Lecture 15
Bose gases 1

06.03.2019

Quantum Gases (Module IV)

Bose statistics of photons

Module IV:Quantum gases

on. 6. mar.	Bose statistics of phonons
fr. 8. mar.	Debye theory of phonons
on. 13. mar.	Ideal Bose atoms, Bose-Einstein condensation
fr. 15. mar.	Weakly interacting atoms, Bose Einstein condensation
on. 20. mar.	Fermi ideal gases (Oblig 1)
fr. 22. mar.	Summary and questions

Quantum gas

Consider a system of $N = \sum_j n_j$ free quantum particles with number n_j of particles in each *quantum* state ϵ_j

Fermions: $n_j = 0,1$

Bosons: $n_j = 0,1,2,\cdots$

Canonical partition function:

Conditioned sum weighted by the Boltzmann factor over all microstates with $\{n_j\}$ partition of particles between the energy levels $\{\epsilon_j\}$, such that energy and number of particles in that microstate are

$$E_N = \sum_j n_j \epsilon_j\,, \qquad N = \sum_j n_j$$

$$Z_N = \sum_{\{n_j\}} e^{-\beta E_N} = \sum_{\{n_j\}} e^{-\beta \sum_j \epsilon_j n_j}\,, with\ \sum_j n_j = N \text{ (fixed)}$$

Quantum gas

Consider a system of $N = \sum_j n_j$ quantum particles with number n_j of particles in each quantum state ϵ_j

Fermions: $n_j = 0,1$

Bosons: $n_j = 0,1,2,\cdots$

Grand-canonical partition function:

Unconditioned sum weighted by the Gibbs factor over all microstates with $\{n_j\}$ partition of particles between the energy levels $\{\epsilon_j\}$,

$$\Xi = \sum_{N=0}^{\infty} \sum_{\{n_j\}} e^{-\beta \sum_j (\epsilon_j - \mu) n_j}$$

$$\Xi = \sum_{\{n_j\}} e^{-\beta \sum_j (\epsilon_j - \mu) n_j} = \prod_j \sum_{n_j} e^{-\beta(\epsilon_j - \mu) n_j}$$

Quantum gas: Thermodynamic properties

Grand-canonical partition function:

$$\Xi = \prod_j \left(\frac{1}{1 \pm e^{-\beta(\epsilon_j - \mu)}}\right)^{\mp 1}, \begin{cases} top\ sign:\ fermions \\ bottom\ sign:\ bosons \end{cases}$$

Landau free energy:

$$\Omega(\mathrm{T}, \mathrm{V}, \mu) = -PV = -kT \log \Xi$$

$$\Omega = \mp kT \sum_j \log\left[1 \pm e^{-\beta(\epsilon_j - \mu)}\right]$$

$\sum_j \equiv$ sum over all quantum states

Quantum gas: Themodynamic properties

Pressure:

$$PV = \pm kT \sum_j \log\left(1 \pm e^{-\beta(\epsilon_j - \mu)}\right)$$

Average number of particles:

$$\langle N \rangle = \sum_j \frac{1}{e^{\beta(\epsilon_j - \mu)} \pm 1}$$

Average energy:

$$\langle E \rangle = \sum_j \frac{\epsilon_j}{e^{\beta(\epsilon_j - \mu)} \pm 1}$$

Quantum Gas: Density of states

Replace the sum over quantum states by an integral over energy weighted by the density of states as a function of energy:

$$\sum_j \equiv \int d\epsilon D(\epsilon)$$

Quantum density of states

Lets take for example the average number of particles

$$\langle N\rangle = \sum_j \frac{1}{e^{\beta(\epsilon_j-\mu)} \pm 1}$$

$$= \int d\epsilon \frac{1}{e^{\beta(\epsilon-\mu)} \pm 1} \sum_j \delta(\epsilon_j - \epsilon)$$

$$= \int d\epsilon \frac{1}{e^{\beta(\epsilon-\mu)} \pm 1} D(\epsilon)$$

Density of states: the number of allowed quantum states per unit energy for a particle at a given energy ϵ:

$$D(\epsilon) \equiv \sum_j \delta(\epsilon_j - \epsilon)$$

Density of states: Thermodynamics of quantum gases

$$\sum_j \equiv \int d\epsilon D(\epsilon)$$

Pressure:

$$PV = \pm kT \int d\epsilon D(\epsilon) \log\left(1 \pm e^{-\beta(\epsilon-\mu)}\right)$$

