Problem set 13

1 1D Ising model with three particles

$$H = -J \sum_{i=1}^{2} \sigma_i \sigma_{i+1} \quad \text{(no periodic boundaries)}$$

State	Energy/(-J)
+++	2
++-	0
+-+	-2
+--	0
-++	0
-+-	-2
--+	0
---	2

a $$Z = \sum_{\text{states}} e^{-\beta \varepsilon_s} = 2 \cdot e^{-2\beta J} + 2 e^{2\beta J} + 4 e^0 = 4\left(\cosh 2\beta J + 1\right)$$

b $$\langle E \rangle = \sum_{\text{states}} \varepsilon_s \frac{e^{-\beta \varepsilon_s}}{Z} = \frac{2 \cdot (2J) e^{-2\beta J} + 2 \cdot (-2J) e^{2\beta J} + 4 \cdot 0 \cdot e^0}{Z}$$

$$= \frac{-2J \, 4 \sinh 2\beta J}{4(\cosh 2\beta J + 1)}, \quad \text{use } \sinh 2x = 2 \sinh x \cosh x, \quad \cosh 2x + 1 = 2\cosh^2 x$$

$$= -2J \tanh \beta J$$

c $$\langle \sigma_2 \rangle = \sum_{\text{states}} \sigma_2 \frac{e^{-\beta \varepsilon_s}}{Z}$$

$$= \frac{1}{Z}\left[e^{2\beta J} + e^0 - e^{-2\beta J} - e^0 + e^0 + e^{-2\beta J} - e^0 - e^{2\beta J}\right]$$

$$= 0$$

$$\langle \sigma_1 \sigma_3 \rangle = \sum_{\text{states}} \sigma_1 \sigma_3 \frac{e^{-\beta \varepsilon_s}}{Z}$$

$$= \frac{1}{Z}\left[e^{2\beta J} - e^0 + e^{-2\beta J} - e^0 - e^0 + e^{-2\beta J} - e^0 + e^{2\beta J}\right]$$

$$= \frac{1}{Z} \cdot 4\left(\cosh 2\beta J - 1\right), \quad \text{use } \cosh 2x - 1 = \cosh^2 x + \sinh^2 x - (\cosh^2 x - \sinh^2 x) = 2\sinh^2 x$$

$$= \frac{4 \cdot 2 \sinh^2 \beta J}{4 \cdot 2 \cosh^2 \beta J}$$

$$= \tanh^2 \beta J$$

2 Ising model

A lattice of $N+1$ ising spins $\sigma_i = \pm 1$, external field B.
Each of the N last spins couple to σ_0, so

$$H = -B\sum_{i=0}^{N}\sigma_i - J\sum_{i=1}^{N}\sigma_i\sigma_0$$

$$= -(B+\sigma_0 J)\sum_{i=1}^{N}\sigma_i - B\sigma_0$$

a $$Z = \sum_{\text{states}} e^{-\beta\varepsilon_s}$$

$$= \sum_{\sigma_0}\sum_{N\text{-states}} e^{-\beta\varepsilon_s}$$

$$= \underbrace{\sum_{N\text{-states}} e^{\beta B} e^{\beta(B+J)\sum_{i=1}^{N}\sigma_i}}_{\sigma_0=+1} + \underbrace{\sum_{N\text{-states}} e^{-\beta B} e^{\beta(B-J)\sum_{i=1}^{N}\sigma_i}}_{\sigma_0=-1}$$

$$= e^{\beta B}\sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\cdots\sum_{\sigma_N=\pm1} e^{\beta(B+J)\sigma_1} e^{\beta(B+J)\sigma_2}\cdots + e^{-\beta B}\sum_{\sigma_1}\sum_{\sigma_2} e^{\beta(B-J)\sigma_1}\cdots$$

$$= e^{\beta B}\Big(\sum_{\sigma_i=\pm1} e^{\beta(B+J)\sigma_i}\Big)^N + e^{-\beta B}\Big(\sum_{\sigma_i=\pm1} e^{\beta(B-J)\sigma_i}\Big)^N$$

$$= e^{\beta B}\left[2\cosh\beta(B+J)\right]^N + e^{-\beta B}\left[2\cosh\beta(B-J)\right]^N$$

b $$\langle\sigma_i\rangle_{i\neq 0} = \frac{1}{Z}\sum_{\text{states}}\sigma_i e^{-\beta\varepsilon_s}$$

