Lecture 19

20.03.2018

Ideal Fermi gas

Ideal Fermi gases: Themodynamic properties

- *Pressure:*

$$PV = kT \int_0^\infty d\epsilon\, D(\epsilon) \ln\left(1 + e^{-\beta(\epsilon-\mu)}\right)$$

- *Average number of particles:*

$$\langle N\rangle(T,V,\mu) = \int_0^\infty d\epsilon\, D(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)}+1} \to$$

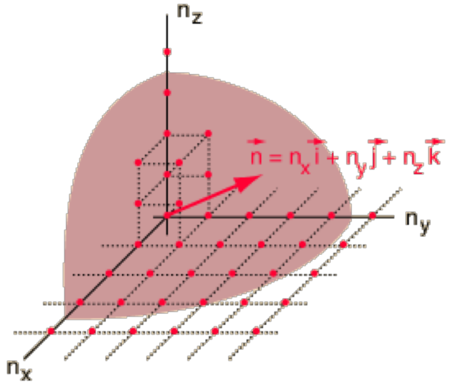
$\frac{\langle N\rangle}{V} = \rho(T,\mu)$ is an indirect equation for finding the chemical potential $\mu(\rho,T)$

- *Average energy:*

$$\langle E\rangle(T,V,\mu) = \int_0^\infty d\epsilon\, D(\epsilon) \frac{\epsilon}{e^{\beta(\epsilon-\mu)}+1} \to$$

Heat capacity $C_V(T) = \left(\frac{\partial\langle E\rangle}{\partial T}\right)_{V,N}$

Ideal Fermi gases: Density of states in 3D



- $\Psi_1(\boldsymbol{r}) = e^{\frac{2\pi i}{L}\boldsymbol{n}\cdot\boldsymbol{r}}$ 1-particle wave function
- Each fermion (i.e. electron) has a spin moment = $\pm\frac{1}{2}$
- Energy levels a fermion in a box $V = L^3$ with periodic boundary conditions:

$$\epsilon_n = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 n^2,$$

Number of available states between a mode with n between n *and* $n + dn$: $D(n)dn = 2\times 4\pi n^2 dn$

$$\sum_{\boldsymbol{n}} := 2\times\int dn\ 4\pi n^2 = \int dn\ D(n)$$

Density of states corresponding to energy ϵ:

$$D(\epsilon) = D(n)\frac{dn}{d\epsilon} \rightarrow D(\epsilon) = 2\,\frac{V}{\sqrt{2}\pi^2}\frac{m^{3/2}}{\hbar^3}\epsilon^{\frac{1}{2}}$$

The difference with respect to the density of states of bosons is the spin degeneracy of the energy levels (hence the extra factor of 2).

Different conventions: Sometimes the spin degeneneracy is not included in the density of states, and appears separately on the integral expressions

Pressure and average energy

- $P = kT\frac{\sqrt{2}}{\pi^2}\frac{m^{3/2}}{\hbar^3}\int_0^\infty d\epsilon\ \epsilon^{1/2}\ln\left(1+e^{-\beta(\epsilon-\mu)}\right)$

$$P = kT\frac{\sqrt{2}}{\pi^2}\frac{m^{\frac{3}{2}}}{\hbar^3}\frac{2}{3}\int_0^\infty d\epsilon\frac{d}{d\epsilon}\left(\epsilon^{\frac{3}{2}}\right)\ln\left(1+e^{-\beta(\epsilon-\mu)}\right)$$

$$P = \frac{2}{3}\frac{\sqrt{2}}{\pi^2}\frac{m^{\frac{3}{2}}}{\hbar^3}\int_0^\infty d\epsilon\frac{\epsilon^{\frac{3}{2}}}{e^{\beta(\epsilon-\mu)}+1} = \frac{2}{3}\frac{\langle E\rangle}{V}$$