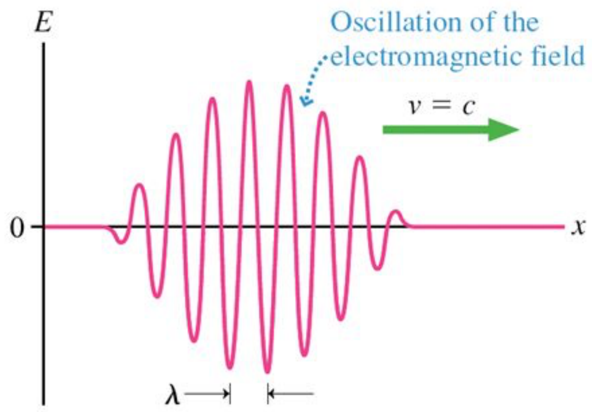
Average number of particles:

$$\langle N \rangle = \frac{1}{\beta}\frac{\partial}{\partial\mu}\log \Xi = \int d\epsilon D(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)} \pm 1} = \int d\epsilon\, D(\epsilon)\langle n \rangle_\epsilon$$

Average energy:

$$\langle E \rangle = \int d\epsilon D(\epsilon) \frac{\epsilon}{e^{\beta(\epsilon-\mu)} \pm 1} = \int d\epsilon\, D(\epsilon)\langle n \rangle_\epsilon \epsilon$$

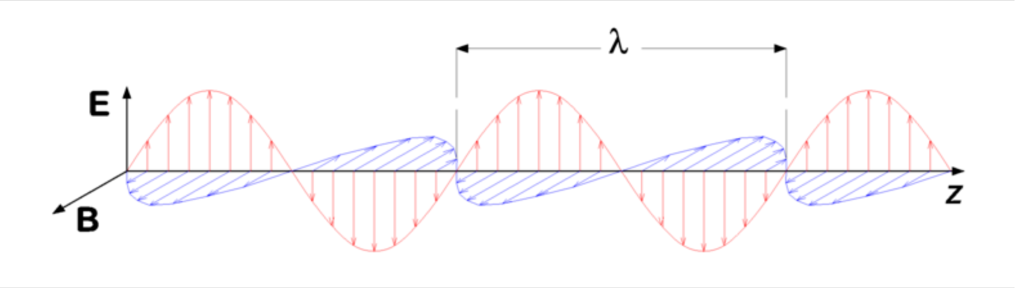
Photons:



Light: traveling electromagnetic (EM) waves

- EM modes are described by
  - wavevector $\vec{k}$, which is restricted to discrete values $\vec{k} = \frac{2\pi}{L}\vec{n}$
  - frequency of an EM mode is $\omega = c|\vec{k}| = ck$
- EM mode has two transverse modes

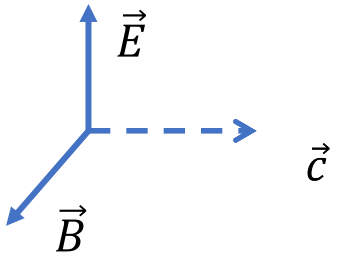
$$\vec{k} \cdot \vec{E} = \mathbf{0}, \qquad \vec{k} \cdot \vec{B} = \mathbf{0}$$



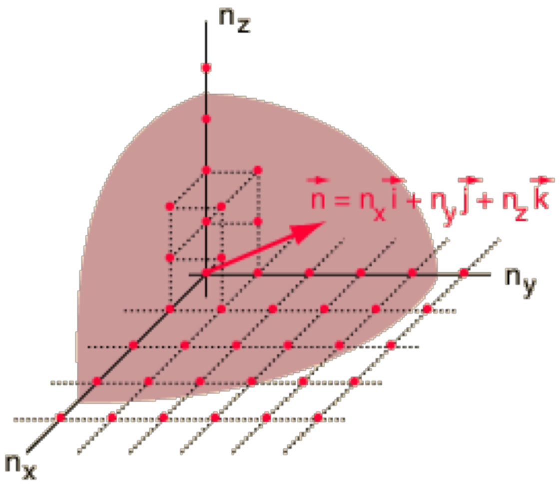
Photons: quanta of light

- Each EM mode is populated by photons, each with a quanta of energy:

$$\epsilon = \hbar kc = \hbar\omega$$



Equilibrium state of a photon gas:



