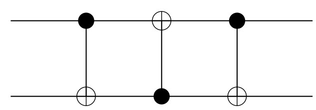
Problem set 8

8.1 SWAP gate

a) A useful quantum gate is the SWAP gate which interchanges the state of two qubits. That is, if the input state is $|\psi\rangle \otimes |\phi\rangle$ the output will be $SWAP|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$. Show that the following quantum circuit on two qubits gives a decomposition of the SWAP gate using three CNOT gates.



b) Write the matrix for the SWAP gate. Describe which basis you use.

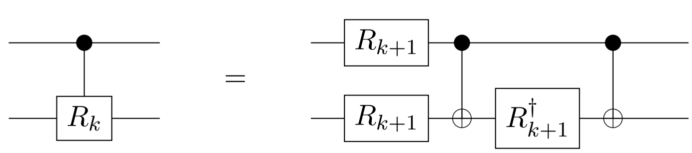
c) Generalize the above circuit to implement the SWAP gate on two n-qubit registers using only CNOT gates. How many CNOT gates do you need?

8.2 Quantum circuit for controlled R_k

a) In the quantum Fourier transformation, we needed to perform a controlled R_k operation. The one-qubit operator

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

is then performed on the target qubit if the control qubit is in the state $|1\rangle$. When the control qubit is in the state $|0\rangle$ no operation is performed on the target qubit. We know that all two-qubit operators can be decomposed in single qubit operators and controlled NOT (CNOT) operations. Show that the following quantum circuit is one such decompostion for the controlled R_k operation



b) We consider now general controlled U operations, with U a one-qubit operator. This means that the operation U is performed on the target qubit if the control qubit is in the state $|1\rangle$. When the control qubit is in the state $|0\rangle$ no operation is performed on the target qubit. In both cases, the control qubit is not changed. If this was a classical system, this would be all the possibilities, but in a quantum system, one can have a control qubit that is in a superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ of the two basis states. In general, the two qubits will be entangled by this operation, so no definite quantum state can be ascribed to any of them. However, a special situation arises if the initial state of the target qubit is an eigenstate of U. Draw a quantum circuit desribing this situation. Show that in this case, the two qubits are not entangled by the operation. Show also that in this case, it is the target qubit that is not changed, while the state of the control qubit is changed. Find the final state of the control qubit in terms of the eigenvalues of U.

c) This result is surprising if we only are used to the classical world, and deserves an explanation. Explain in words why the target is not changed while the state of the control does change.

8.3 Approximate quantum cloning

The no-cloning theorem tells us that it is impossible to copy an unknown quantum state. In this problem we will study a protocol which takes the quantum state of a two-level system and produces two two-level systems with the state of both approximating as well as possible the original state.

We consider three two-level systems. The first (system A) is the original to be copied, the second (system B) is the system that will receive a copy of the state, and the third (system C) is an auxiliary system (often called an ancilla). We label the states in the usual way, $|000\rangle = |0\rangle_A|0\rangle_B|0\rangle_C = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C$ and similar for the other basis states. We define a unitary operation through the equations

$$U|000\rangle = \sqrt{\frac{2}{3}}|000\rangle + \sqrt{\frac{1}{6}}\left(|011\rangle + |101\rangle\right) \tag{1}$$

$$U|100\rangle = \sqrt{\frac{2}{3}}|111\rangle + \sqrt{\frac{1}{6}}\left(|010\rangle + |100\rangle\right) \tag{2}$$

Let the initial state of system A be $|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$, and apply the above operation to the system if the inital state of B and C is $|00\rangle_{BC}$.

a) Calculate the reduced density matrices ρ_A and ρ_B of systems A and B.

b) Determine the Bloch vector of both the final states ρ_A and ρ_B and find how they are related to the Bloch vector of the initial state $|\psi\rangle$.

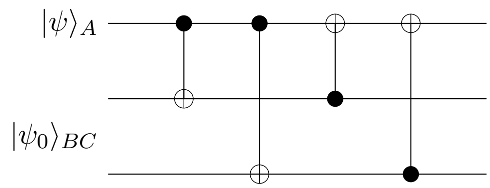
c) We define the fidelity of the copying operation as the overlap of the copied state with the original

$$F = \langle\psi|\rho_B|\psi\rangle.$$

Calculate the fidelity for this operation.

d) The relations (1) and (2) do not define the operation U completely, and it is necessary to show that it can be completed as a unitary operation on all the basis vactors. We can do this by demonstrating that it is produced by a quantum circuit.

Show that the following quantum circuit will implement the unitary operation U on the required input states.



The systems B and C must initially be prepared in the state

$$|\psi_0\rangle_{BC} = \sqrt{\frac{2}{3}}|00\rangle_{BC} + \sqrt{\frac{1}{6}}\left(|01\rangle_{BC} + |11\rangle_{BC}\right).$$

There is a simple circuit to do this step also, starting from the state $|00\rangle_{BC}$, but for this problem we just assume that it has been prepared.