

## Problem set 3

### 3.1 Ladder operators in the Heisenberg picture

Consider a harmonic oscillator with Hamiltonian

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \quad (1)$$

expressed in terms of the ladder operators  $\hat{a}^\dagger$  and  $\hat{a}$ . Show that these two operators take the following time dependent form in the Heisenberg picture

$$\hat{a}^\dagger(t) = e^{i\omega t}\hat{a}^\dagger, \quad \hat{a}(t) = e^{-i\omega t}\hat{a} \quad (2)$$

### 3.2 Displacement operators in phase space

For a particle moving in one dimension the position coordinate  $x$  and the momentum  $p$  define the coordinates of the two-dimensional classical *phase space*.

In the quantum description of the one-dimensional harmonic oscillator non-hermitian lowering operator is defined as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} + i\hat{p}) \quad (3)$$

We may consider this as the operator of a complex phase space variable, with position as the real part and momentum as the imaginary part. It has a dimensionless form due to the constants introduced in the expression.

A coherent state, in a similar way is characterized by a complex number  $z$ , the eigenvalue of  $\hat{a}$ , which we may interpret as a complex phase space coordinate,

$$z = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x_c + ip_c) \quad (4)$$

The following operator

$$\hat{\mathcal{D}}(z) = e^{(z\hat{a}^\dagger - z^*\hat{a})} \quad (5)$$

acts as a displacement operator in phase space, in the sense

$$\hat{\mathcal{D}}(z)^\dagger\hat{x}\hat{\mathcal{D}}(z) = \hat{x} + x_c, \quad \hat{\mathcal{D}}(z)^\dagger\hat{p}\hat{\mathcal{D}}(z) = \hat{p} + p_c \quad (6)$$

Show that displacements in two different directions in general will not commute but rather satisfy a relation of the form

$$\hat{\mathcal{D}}(z_a)\hat{\mathcal{D}}(z_b) = e^{i\alpha(z_a, z_b)}\hat{\mathcal{D}}(z_b)\hat{\mathcal{D}}(z_a) \quad (7)$$

with  $\alpha(z_a, z_b)$  as a complex phase. Determine the phase as a function of  $z_a$  and  $z_b$ . What is the condition for the two operators to commute?

### 3.3 Eigenvectors for $\hat{a}^\dagger$ ?

The coherent states  $|z\rangle$  are defined as eigenvectors of the lowering operator  $\hat{a}$ . Assume  $|\bar{z}\rangle$  to be eigenvector of the raising operator  $\hat{a}^\dagger$ ,

$$\hat{a}^\dagger|\bar{z}\rangle = \bar{z}|\bar{z}\rangle \quad (8)$$

Show that no normalizable vector exists that satisfies this equation by expanding the state  $|\bar{z}\rangle$  in the energy eigenstates  $|n\rangle$ .

### 3.4 A driven harmonic oscillator (Exam 2010)

A quantum mechanical, driven harmonic oscillator is described by the following Hamiltonian

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar\lambda(\hat{a}^\dagger e^{-i\omega t} + \hat{a}e^{i\omega t}) \quad (9)$$

where  $\hat{a}$  og  $\hat{a}^\dagger$  satisfy the standard commutation relations for lowering and raising operators, and where  $\omega_0, \omega$  og  $\lambda$  are three constants. We introduce the following dimensionless position and momentum operators,

$$\hat{x} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = -\frac{i}{2}(\hat{a} - \hat{a}^\dagger) \quad (10)$$

a) As a reminder, Heisenberg's equation of motion has the form

$$\frac{d}{dt}\hat{A} = \frac{i}{\hbar} [\hat{H}, \hat{A}] + \frac{\partial}{\partial t}\hat{A} \quad (11)$$

for any given observable  $\hat{A}$ . Apply this to the lowering operator  $\hat{a}$ , and show that it satisfies an equation of the form

$$\frac{d^2\hat{x}}{dt^2} + \omega_0^2\hat{x} = C \cos \omega t \quad (12)$$

Determine the constant  $C$ .

b) By use of the the time dependent unitary transformation

$$\hat{T}(t) = e^{i\omega t \hat{a}^\dagger \hat{a}} \quad (13)$$

the new Hamiltonian,  $\hat{H}_T(t)$ , which determines the time evolution of the transformed state vectors  $|\psi_T(t)\rangle = \hat{T}(t)|\psi(t)\rangle$ , will take a time independent form. Find the expression for the transformed Hamiltonian.

c) A coherent state is defined as an eigenstate of the lowering operator  $\hat{a}$ ,

$$\hat{a}|z\rangle = z|z\rangle \quad (14)$$

Assume at time  $t = 0$  the oscillator is in the ground state for the  $\lambda$ -independent part of the Hamiltonian, that is

$$|\psi(0)\rangle = |0\rangle, \quad \hat{a}|0\rangle = 0 \quad (15)$$

Show that, during the time evolution, it will continue as a coherent state,

$$|\psi(t)\rangle = e^{i\alpha(t)}|z(t)\rangle \quad (16)$$

with  $\alpha(t)$  as a time dependent phase and  $z(t)$  as a complex-valued function of time.

Find the function  $z(t)$  and give a qualitative description of the motion in the complex  $z$ -plane. Show that the real part  $x(t) = (z(t) + z(t)^*)/2$  satisfies the same equation of motion (12) as the position operator  $\hat{x}(t)$ .

As a reminder we give the following operator relation

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots \quad (17)$$

which applies generally to two operators  $\hat{A}$  og  $\hat{B}$ .