EM modes in an L^3 periodic box

- EM modes are described by
  - wavevector $\vec{k}$, which is restricted to discrete values $\vec{k} = \frac{2\pi}{L}\vec{n}$
  - frequency of an EM mode is $\omega = c|\vec{k}| = ck$
- EM mode is populated by photons, each with a quanta of energy:

$$\epsilon_k = \hbar kc = \hbar\omega$$

Number of photons $\mathrm{n_k}$ occupying EM mode with wavevector $\vec{k} \equiv$ quantum state of the electromagnetic field (quantum state is representated by a state vector $\vec{n}$ or wavevector $\vec{k}$)

Photon gas: uncountable, $\mu \equiv 0$

Grand-canonical partition function:

$$\Xi = \prod_j \sum_{n_j} e^{-\beta(\epsilon_j - \mu)n_j} \rightarrow \Xi = \prod_{\vec{n}} \sum_{n_k} e^{-\beta \epsilon_k n_k}$$

Grand-canonical partition function at a given wavenumber k:

$$\Xi(k) = \sum_{n_k=0}^{\infty} e^{-n_k \beta \hbar c k} = \frac{1}{1 - e^{-\beta \hbar c k}}$$

$$\Xi = \prod_{\vec{n}} \Xi(k) = \prod_{\vec{k}} \frac{1}{1 - e^{-\beta \hbar c k}}$$

Photon gas: $\mu = 0$

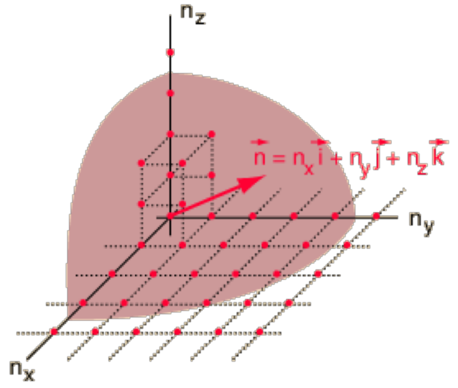
Grand-canonical partition function

$$\Xi = \prod_{\vec{n}} \Xi(k) = \prod_{\vec{n}} \frac{1}{1 - e^{-\beta\hbar c k}}$$

Landau potential:

$$\Omega(\mathrm{T}, \mathrm{V}) = -kT \sum_{\vec{n}} \log \Xi(k) = kT \sum_{\vec{n}} \log\left(1 - e^{-\beta\hbar c k}\right)$$

Photon Density of states



$\vec{k} = \frac{2\pi}{L}\vec{n}$ *is the quantization of the wavevector with the size of the box*

$\vec{n}$ *is a state vector.*

$$\sum_{\vec{n}} = \sum_{n_x}\sum_{n_y}\sum_{n_z} \approx \mathbf{2}\int d\vec{n} = \int dn\, D_n(n)\,, \qquad D_n(n) = 2\times 4\pi n^2$$

$$\Sigma_{\vec{n}} = 2\frac{L^3}{(2\pi)^3}\int d\vec{k} = \int dk\, D(k), \quad D(k) = \frac{V}{\pi^2}k^2$$

- Sum over modes is replaced by an integral in the limit of sufficiently large volume or high enough T, such that there are many states with $\epsilon_n = \frac{2\pi\hbar c}{L} n \leq kT$ contributing to the sum
- Factor 2 accounts for the two transverse polarizations

Density of states

$D_n(n)dn = 2\times 4\pi n^2 dn$ number of modes with quantum number between n and n+dn

- Number of modes with wavenumber between k and $k + dk$ = Number of modes with frequency between ω and $\omega + d\omega$