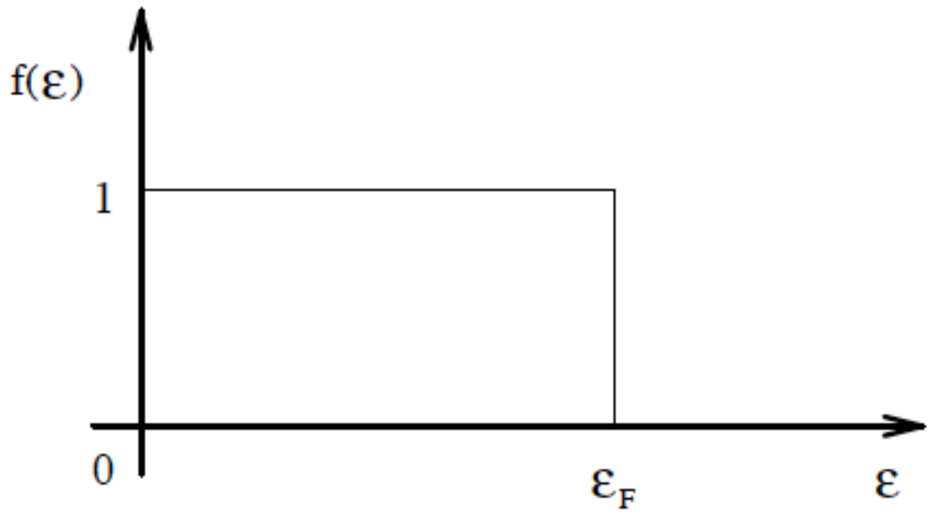
- $\langle E\rangle = \frac{3}{2}PV$

  The same relationship between energy density and pressure holds for the ideal bose gas

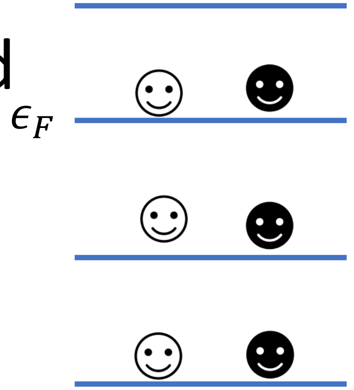
  General expression for non-relativistic quantum ideal gas (independent of $\langle n\rangle(\epsilon)$)

Fermi distribution at T=0 K

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \rightarrow_{T\rightarrow 0} \begin{cases} 1, & \epsilon < \mu \\ 0, & \epsilon > \mu \end{cases}$$

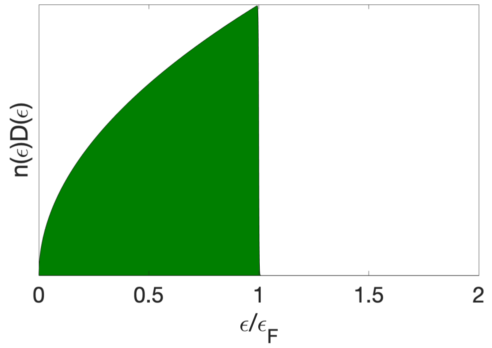


$\epsilon_F \equiv \mu$ Fermi energy level below which all states are occupied (degenerate gas– degenerate ground state)



Determined by the gas density. $\epsilon_F = \epsilon_F(\rho)$

Fermi energy at T=0 K



$$\rho = \frac{1}{V}\int_0^{\infty} d\epsilon\, D_\epsilon(\epsilon)\langle n\rangle_\epsilon =_{T=0K} \frac{\sqrt{2}}{\pi^2}\frac{m^{3/2}}{\hbar^3}\int_0^{\epsilon_F} d\epsilon\, \epsilon^{\frac{1}{2}}$$

$$\rho(\epsilon) = \frac{1}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon_F^{\frac{3}{2}}$$

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2\rho)^{\frac{2}{3}}$$

$$\epsilon_F = kT_F \rightarrow T_F = \frac{\hbar^2}{2mk}(3\pi^2\rho)^{\frac{2}{3}}$$ Fermi temperature

Average energy at T=0 K

$$\frac{\langle E\rangle_0}{V} = \frac{\sqrt{2}}{\pi^2}\frac{m^{3/2}}{\hbar^3}\int_0^{\epsilon_F} d\epsilon\, \epsilon^{\frac{3}{2}} = \frac{1}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon_F^{\frac{5}{2}}$$

$$\frac{\langle E\rangle_0}{V} = \frac{\hbar^2}{10\pi^2 m^2}(3\pi^2\rho)^{\frac{5}{3}}$$

