Problem set 6

6.1 Pure and mixed states

a) Explain what is the difference between pure and mixed quantum states. How are they represented mathematically?

b) An ensemble of spin-$\frac{1}{2}$ particles are produced by some (to you) unknown procedure. You are informed that the particles will be either (ensemble A) in the state $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ or (ensemble B) in a random statistical mixture with 50% of the particles in the state $|\uparrow\rangle$ and 50% of the particles in the state $|\downarrow\rangle$. You are allowed to measure the spin of each particle along an axis of your choice (you do not have to choose the same axis for each particle). Describe an experiment which would reveal whether the particles are prepared in ensemble A or ensemble B. Explain what will be the probabilities of different measurement oucomes for both ensembles when using your measurement procedure.

c) Consider now a third enemble (ensemble C), where the particles are prepared in a random statistical mixture with 50% of the particles in the state $|\rightarrow\rangle$ and 50% of the particles in the state $|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$. Prove that we can not distiguish ensembles B and C by any measurements on the particles.

Instead of direct preparation as described above, we can prepare the ensembles B or C remotely by entanglement in the following way. Person 1 (the preparer) prepares an ensemble of pairs of entangled particles in the state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. He keeps one particle from each pair to himself and sends the other particle from each pair to person 2 (you). By doing appropriate measurements on his particles, person 1 can now decide at a later point if he would like your particles to belong to ensemble B or C.

d) Which measurement should person 1 perform to generate ensemble B and which to generate ensemble C? Justify your answer.

e) Even if the ensembles B and C are indistiguishable by local measurements by person 2, as you showed in question c), they can be distiguished by the correlations between the measurement outcomes of persons 1 and 2. Explain which measurements person 2 should do, and how the difference between ensembles B and C are visible in the correlations. Assume that the pairs are labeled, so that we can compare the measurement oucomes for the two particles belonging to the same pair. What changes if person 1 decides to wait with his measurements until after person 2 makes the measurements, so that the two ensembles are not prepared until after they are measured.

6.2 Entanglement

Two persons A and B communicate with the help of quantum entanglement. They share a set of pairs of particles with spins in the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle) \tag{1}$$

where $|++\rangle = |+\rangle \otimes |+\rangle$ is a state where both particles of the pair have *spin up* in the z-direction, and similarly $|--\rangle = |-\rangle \otimes |-\rangle$ is the state where both particles have *spin down* in the z-direction.

a) What is the quantity used as measure for the degree of entanglement in such a two-partite system, and what is the degree of entanglement in the given spin state?

b) Assume A and B perform independent spin operations on their particles in a given pair, each operation described by a unitary operator, $\hat{U}_A$ or $\hat{U}_B$. What happens to the entanglement of the two-particle system under such an operation.

c) Assume A performs an ideal measurement of the spin component in the x- direction, which projects the spin to an eigenstate of the x-component of the spin operator. What happens to the entanglement in this case?

6.3 Matrix representation of tensor products

Assume $|a\rangle = \sum_{i=1}^{2} a_i |i\rangle_A$ is a vector in an 2-dimensional Hilbert space $\mathcal{H}_A$ and $|b\rangle = \sum_{j=1}^{2} b_j |j\rangle_B$ is a vector in another 2-dimensional Hilbert space $\mathcal{H}_B$, with $\{|i\rangle_A\}$ as an orhonormal basis set in $\mathcal{H}_A$ and $\{|j\rangle_B\}$ as a similar vector set in $\mathcal{H}_B$. The composite vector $|c\rangle = |a\rangle \otimes |b\rangle$ is a product vector in the tensor product space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Expanded in the product basis it has the form $|c\rangle = \sum_{ij} a_i b_j |ij\rangle$ with $|ij\rangle = |i\rangle_A \otimes |j\rangle_B$.

We consider the matrix representation of the vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{2}$$

The vector $|c\rangle$ can be representet as a single column matrix of dimension 4. We define the matrix elements $c_k, k = 1, ..., 4$, of such a matrix by the following relation

$$c_{j+2(i-1)} = a_i b_j \tag{3}$$

a) Express the column matrix $\mathbf{c}$ (4×1 matrix) in terms of the matrix elements of $\mathbf{a}$ and $\mathbf{b}$, and show that it can be written in a compact form as

$$\mathbf{c} = \begin{pmatrix} a_1 \mathbf{b} \\ a_2 \mathbf{b} \end{pmatrix} \tag{4}$$

What are the vector representations for the four basis vectors $|ij\rangle$?