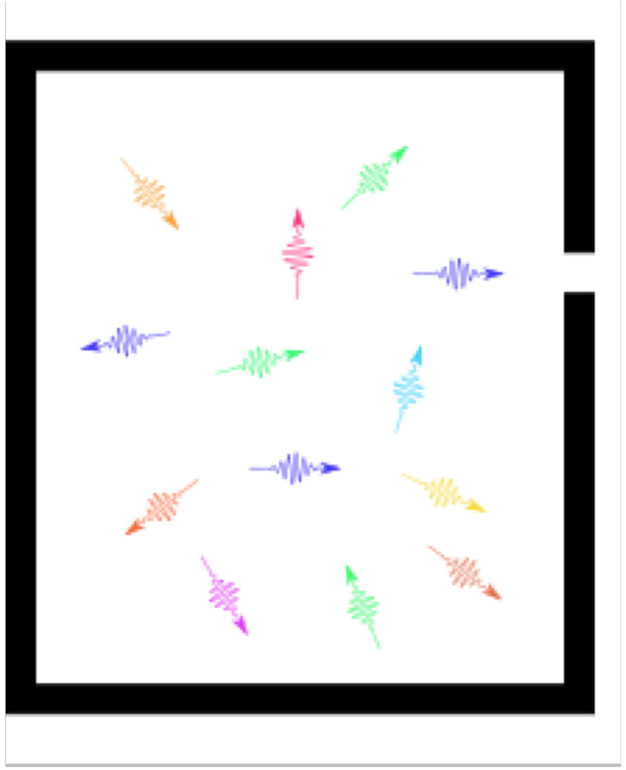
$$D(k)dk = D_\omega(\omega)d\omega \rightarrow D_\omega(\omega) = \frac{V}{\pi^2 c^3}\omega^2$$

- Number of modes with frequency between ω and $\omega + d\omega$ = Number of modes with energy between ϵ and $\epsilon + d\epsilon$

$$D_\omega(\omega)d\omega = D_\epsilon(\epsilon)d\epsilon \rightarrow D_\epsilon(\epsilon) = \frac{V}{\pi^2 \hbar^3 c^3}\epsilon^2$$

Photon gas: thermodynamic properties

Photon Gas in thermodynamic equilibrium: thermal radiation



Blackbody:

Idealized body that absorbs/emits photons of any wavelength and reflects none.

Absorbed energy = emitted energy (at each frequency)

Blackbody radiation:

Backbody radiation is the radiatien emitted by a black body at temperature T

At finite temperature, a blackbody will glow with a «colour» depending on T

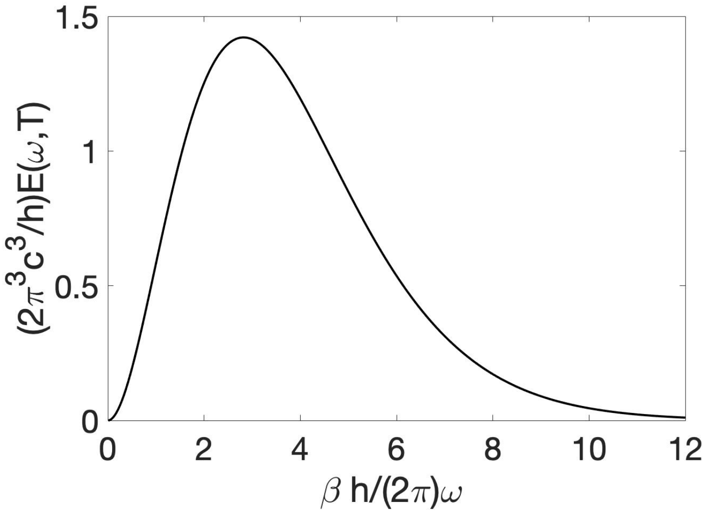
Planck distribution:

Spectral energy distribution of a photon gas

Average energy of a photon gas:

$$\langle E\rangle(T,V) = \int d\epsilon \frac{D(\epsilon)\epsilon}{e^{\beta\epsilon}-1} = \int d\omega \frac{D_\omega(\omega)\hbar\omega}{e^{\beta\hbar\omega}-1}$$

$$= \frac{V\hbar}{\pi^2 c^3}\int_0^\infty d\omega \frac{\omega^3}{e^{\beta\hbar\omega}-1}$$