Average kinetic energy of the Fermi gas is nonzero even at $\mathrm{T} = 0K$

In a Fermi gas, the fermi particles must occupy excited states even at $\mathrm{T} = 0K$ due to the Pauli exclusion principle

Exclusion Pressure at T=0 K

- *Determined directly from the energy density*

$$\frac{\langle E\rangle_0}{V} = \frac{3}{2}P_0 = \frac{\hbar^2}{10\pi^2 m^2}(3\pi^2\rho)^{\frac{5}{3}}$$

$$P_0 = \frac{\hbar^2}{15\pi^2 m^2}(3\pi^2\rho)^{\frac{5}{3}} > 0$$

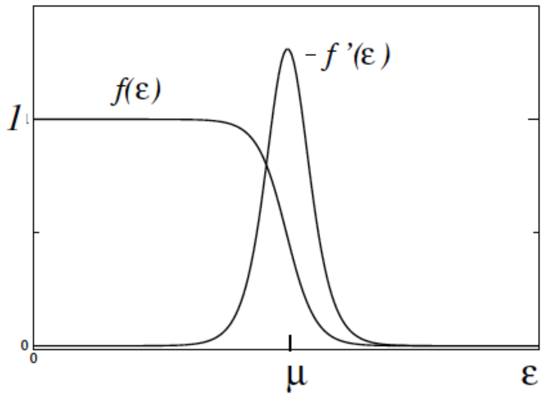
Quantum pressure of a fermi gas: it keeps degenerate starts ($T < T_F$) from collapsing under the gravitational pull

Denenerate ideal Fermi gas $T < T_F$

The Fermi temperature is most often much larges than the gas temperature

Therefore, even though the fermi gas is at finite temperature, it behaves as if it was a near zero temperature when $T \ll T_F$

$$\rho = \frac{\sqrt{2}}{\pi^2}\frac{m^{\frac{3}{2}}}{\hbar^3}F\left(\frac{1}{2}\right)$$



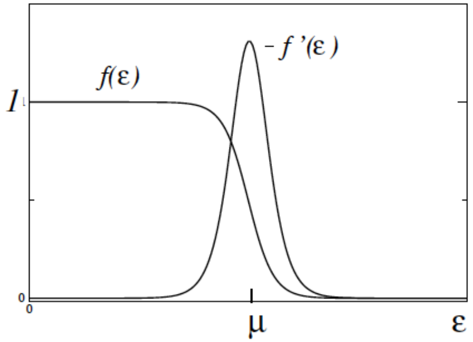
$$\frac{\langle E\rangle}{V} = \frac{\sqrt{2}}{\pi^2}\frac{m^{\frac{3}{2}}}{\hbar^3}F\left(\frac{3}{2}\right)$$

$$F(a) = \int_0^\infty d\epsilon\, \epsilon^a f(\epsilon) = -\frac{1}{a+1}\int_0^\infty d\epsilon\, \epsilon^{a+1} f'(\epsilon) = \frac{\beta}{a+1}\int_0^\infty d\epsilon\, \frac{\epsilon^{a+1} e^{\beta(\epsilon-\mu)}}{(e^{\beta(\epsilon-\mu)}+1)^2}$$

$f'(\epsilon)$ *is peaked around* $\epsilon = \mu > 0$

$$F(a) = -\frac{1}{a+1}\int_{-\infty}^\infty d\epsilon\, \epsilon^a f'(\epsilon) =_{x=\beta(\epsilon-\mu)} \frac{1}{a+1}\int_{-\infty}^\infty dx\, \frac{(\mu+kTx)^{a+1}e^x}{(e^x+1)^2}$$

Denenerate ideal Fermi gas $T < T_F$



Sommerfeld expansion: $\frac{kT}{\mu} \ll 1$

$$F(a) = -\frac{1}{a+1}\int_{-\infty}^{\infty} d\epsilon\, \epsilon^a f'(\epsilon) =_{x=\beta(\epsilon-\mu)} \frac{1}{a+1}\int_{-\infty}^{\infty} dx\, (\mu + kTx)^{a+1} \frac{e^x}{(e^x+1)^2}$$

