Lecture 20

22.03.2018

Ideal quantum gases

Recap module IV

Ideal quantum gas

Consider a system of $N = \sum_j n_j$ fre. quantum particles with number n_j of particles in each energy state ϵ_j

Fermions: $n_j = 0,1$

Bosons: $n_j = 0,1,2, \cdots$

Grand-canonical partition function:

Unconditioned sum weighted by the Gibbs factor over all microstates with $\{n_j\}$ partition of particles between the energy levels $\{\epsilon_j\}$,

$$\Xi = \sum_{N=0}^{\infty} \sum_{\{n_j\}} e^{-\beta \sum_j (\epsilon_j - \mu) n_j}$$

$$\Xi = \sum_{\{n_j\}} e^{-\beta \sum_j (\epsilon_j - \mu) n_j} = \prod_j \sum_{n_j} e^{-\beta (\epsilon_j - \mu) n_j}$$

Thermodynamic properties

Grand-canonical partition function:

$$\Xi = \prod_j \left(\frac{1}{1 \pm e^{-\beta(\epsilon_j - \mu)}} \right)^{\mp 1}, \begin{cases} top \; sign: \; fermions \\ bottom \; sign: \; bosons \end{cases}$$

Landau free energy:

$$\Omega(\mathrm{T}, \mathrm{V}, \mu) = -PV = -kT \log \Xi$$

$$\Omega = \mp kT \sum_j \log \left[1 \pm e^{-\beta(\epsilon_j - \mu)} \right]$$

$\sum_j \equiv \int_0^\infty d\epsilon \, D_\epsilon(\epsilon), \quad D_\epsilon(\epsilon)$ is the density of state in a unit energy interval and determined by the system Hamiltonian and spacial dimension

Thermodynamics of ideal quantum gases

Pressure $P(T, \mu)$

$$PV = \pm kT \int d\epsilon D(\epsilon) \log\left(1 \pm e^{-\beta(\epsilon-\mu)}\right) \qquad (1)$$

Equation for the average number of particles provides the relationship between the density ρ and chemical potential μ:

$$\langle N \rangle(T, V, \mu) = \int d\epsilon D(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)} \pm 1} = \int d\epsilon\, D(\epsilon)\langle n \rangle_\epsilon \qquad (2)$$

Average energy:

$$\langle E \rangle(T, V, \mu) = \int d\epsilon D(\epsilon) \frac{\epsilon}{e^{\beta(\epsilon-\mu)} \pm 1} = \int d\epsilon\, D(\epsilon)\langle n \rangle_\epsilon \epsilon \qquad (3)$$

For systems with non-zero μ, we use Eq. (2) as the equation that determines $\mu(\rho)$ and eliminate the μ dependence in Eqs. (1) and (3), so that we can find the equation of state $P(T, \rho)$ and the energy $E(T, \rho)$

Photons:

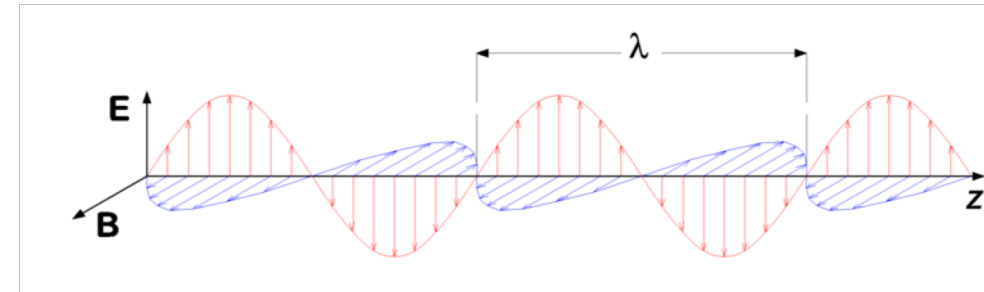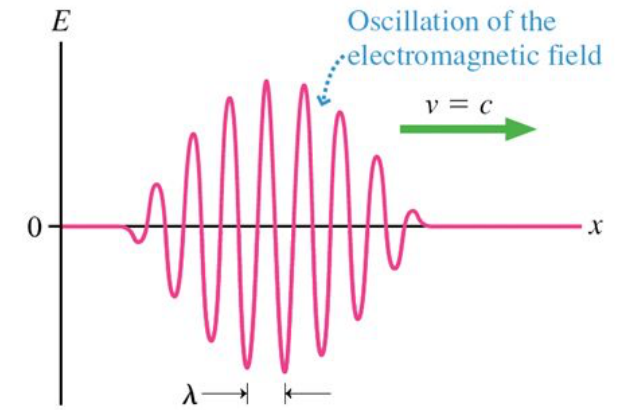Light: traveling electromagnetic (EM) waves

- EM modes are described by

  ➤ wavevector $\vec{k}$, which is restricted to discrete values $\vec{k} = \frac{2\pi}{L}\vec{n}$

  ➤ frequency of an EM mode is $\omega = c|\vec{k}| = ck$
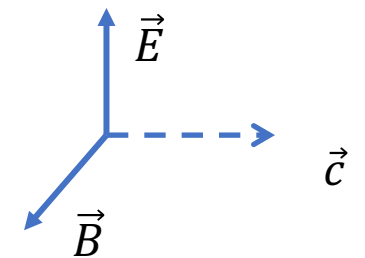
- EM mode has two transverse modes
$$\vec{k} \cdot \vec{E} = \mathbf{0}, \qquad \vec{k} \cdot \vec{B} = \mathbf{0}$$