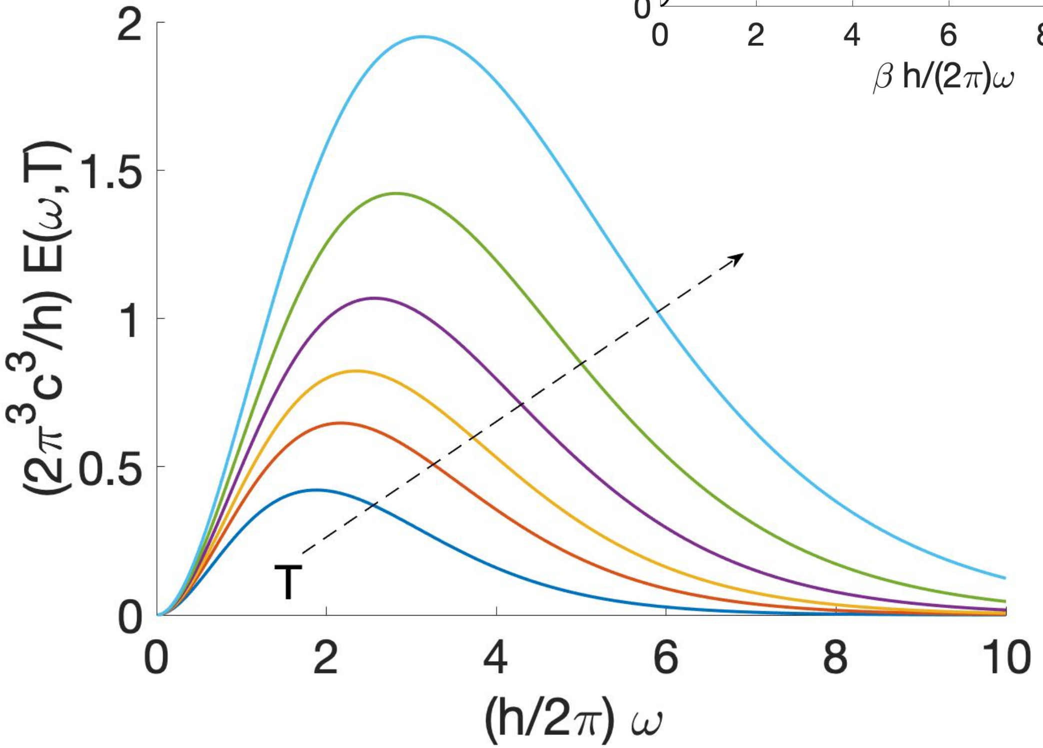


Energy per unit volume at a given frequency ω

(spectral energy density)

$$\frac{\langle E\rangle}{V} = \int d\omega \mathcal{E}(\omega,T)$$

$$\mathcal{E}(\omega,T) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\beta\hbar\omega}-1}$$



