FYS4130 problem set 9 (2014) = problem set 10 (2010) : solutions

Jørgen Kjoshagen Trømborg 2014-03-18

1 Boson gas

(The calculation is given in section 5.4 of the compendium as well.)

Definitions or tools that we will need:

Thermal wavelength $\Lambda = \sqrt{\dfrac{h^2}{2\pi m k_B T}} = \sqrt{\dfrac{2\pi\hbar^2}{m k_B T}}$

Fugacity $\lambda = e^{\beta\mu}$, $\beta = (k_B T)^{-1}$, μ: chemical potential

Poly-logarithmic function $g_p(\lambda) = \sum_{n=1}^{\infty} \dfrac{\lambda^n}{n^p}$

Full density of energy states when $H = \sum_{\text{particles}} \dfrac{\vec{p}_i^{\,2}}{2m}$:

$$\sum_{\substack{\text{single particle}\\ \text{states}}} \rightarrow V\int \frac{d^3p}{(2\pi\hbar)^3}$$

$$\parallel$$

$$\frac{V}{4\pi^2\hbar^3}(2m)^{3/2}\int_0^\infty \varepsilon^{1/2}\,d\varepsilon$$

3D:

$4\pi p^2 dp = d^3p$

$\varepsilon = \dfrac{p^2}{2m} \Rightarrow p\,dp = m\,d\varepsilon$

$p = \sqrt{2m\varepsilon}$

a $\Xi = \sum_{\substack{\text{full system}\\ \text{states}}} e^{-\beta(E_S - \mu N_S)}$, $E_S = \sum_{\substack{\text{single particle}\\ \text{states}}} \varepsilon_s n_s$

, $N_S = \sum_{\text{s.p.s.}} n_s$

$$= \sum_{\text{f.s.s.}} e^{-\beta \sum_{\text{s.p.s}} (\varepsilon_s - \mu) n_s}$$

$$= \sum_{\text{f.s.s.}} \prod_{\text{s.p.s.}} e^{-\beta(\varepsilon_s - \mu) n_s}$$

$$= \prod_{\text{s.p.s.}} \sum_{n_s = 0}^{\infty} e^{-\beta(\varepsilon_s - \mu) n_s}$$

This step works for bosons, but requires some thought. Considering how you would write down the full system states will help. You can also make yourself an example with only 3 different single particle states

$$= \prod_{\text{s.p.s.}} \left(1 - e^{-\beta(\varepsilon_s - \mu)}\right)^{-1}$$, geometric series converges if $\mu \le \varepsilon_s \ \forall$ s.p.s.

$$\ln \Xi = -\sum_{\text{s.p.s.}} \ln\left(1 - e^{-\beta(\varepsilon_s - \mu)}\right) \quad , \quad \ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad (1)$$

$$\ln \Xi = \sum_{n=1}^{\infty} \frac{e^{\beta\mu n}}{n} \underbrace{\sum_{\text{s.p.s.}} e^{-\beta\varepsilon_s n}}_{\approx \frac{V}{4\pi^2\hbar^3}(2m)^{3/2}\underbrace{\int_0^\infty d\varepsilon\, \varepsilon^{1/2} e^{-n\beta\varepsilon}}_{= \frac{\sqrt{\pi}}{2n^{3/2}}(k_BT)^{3/2}}}$$

, good approximation away from B-E condensation conditions

$$= V\left(\frac{2mk_BT}{h^2}\right)^{3/2} \frac{\sqrt{\pi}}{2} \frac{(2\pi)^3}{4\pi^2} \sum_{n=1}^{\infty} \frac{\lambda^n}{n^{5/2}}$$

$$= \underline{\underline{\frac{V}{\Lambda^3} g_{5/2}(\lambda)}}, \text{ QED}$$

b If the lowest energy state is $\varepsilon_0 = 0$ then the sum over single particle states in eq. (1) has as its first term the contribution

$$-\ln(1 - e^{\beta\mu} e^0) = -\ln(1-\lambda).$$

The approximation of the sum as an integral for the rest of the states remains the same, so now we find

$$\underline{\underline{\ln \Xi = \frac{V}{\Lambda^3} g_{5/2}(\lambda) - \ln(1-\lambda)}}$$

c The Landau free energy is $-PV = \Omega = -kT \ln \Xi$, so the equation of state is

$$\underline{\underline{PV = kT\left(\frac{V}{\Lambda^3} g_{5/2}(\lambda) - \ln(1-\lambda)\right)}}$$

2 Boson gas

(See also section 5.5 in the compendium)

a Above and down to the critical temperature we the result from 1a:

$$N = kT \frac{\partial}{\partial \mu} \ln \Xi = kT \frac{V}{\Lambda^3} \frac{\partial}{\partial \mu} \sum_{n=1}^{\infty} \frac{e^{n\beta\mu}}{n^{5/2}} = \frac{V}{\Lambda^3} \sum_{n=1}^{\infty} \frac{e^{n\beta\mu}}{n^{3/2}}$$

$$= \frac{V}{\Lambda^3} g_{3/2}(\lambda)$$

At the critical temperature $\mu \to 0$, $\lambda \to 1$, $g_{3/2}(\lambda) \to g_{3/2}(1) = \zeta(3/2) \approx 2.612$.

$$\frac{1}{\Lambda_c^3} = \frac{\varrho}{2.612}, \quad \varrho = \frac{N}{V}$$

$$\underline{T_c = \frac{2\pi\hbar^2}{k_B m} \left(\frac{\varrho}{2.612}\right)^{2/3}}, \quad \text{QED}$$

b For $T < T_c$ the result in a only counts the excited states N_e.

$$\frac{N_e}{N} = \frac{\frac{V}{\Lambda^3} g_{3/2}(\lambda)}{\frac{V}{\Lambda_c^3} g_{3/2}(\lambda_c)}$$, both λ for $T<T_c$ and λ_c for $T=T_c$ are very close to 1

$$= \underline{\left(\frac{T}{T_c}\right)^{3/2}}$$

The rest of the particles are in the ground state:

$$N_0 = N - N_e$$

$$\underline{\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}}, \quad \text{QED}$$