We consider next operators $\hat{A}$, $\hat{B}$ and $\hat{C} = \hat{A} \otimes \hat{B}$, which act in $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}$ respectively. The corresponding 2×2 matrix $\mathbf{A}$ represents $\hat{A}$ in the basis $\{|i\rangle_A\}$ and the 2×2 matrix $\mathbf{B}$ represents $\hat{B}$ in the basis $\{|j\rangle_B\}$. The tensor product of the operators can be represented as a 4×4 matrix $\mathbf{C}$, with elements

$$C_{j+2(i-1), j'+2(i'-1)} = A_{ii'} B_{jj'} \tag{5}$$

similar to the column matrix c_i, defined in (3).

b) Show that the matrix $\mathbf{C}$ can be written in a form similar to (4). in terms of the components of $\mathbf{A}$ and the matrix $\mathbf{B}$.

c) Find the 4×4 matrix representations of the tensor products $\sigma_k \otimes \sigma_l$, where $\sigma_k, k = 1, 2, 3$, are the Pauli matrices. Write them in a form similar to (4). It is sufficient to do this for three different choices of the Pauli matrices.

d) Show that the matrix representations are consistent with normal matrix multiplication in the sense that the matrix product $\mathbf{Cc}$ gives the vector representing the state $\hat{A} \otimes \hat{B}|a\rangle \otimes |b\rangle$.

6.4 Schmidt decomposition 1

We have a system consisting of two spin-$\frac{1}{2}$ particles. For each of the following states, study the reduced density matrix of of one of the particles and determine if the state is entangled or not. For the states which are not entangled, find a factorization of the state as a tensor product of one state for each particle. For the entagled states, find the Schmidt decomposition of the state.

$$\begin{aligned}
|\psi_1\rangle &= \frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\
|\psi_2\rangle &= \frac{1}{2}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\
|\psi_3\rangle &= a_+|\uparrow\uparrow\rangle + a_-|\uparrow\downarrow\rangle + a_-|\downarrow\uparrow\rangle + a_+|\downarrow\downarrow\rangle \\
|\psi_4\rangle &= a_-|\uparrow\uparrow\rangle + a_+|\uparrow\downarrow\rangle + a_+|\downarrow\uparrow\rangle + a_-|\downarrow\downarrow\rangle
\end{aligned}$$

where

$$a_\pm = \frac{\sqrt{3} \pm 1}{4}$$

6.5 Schmidt decomposition 2

Entanglement can occur not only between distinct particles, but also between different observables fot the same particle, like position and spin. Here we will find the Schmidt decomposition of one continuos and one discrete Hilbert space. A spin-half particle moving in one dimension is described by a two-component wave function

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{6}$$

where the upper matrix position is assumed to correspond to ”spin up” in the z-direction and the lower matrix position to ”spin down” in the same direction. The scalar product of the two wave functions will generally be different from zero, and we write it as

$$\langle\psi_1|\psi_2\rangle = \int dx\, \psi_1^*(x)\psi_2(x) \equiv \Delta \tag{7}$$

a) The Schmidt decomposition of the two-component wave function has the form

$$\Psi(x) = c_1\chi_1\,\phi_1(x) + c_2\chi_2\,\phi_2(x) \tag{8}$$

where c_1 and c_2 are expansion coefficients, χ_1 and χ_2 are normalized, two-component spinors, and $\phi_1(x)$ and $\phi_2(x)$ are normalized, scalar (one-component) wave functions. What are the conditions that the spinors and wave functions should satisfy?

b) Assume the two wave functions of (6) are real Gaussian functions of the form

$$\psi_1(x) = Ne^{-\lambda(x-x_0)^2}\,, \quad \psi_2(x) = Ne^{-\lambda(x+x_0)^2} \tag{9}$$

Determine the normalization factor N and the overlap Δ, expressed in terms of λ and x_0.

c) Determine the coefficients, spinors and wave functions in (8). (Since the wave function $\Psi(x)$ is real, you may assume the variables in Eq.(8) all to be real.)