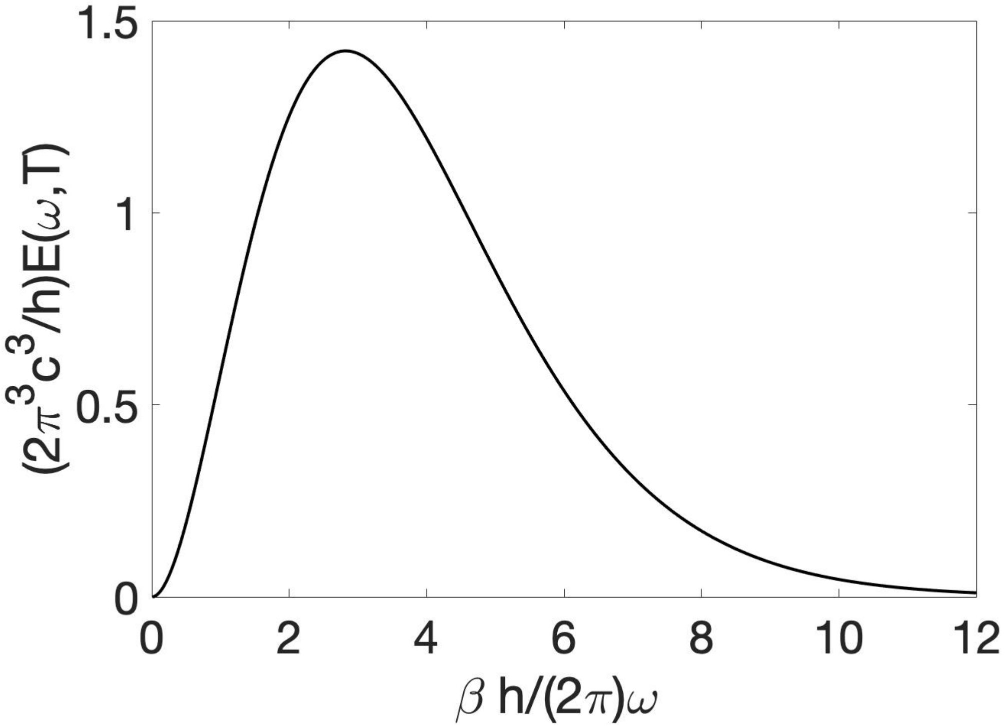
Wien's displacement law

$$\mathcal{E}(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

$$\omega_{max}(T) \approx \zeta \frac{kT}{\hbar}, \zeta \approx 2.822$$

Frequency with the max spectral density

(frequency with maximum light intensity)