$$(\mu + kTx)^{a+1} = \mu^{a+1}\left(1 + \frac{kT}{\mu}x\right)^{a+1} \approx \mu^{a+1}\left(1 + (a+1)\frac{kT}{\mu}x + \frac{a(a+1)}{2}\left(\frac{kT}{\mu}\right)^2 x^2 + \cdots\right)$$

$$F(a) = \frac{\mu^{a+1}}{a+1}\left(\int_{-\infty}^{\infty} dx\, f'(x) + (a+1)\frac{kT}{\mu}\int_{-\infty}^{\infty} dx\, \underbrace{x f'(x)}_{odd\ fct} + \frac{a(a+1)}{2}\left(\frac{kT}{\mu}\right)^2 \int_{-\infty}^{\infty} dx\, x^2 f'(x) + \cdots\right)$$

$$F(a) = \frac{\mu^{a+1}}{a+1}\left(1 + \frac{\pi^2}{6}a(a+1)\left(\frac{kT}{\mu}\right)^2 + \cdots\right)$$

Denenerate ideal Fermi gas $T < T_F$

Sommerfeld expansion: $\frac{kT}{\mu} \ll 1$

$$F(a) = \frac{\mu^{a+1}}{a+1}\left(1 + \frac{\pi^2}{6} a(a+1)\left(\frac{kT}{\mu}\right)^2 + \cdots\right)$$

Applying this expansion to density and mean energy

$$\rho = \frac{1}{3\pi^2}\frac{(2m)^{\frac{3}{2}}}{\hbar^3}\mu^{3/2}\left(1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right)$$

$$\frac{\langle E\rangle}{V} = \frac{1}{5\pi^2}\frac{(2m)^{\frac{3}{2}}}{\hbar^3}\mu^{5/2}\left(1 + \frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right)$$

Denenerate ideal Fermi gas: chemical potential μ

Sommerfeld expansion: $\frac{kT}{\mu} \ll 1$

$$\rho = \frac{1}{3\pi^2}\frac{(2m)^{\frac{3}{2}}}{\hbar^3}\mu^{3/2}\left(1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right)$$

Using that $\rho = \frac{1}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon_F^{\frac{3}{2}}$

$$\epsilon_F^{\frac{3}{2}} = \mu^{3/2}\left(1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right)$$

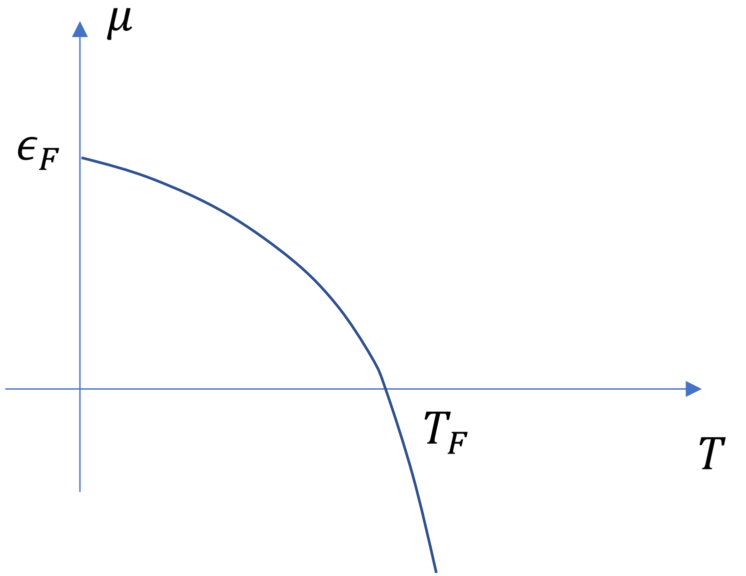
Denenerate ideal Fermi gas: chemical potential μ

Sommerfeld expansion: $\frac{kT}{\mu} \ll 1$

$$\epsilon_F^{\frac{3}{2}} = \mu^{3/2}\left(1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right)$$

$$\mu = \epsilon_F\left(1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right)^{-2/3}$$

$$\mu = \epsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \cdots\right)$$



Denenerate ideal Fermi gas: average energy

Sommerfeld expansion: $\frac{kT}{\mu} \ll 1$

$$\mu = \epsilon_F \left(1 - \frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \cdots\right)$$

$$\frac{\langle E \rangle}{V} = \frac{1}{5\pi^2}\frac{(2m)^{\frac{3}{2}}}{\hbar^3}\mu^{5/2}\left(1 + \frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right)$$

$$= \frac{1}{5\pi^2}\frac{(2m)^{\frac{3}{2}}}{\hbar^3}\epsilon_F^{5/2}\left(1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \cdots\right)$$

