_0 c^5}.
\end{aligned}
$$

This gives the lifetime

$$
\tau = \frac{1}{w_{BA}} = \frac{6\pi m^2\epsilon_0 c^5}{e^2\hbar\omega_B^3}.
$$