$$\frac{d\mathcal{E}(\omega, T)}{d\omega} = 0 \rightarrow$$

$$3 - \zeta = 3e^{-\zeta}, \zeta = \beta\hbar\omega_{max}$$

Planck distribution

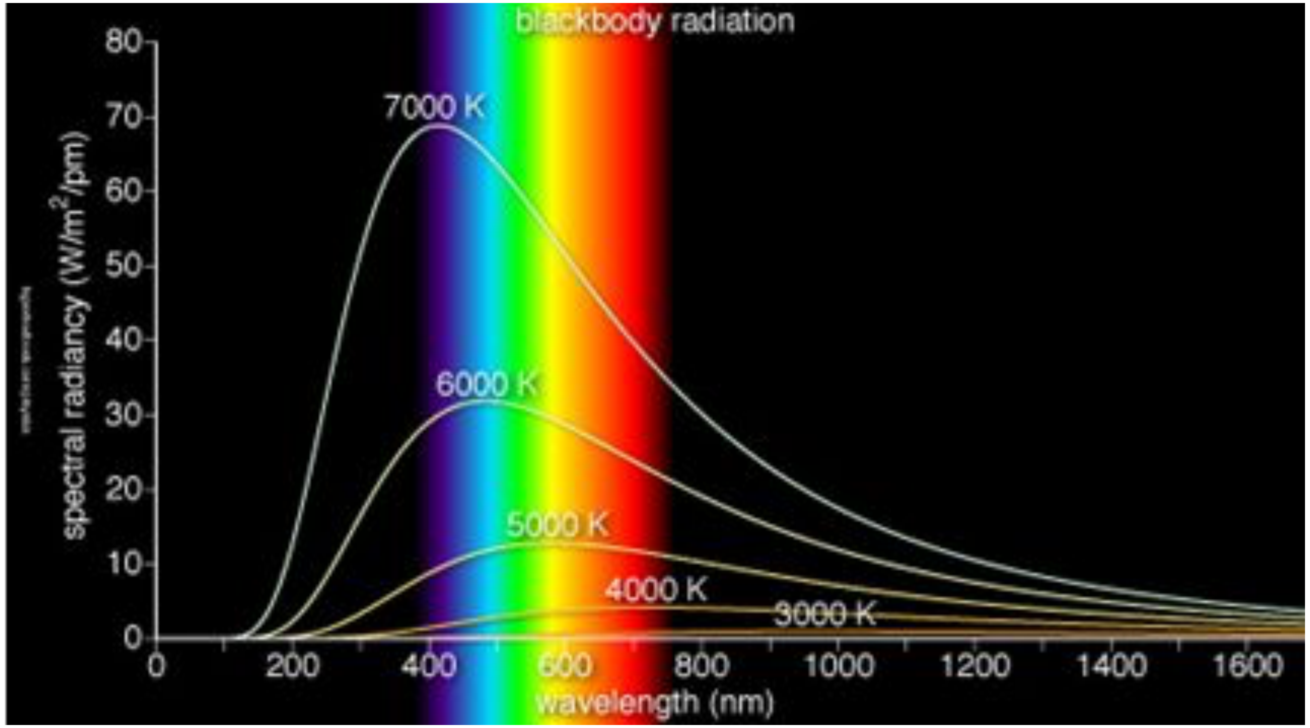
Spectral energy density

$$\frac{\langle E\rangle}{V} = \int d\omega \mathcal{E}(\omega, T)$$

$$\mathcal{E}(\omega, T) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

$$\mathcal{E}(\omega, T) d\omega = u(\lambda) d\lambda$$

$$u(\lambda) = \frac{8h\pi c}{\lambda^5}\frac{1}{e^{\beta hc/\lambda} - 1}$$



Rayleigh-Jeans ultraviolet catastrophe

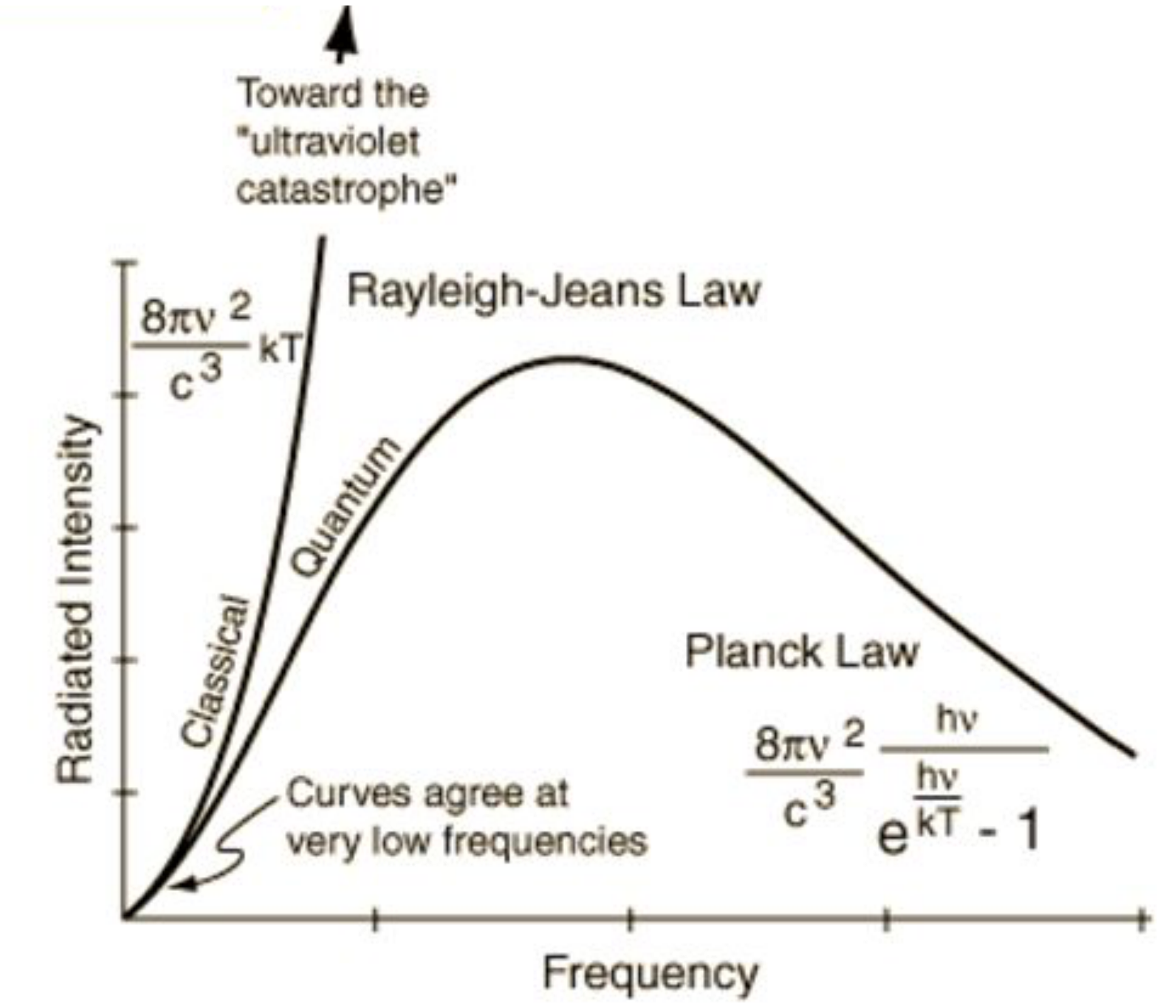
Classical limit $\hbar\omega \ll kT$

$$\mathcal{E}(\omega,T) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\beta\hbar\omega}-1}$$

$$\mathcal{E}(\omega,T) \approx \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{\beta\hbar\omega} \approx kT\,\omega^2$$