Denenerate ideal Fermi gas: heat capacity

Sommerfeld expansion: $\frac{kT}{\mu} \ll 1$

$$\frac{\langle E \rangle}{V} = \frac{1}{5\pi^2}\frac{(2m)^{\frac{3}{2}}}{\hbar^3}\epsilon_F^{5/2}\left(1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \cdots\right)$$

$$\mathrm{C_V} = V\,\frac{(2m)^{\frac{3}{2}}}{\hbar^3}\epsilon_F^{\frac{1}{2}}\frac{k^2}{6}T$$

Using $\frac{\langle N \rangle}{V} = \rho = \frac{1}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon_F^{\frac{3}{2}} \rightarrow$

$$\mathrm{C_V} = \frac{1}{2}\langle N \rangle \pi^2 k\,\frac{T}{T_F}$$

Degenerate electron gas in most metals has a Fermi temperature $T_F = \frac{\epsilon_F}{k} \sim 10^4 K$ *is much larger than the room temperature*

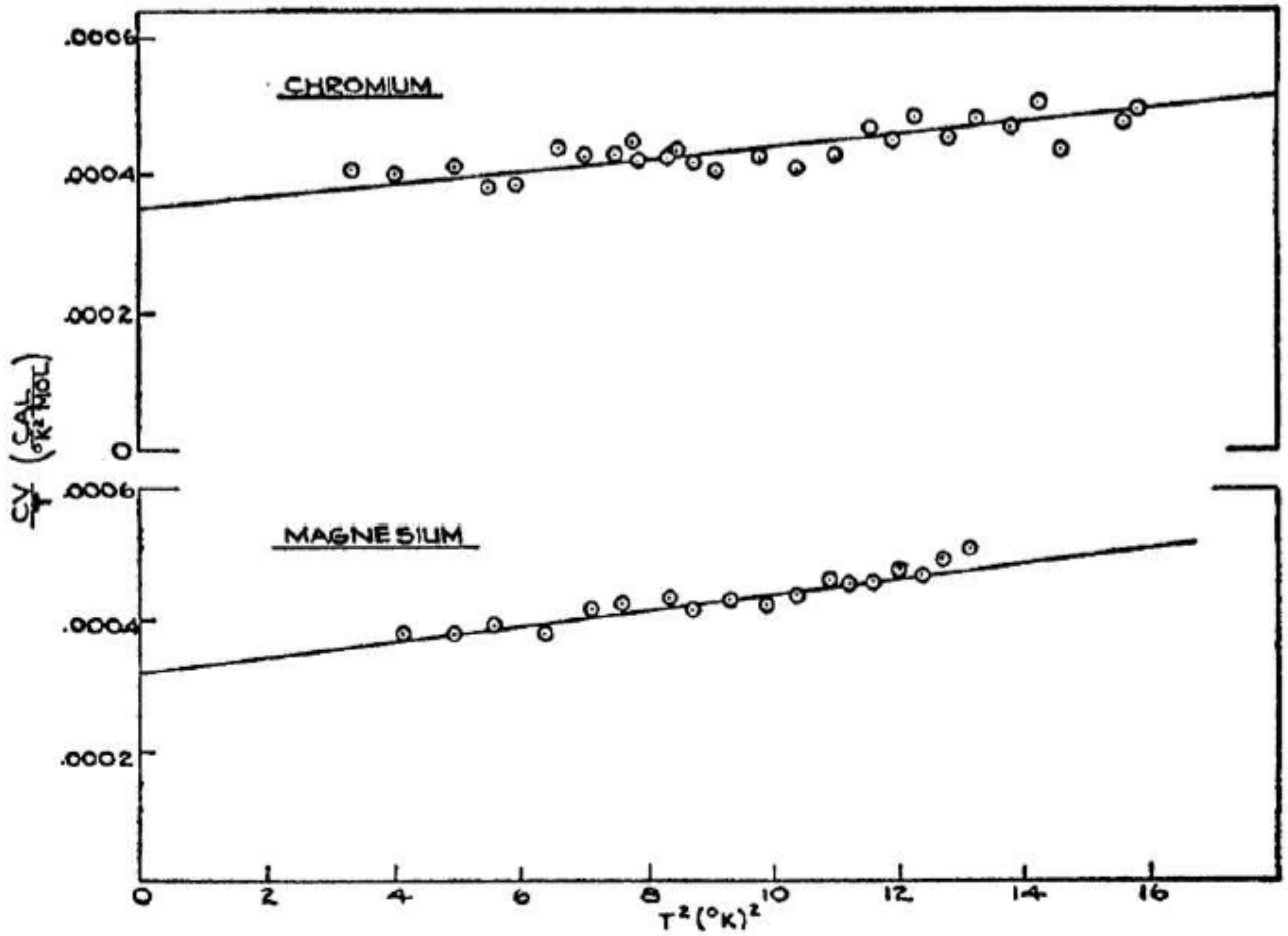


Figure 5.20: The specific heat C_V/T of a solid as function of T^2 has an intercept determined by the electrons and a slope determined by the phonons. This specific heat is measured for Chromium and Magnesium by *S. A. Friedberg, I. Estermann, and J. E. Goldman* 1951.

Denenerate ideal Fermi gas: pressure

Sommerfeld expansion: $\frac{kT}{\mu} \ll 1$

$$\frac{\langle E \rangle}{V} = \frac{1}{5\pi^2}\frac{(2m)^{\frac{3}{2}}}{\hbar^3}\epsilon_F^{5/2}\left(1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \cdots\right) = \frac{3}{2}PV$$

