Answers to problem set 5
FYS4130 at UiO, Spring 2012

Jørgen Trømborg

February 2012

5.12

a) $\frac{S}{k_{\mathrm{B}}} = \ln(N!) - \ln\left(\frac{N}{2} + \frac{L}{2d}\right) - \ln\left(\frac{N}{2} - \frac{L}{2d}\right)$

b) $F = -T\frac{\partial S_{\mathrm{bath}}}{\partial L} = T\frac{\partial S_{\mathrm{band}}}{\partial L}$

c) $K = \frac{k_{\mathrm{B}}T}{Nd^2}$

d) The stiffness increases and the band contracts. One can argue this in detail from $\frac{\partial L}{\partial T}\big|_F$.

e) FFTFTT

5.13

a) $S_{\mathrm{sorted}} = 0$, $S_{\mathrm{random}} = \frac{\ln(52!)}{\ln 2} \approx 226$

b) $S_{\mathrm{shuffle}} = 52$

c) 52 shuffles leave the ordering unchanged, namely the 52 ways to cut the deck in “halves” (53 if you allow one of the “halves” to be empty).

d) 5. The discrepancy with the exercise introduction arises as ignoring overlaps between riffles becomes a crude approximation.

5.14

a) $S_{\mathrm{max}} = 8000$

b) $\Delta S_{\mathrm{Shannon}} = -4$ bits. You need 6 characters to encode the 12 messages.

5.15

a) $S_{\mathrm{letter}} = \frac{3}{2}$

b) –

c) The maximum entropy per bit is 1, so you need at least S_{Shannon} bits to send entropy S_{Shannon}. On average you need $\frac{3}{2}$ bits per letter in the A’bç! channel.

d) $A \to 10$, $B \to 11$, $C \to 0$.

6.2

a)

$$\frac{\rho_2}{\rho_1} = \begin{cases} e^{-(E_2-E_1)\beta} & \\ 0 & T \to 0 \\ 1 & T \to \infty \end{cases}$$

b)

$$\begin{aligned} Z(\beta) &= e^{-E_1\beta} + e^{-E_2\beta} \\ \rho_i &= \frac{e^{-E_i\beta}}{Z} \\ \langle E \rangle &= E_1\rho_1 + E_2\rho_2 \end{aligned}$$

6.3

a) $S_{\text{micro}}(E) = k_{\text{B}}[N \ln N - m \ln m - (N-m)\ln(N-m)]$

b)

$$T = \frac{\varepsilon}{k_{\text{B}}} \frac{1}{\ln(N/m - 1)} = \frac{\varepsilon}{k_{\text{B}}} \left[\ln\left(\frac{N/2 - E/\varepsilon}{N/2 + E/\varepsilon} \right) \right]^{-1}$$

For $E > 0$, $T < 0$.

c) (i)

$$\begin{aligned} Z_{\text{canon}} &= 2\cosh\left(\frac{\beta\varepsilon}{2}\right) \\ E_{\text{canon}} &= -\frac{\varepsilon}{2}\tanh\left(\frac{\beta\varepsilon}{2}\right) \\ S_{\text{canon}} &= -k_{\text{B}}\left[\frac{e^{\beta\varepsilon/2}}{Z} \ln \frac{e^{\beta\varepsilon/2}}{Z} + \frac{e^{-\beta\varepsilon/2}}{Z} \ln \frac{e^{-\beta\varepsilon/2}}{Z} \right] \end{aligned}$$

(ii) –

d) As $T \to \infty$, $E \to 0^-$. You cannot get into the regime where $E > 0$.

e)

$$S_N = \begin{cases} NS_1 & \\ 0 & T \to 0 \\ Nk_{\text{B}}\ln 2 & T \to \infty \end{cases}$$

In general the canonical entropy is greater than the microcanonical entropy of the information we have about the state of the system. The micro-canonical ensemble has all states on an energy surface, whereas in the canonical we average over all energy surfaces. For large N the fluctuations away from the mean energy become negligible, and the two ensembles agree.

f)

$$\begin{aligned} \sigma_E &= \frac{\varepsilon}{\sqrt{N}}[m(N-m)]^{1/2} \\ \frac{\sigma_E}{E} &\to \frac{2\sqrt{m}}{N} \end{aligned}$$