Equipartition of the average energy density for each mode →

Energy density increases with frequency leading to the ultraviolet catastrope



Stephan's law: Emission energy density $\sim T^4$

Total energy per unit volume emitted at T

$$\frac{\langle E\rangle}{V} = \frac{\hbar}{\pi^2 c^3}\int_0^\infty d\omega\,\frac{\omega^3}{e^{\beta\hbar\omega}-1}$$

Substitute: $x = \beta\hbar\omega$

$$\frac{\langle E\rangle}{V} = \frac{(kT)^4}{\pi^2 c^3 \hbar^3} I_3, \qquad I_3 = \int_0^\infty dx\,\frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

$$\boldsymbol{\frac{\langle E\rangle}{V} = \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4}$$

Free energy of a photon gas

$Because\, \mu = 0\ for\, photons\ \ F(T,V) \equiv \Omega(\mathrm{T,V})$

$$F(T,V) = kT \int d\omega D(\omega) \log\left(1 - e^{-\beta\epsilon}\right) = \frac{VkT}{\pi^2 c^3} \int d\omega \omega^2 \log\left(1 - e^{-\beta\hbar\omega}\right)$$

Integration by parts

$$F(T,V) = -\frac{V\hbar}{\pi^2 c^3} \int d\omega \omega^3 \frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}$$

$$F(T,V) = -\frac{V\pi^2}{45\hbar^3 c^3} (kT)^4$$

Equation of state

Radiation Pressure

$$PV = -kT \int d\omega D(\omega) \log\left(1 - e^{-\beta\hbar\omega}\right) = -F(T,V)$$

$$F(T,V) = -\frac{V\pi^2}{45\hbar^3 c^3}(kT)^4$$

$$\boldsymbol{P = \frac{\pi^2}{45\hbar^3 c^3}(kT)^4}$$

$$P = \frac{\langle E \rangle}{3V} \rightarrow \boldsymbol{\langle E \rangle = 3\mathrm{PV}}$$

Entropy

$$F(T,V) = -\frac{V\pi^2}{45\hbar^3c^3}(kT)^4$$

$$S(V,T) = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{4V\pi^2k^4}{45\hbar^3c^3}T^3 \sim T^3$$

Heat capacity

$$\frac{\langle E \rangle}{V} = \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4$$

$$\boldsymbol{C_V = \left(\frac{\partial \langle E \rangle}{\partial T}\right)_V = \frac{4V\pi^2 k^4}{15\hbar^3 c^3} T^3 \sim T^3}$$

By comparison with the ideal gas, where $\boldsymbol{C_V = \frac{3Nk}{2}}$ is independent of T

Density of photons at a given T

Average occupation number for the state with frequency ω

$$\langle n_\omega \rangle = \frac{1}{e^{\beta\hbar\omega} - 1}$$

Average number of photons at temperature T

$$\langle N \rangle(T,V) = \int d\omega D(\omega)\langle n_\omega \rangle = \frac{V}{\pi^2 c^3}\int_0^\infty d\omega \frac{\omega^2}{e^{\beta\hbar\omega} - 1}$$

Density of photons at T:

$$\boldsymbol{\rho}(\boldsymbol{T}) = \frac{\langle \boldsymbol{N} \rangle}{\mathbf{V}} = \left(\frac{\boldsymbol{kT}}{\hbar \boldsymbol{c}}\right)^{\mathbf{3}} \frac{\boldsymbol{I_2}}{\boldsymbol{\pi^2}}, \qquad I_2 = \int_0^\infty dx \frac{x^2}{e^x - 1} \approx 2.404$$