$$\mathrm{P} = \frac{2}{15\pi^2 V}\frac{(2m)^{\frac{3}{2}}}{\hbar^3}\epsilon_F^{5/2}\left(1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \cdots\right)$$

$$P_0 = \frac{\hbar^2}{15\pi^2 m^2}(3\pi^2\rho)^{\frac{5}{3}}$$

High temperature limit (classical ideal gas): $T > T_F$

Pressure equation of state:

$$P = \frac{1}{3\pi^2}\frac{(2m)^{3/2}}{\hbar^3}\int_0^\infty d\epsilon\,\frac{\epsilon^{\frac{3}{2}}}{\lambda^{-1}e^{\beta\epsilon}+1}, \qquad \lambda = e^{\beta\mu} < 1$$

$$\frac{P}{kT} = \Lambda^{-3}(T)\,\frac{8}{3\sqrt{\pi}}\int_0^\infty dx\,\frac{x^{\frac{3}{2}}}{\lambda^{-1}e^{x}+1}, \qquad x = \beta\epsilon$$

Similarily, we can write the density equation with a dimensionless integral form

$$\rho = \frac{\sqrt{2}}{\pi^2}\frac{m^{3/2}}{\hbar^3}\int_0^\infty d\epsilon\,\frac{\epsilon^{\frac{1}{2}}}{\lambda^{-1}e^{\beta\epsilon}+1}$$

$$\rho(T,\lambda) = \Lambda^{-3}(T)\,\frac{4}{\sqrt{\pi}}\int_0^\infty dx\,\frac{x^{\frac{1}{2}}}{\lambda^{-1}e^{x}+1}$$

Density and chemical potential: $T > T_F$

The density equation determines the chemical potential as a function of temperature and density

$$\rho(T,\lambda) = \Lambda^{-3}(T)\frac{4}{\sqrt{\pi}}\int_0^\infty dx \frac{x^{\frac{1}{2}}}{\lambda^{-1}e^x + 1}, \qquad \lambda = e^{\beta\mu} < 1$$