The calculation as above, except the sum with σ_i now becomes a sinh:

$$m = \frac{1}{Z}\left(e^{\beta B}\sum_{\sigma_1=\pm1} e^{\beta(B+J)\sigma_1}\cdots\sum_{\sigma_i=\pm1}\sigma_i e^{\beta(B+J)\sigma_i}\cdots\sum_{\sigma_N=\pm1} e^{\beta(B+J)\sigma_N} + e^{-\beta B}\sum\sum\cdots\right)$$

Define $A = e^{\beta B}\left[2\cosh\beta(B+J)\right]^N$

$C = e^{-\beta B}\left[2\cosh\beta(B-J)\right]^N$.

$$m = \frac{1}{A+C}\Big(A\tanh\beta(B+J) + C\tanh\beta(B-J)\Big)$$

$$\langle \sigma_0 \sigma_i \rangle = \frac{1}{Z} \frac{\partial Z}{\partial(\beta J_i)}$$

$$= \frac{1}{A+C} \Big(e^{\beta B} \left[2\cosh\beta(B+J) \right]^{N-1} 2\sinh\beta(B+J) - e^{-\beta B} \left[2\cosh\beta(B-J) \right]^{N-1} 2\sinh\beta(B-J) \Big)$$

$$= \frac{1}{A+C} \Big(A\tanh\beta(B+J) - C\tanh\beta(\beta-J) \Big)$$

C $\quad H_{ent} = \begin{cases} -\frac{1}{Z}\frac{\partial Z}{\partial \beta} & \text{, def. 1} \\ -\frac{1}{Z}\sum_s \left(B\sum_{i=0}^{N}\sigma_i + J\sum_{i=1}^{N}\sigma_0\sigma_i \right) e^{-\beta E_s} & \text{, def. 2} \end{cases}$

$$= -B\sum_{i=0}^{N}\langle\sigma_i\rangle - J\sum_{i=1}^{N}\langle\sigma_0\sigma_i\rangle$$

$$= -B\langle\sigma_0\rangle - BN\langle\sigma_i\rangle_{i\neq 0} - JN\langle\sigma_0\sigma_i\rangle$$

$$\langle E\rangle = \frac{1}{N+1} H_{ent} \approx \frac{H_{ent}}{N} = -B\langle\sigma_i\rangle_{i\neq 0} - J\langle\sigma_0\sigma_i\rangle$$

Limit $B \to 0$ with J finite:

$A \to [2\cosh\beta J]^N$

$C \to [2\cosh\beta J]^N$

$$\langle\sigma_i\rangle = \langle\sigma_0\rangle + \langle\sigma_i\rangle_{i\neq 0} = \frac{A-C}{A+C} + \frac{A\tanh\beta J + C\tanh(-\beta J)}{A+C} = 0+0 = 0$$

↳ as expected

$$\langle\sigma_0\sigma_i\rangle = \frac{1}{A+C}\left[A\tanh\beta J - C\tanh(-\beta J) \right] = \tanh\beta J$$

Limit B finite, $J \to 0$:

$A \to e^{\beta B}[2\cosh\beta B]^N$

$C \to e^{-\beta B}[2\cosh\beta B]^N$

$\langle\sigma_0\rangle = \frac{A-C}{A+C} = \tanh\beta B$ } equal, as expected

$\langle\sigma_i\rangle_{i\neq 0} = \tanh\beta B$

$$\langle\sigma_0\sigma_i\rangle = \frac{1}{A+C}\left(A\tanh\beta B - C\tanh\beta B \right) = \frac{A-C}{A+C}\tanh\beta B = \tanh^2\beta B$$

↳ correlation through common response to B-field