1 Ising model in 1d: Exact versus mean-field solution

Consider a magnetic system with N spins of $s_i = \pm 1/2$ on a one-dimensional lattice with periodic boundary conditions $s_{N+1} = s_1$. The spins interact with their nearest neighbor and an applied field, h, with a Hamiltonian given by

$$\mathcal{H} = -J \sum_{i=1}^{N} s_i s_{i+1} - h \sum_{i=1}^{N} s_i, \tag{1}$$

where $J > 0$.

a) Using the transfer matrix determine the free energy per spin f_s. Compare it with the free energy per spin, f_s^{MF}, derived in the mean field approximation.

b) Determine the susceptibility χ from the exact free energy.

c) Find the critical point from the divergent behaviour of χ near it. Is it close to the mean field critical point?

c) Determine the correlation length-scale ξ at $h = 0$ as a function of $(T - T_c)$. How does it compare with the mean field prediction? (*Hint: Use that* $\xi^{-1} = -ln\frac{\lambda_-}{\lambda_+}$, *where* $\lambda_\pm$ *are the eigenvalues of the transfer matrix.*)

2 Ginzburg-Landau free energy: Metastability, Susceptibility, Correlations

Consider a one-dimensional system in an external field, very close to T_c, and described by a Ginzburg-Landau free energy given as

$$\mathcal{F} = \int d\mathbf{r} \left[\frac{1}{2} \left(\nabla m(\mathbf{r}) \right)^2 + \frac{t}{2} m(\mathbf{r})^2 + \frac{1}{4} m(\mathbf{r})^4 - h(\mathbf{r}) m(\mathbf{r}) \right] \tag{2}$$

where $t = (T - T_c)/T_c$, $h(\mathbf{r}) > 0$ is the applied field, and $m(\mathbf{r})$ is the real order parameter.

a) Plot the homogeneous part of the free energy density as a function of m for $h > 0$ and different values of t above and below the critical point.

b) Determine the equation of state that minimizes the homogeneous free energy density. Plot h as a function of m below T_c and compare with Fig. 11.3 (Sethna's book, pg. 244).

c) Find the metastable state and the value of h at which the metastable state becomes unstable.

d) Find the equation of state that minimizes the free energy in Eq. (2) with respect to the order parameter.

e) Determine the equation satisfied by the static susceptibility $\chi(\mathbf{r} - \mathbf{r}')$ above, but close to T_c, and solve it one dimension $d = 1$. Show that the uniform static susceptibility χ_0 diverges as $1/t$. (*Hints: Compute the Fourier transform $\tilde{\chi}(k)$ of the susceptibility and solve the inverse Fourier transform in the complex plane. $\chi_0 = \tilde{\chi}(k = 0)$.*)

d) Using the fluctuation-response relation, find the two-point correlation function for the one-dimensional case. How does the correlation length-scale diverge near T_c?