Taylor expand the integrand with respect to the fugacity as the expansion parameter $\lambda < 1$

$$\rho(T,\lambda) = \Lambda^{-3}(T)\frac{4}{\sqrt{\pi}}\int_0^\infty dx\, \lambda x^{\frac{1}{2}} e^{-x}(1 - \lambda e^{-x} + \cdots)$$

$$\rho = \Lambda^{-3}(T)\frac{4}{\sqrt{\pi}}\lambda\left[\int_0^\infty dx\, x^{\frac{1}{2}} e^{-x} - \lambda\int_0^\infty dx\, x^{\frac{1}{2}} e^{-2x} + \cdots\right]$$

$$\rho = \Lambda^{-3}(T)\frac{4}{\sqrt{\pi}}\lambda\left[\frac{\sqrt{\pi}}{2} - 2^{-\frac{5}{2}}\sqrt{\pi}\lambda + \cdots\right]$$

$$\rho = 2\Lambda^{-3}(T)\,\lambda\left[1 - 2^{-3/2}\lambda + \cdots\right]$$

Invert the series expansion to find the fugacity in terms of density:

$$\lambda(T) = \frac{\Lambda^3(T)\rho}{2}\left[1 + 2^{-\frac{5}{2}}\Lambda^3(T)\rho - \cdots\right]$$

High temperature limit of pressure: $T > T_F$

In the high T limit, the pressure can be written as a virial expansion with respect to density dependency.

Expand the pressure in powers of fugacity:

$$\frac{P}{kT} = \Lambda^{-3}(T)\frac{8}{3\sqrt{\pi}}\lambda\left[\int_0^\infty dx\, x^{\frac{3}{2}}e^{-x} - \lambda\int_0^\infty dx\, x^{\frac{3}{2}}e^{-2x} + \cdots\right]$$

$$\frac{P}{kT} = \Lambda^{-3}(T)\frac{8}{3\sqrt{\pi}}\lambda\left[\frac{3\sqrt{\pi}}{4} - \lambda\frac{3\sqrt{\pi}}{16\sqrt{2}} + \cdots\right]$$

$$\frac{P}{kT} = 2\Lambda^{-3}(T)\lambda\left[1 - 2^{-\frac{5}{2}}\lambda + \cdots\right]$$

Inserting the dependence of fugacity on density and keeping only the first two terms:

$$\lambda(T) = \frac{\Lambda^3(T)\rho}{2}\left[1 + 2^{-5/2}\Lambda^3\rho - \cdots\right]$$

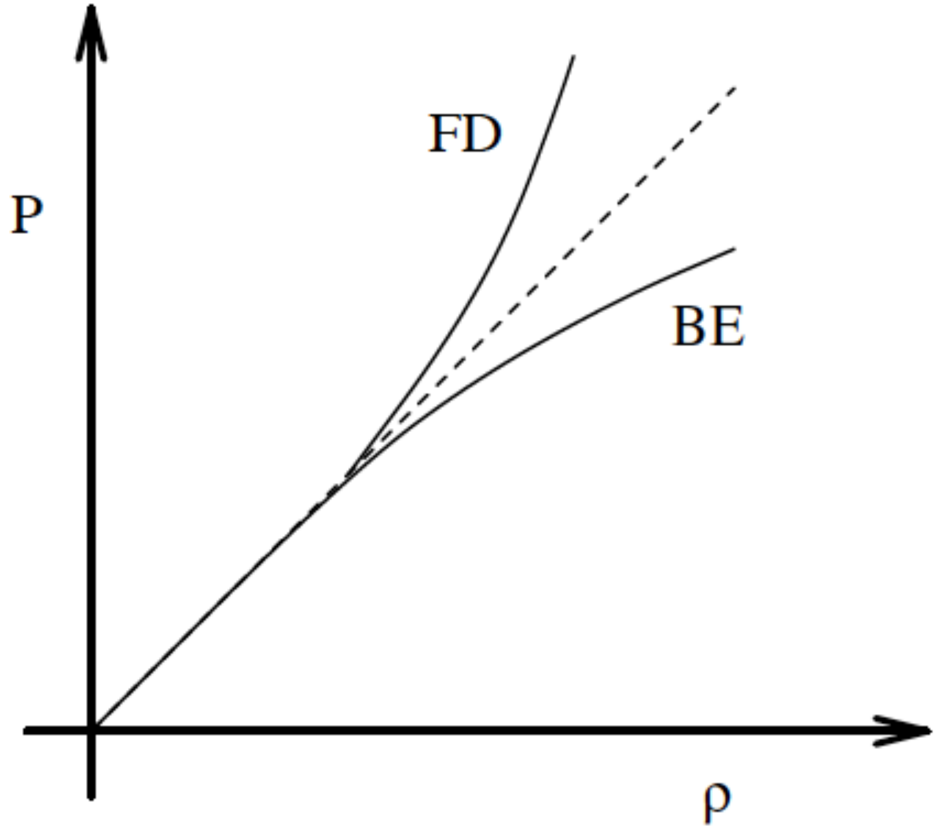
$$\frac{\mathrm{P}}{kT} = \rho\left(1 + 2^{-\frac{7}{2}}\Lambda^3(T)\rho\right), \quad B_2(T) = 2^{-\frac{7}{2}}\Lambda^3(T) > 0$$

The positive second virial coefficient means that pressure is larger than the ideal gas pressure due to statistical repelling forces

Equation of state for quantum gases: $high\,T$

$$P_{\text{fermions}} \approx kT\rho\,\left(1 + 2^{-\frac{7}{2}}\Lambda^3\rho\right)$$

$$P_{\text{boso}ns} \approx kT\rho\,\left(1 - 2^{-\frac{5}{2}}\Lambda^3\rho\right)$$



Nonzero second virial coeff. $B_2(T) \neq 0$

Bosons: $B_2(T) < 0$ *statistical attraction*

Fermions: $B_2(T) > 0$ *statistical repulsion*

Example of degerate Fermi gas: Degenerate dwarf

Consider dwarf start of radius R and mass $M \approx M_{Sun}$ (dominated by nucleons)

mass density $\rho = \frac{3M}{4\pi R^3}$ is very high (like Sun's mass collapsed within Earth's volume).

Density of electrons $n_e \approx \frac{\rho}{m_N} \approx 10^{30}\ cm^{-3}, m_M$= nucleon mass

The corresponding Fermi temperature is

$T_F = \frac{\hbar^2}{2m_e k}(3\pi^2 n_e)^{\frac{2}{3}} \sim 4.3\times 10^9 K \gg 10^7 K$, the typical dwarf T

$T \ll T_F$ regime where the electron gas is degenerate; we can neglect finite temperatyre corrections and treat the fermi gas as if it were at 0K

Degenerate pressure of a dwarf

Suppose the electrons are non-relativisitic, then the pressure (assumed uniform) is

$$P_e = \frac{2}{3}\frac{\langle E\rangle}{V} = \frac{2}{15\pi^2}\frac{(2m_e)^{\frac{3}{2}}}{\hbar^3}\epsilon_F^{5/2}$$

$$\epsilon_F = \frac{\hbar^2}{2m_e}(3\pi^2 n_e)^{\frac{2}{3}}$$

$$P_e = \frac{\hbar^2}{15\pi^2 m_e}(3\pi^2 n_e)^{\frac{5}{3}}, \qquad \rho = \frac{3M}{4\pi R^3}$$

$$n_e = \frac{3}{4\pi R^3}\frac{M}{m_N}$$

$$P_e = \frac{3\hbar^2}{20\pi m_e}(9\pi)^{\frac{3}{2}}\left(\frac{M}{m_N}\right)^{\frac{5}{3}} R^{-5}$$

This quantum pressure has to balance the gravitational inwards pressure

Gravitational pressure

gravitational inwards pressure

$$P_g = -\frac{\partial U_g}{\partial V} = -\frac{dU_g}{dR}\frac{dR}{dV} = -\frac{1}{4\pi R^2}\frac{dU_g}{dR}$$

The gravitational potential is $U_g = -\frac{3}{5}\frac{GM}{R}$

$$P_g = \frac{3GM^3}{4\pi} R^{-4}$$

Size of a dwarf star

$T < T_F$ regime where the electron gas is degenerate and we can neglect high T corrections

Equilibrium size of a dwarf

$$P_g = P_e$$

$$\frac{3GM^3}{4\pi}R^{-4} = \frac{3\hbar^2}{20\pi m_e}(9\pi)^{\frac{3}{2}}\left(\frac{M}{m_N}\right)^{\frac{5}{3}}R^{-5} \rightarrow$$

$$R = \frac{1}{5}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\frac{\hbar^2}{Gm_e m_N^2}\left(\frac{m_N}{M}\right)^{\frac{1}{3}} \sim 5100\ km\ (\sim 6300\ km\ for\ Earth)$$