Photons: quanta of light

- Each EM mode is populated by photons, each with a quanta of energy:

$$\epsilon = \hbar kc = \hbar\omega$$

Photon gas: uncountable photons $\mu \equiv 0$

Grand-canonical partition function:

$$\Xi = \prod_{\vec{n}} \frac{1}{1 - e^{-\beta\hbar ck}}$$

Landau potential:

$$\Omega(\mathrm{T,V}) = -PV = kT \sum_{\vec{n}} \ln\left(1 - e^{-\beta\hbar ck}\right) = kT \int dn\, D(n) \ln\left(1 - e^{-\frac{\beta hc}{L}n}\right)$$

Density of states

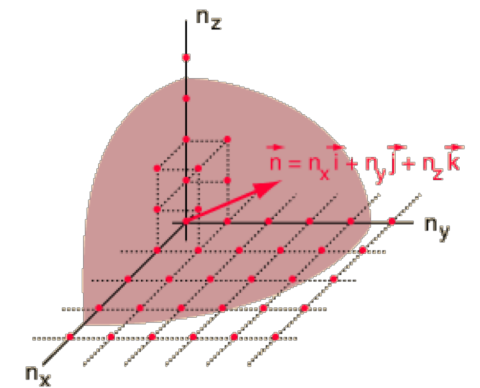$D(n)dn = 2 \times 4\pi n^2 dn$ number of modes with quantum number between n and n+dn

$D(n)dn = D_k(k)dk = D_\omega(\omega)d\omega = D_\epsilon(\epsilon)\epsilon$

- $D_k(k) = \frac{V}{\pi^2}k^2, \quad D_\omega(\omega) = \frac{V}{\pi^2 c^3}\omega^2, \quad D_\epsilon(\epsilon) = \frac{V}{\pi^2\hbar^3 c^3}\epsilon^2$

Pressure: $\quad PV = -kT \int d\epsilon D(\epsilon) \log\left(1 - e^{-\beta\epsilon}\right)$

Average number of particles: $\quad \langle N \rangle = \int d\epsilon\, D(\epsilon)\langle n \rangle_\epsilon = \int d\epsilon D(\epsilon) \frac{1}{e^{\beta\epsilon} - 1}$

Average energy: $\quad \langle E \rangle = \int d\epsilon\, D(\epsilon)\langle n \rangle_\epsilon \epsilon = \int d\epsilon D(\epsilon) \frac{\epsilon}{e^{\beta\epsilon} - 1}$
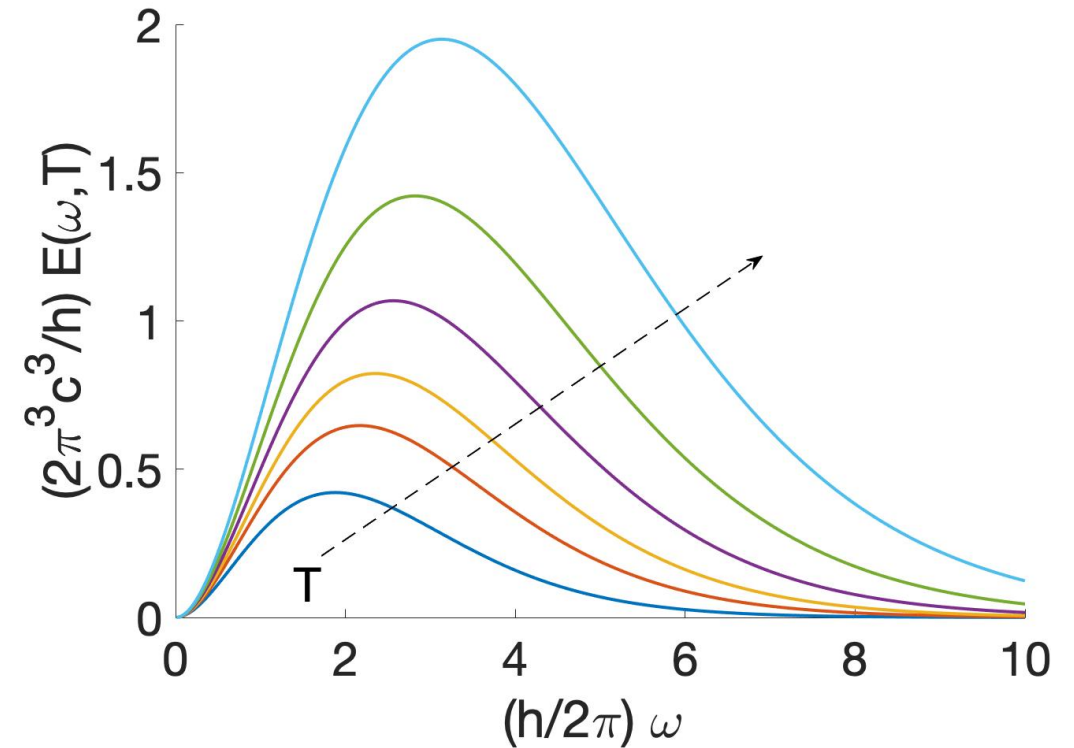
Planck distribution:

Spectral energy distribution of a photon gas

Average energy density of a photon gas:

$$\frac{\langle E \rangle (T,V)}{V} = \frac{1}{V} \int d\omega \, \frac{D_\omega(\omega) \hbar \omega}{e^{\beta \hbar \omega} - 1}$$

$$= \frac{\hbar}{\pi^2 c^3} \int_0^\infty d\omega \, \frac{\omega^3}{e^{\beta \hbar \omega} - 1} = \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4$$

Energy per unit volume at a given frequency ω

$$\frac{\langle E \rangle}{V} = \int d\omega \, \mathcal{E}(\omega, T)$$

$$\mathcal{E}(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta \hbar \omega} - 1}$$

Radiation pressure and photon density

Equation of state

$$PV = -kT \int d\omega D(\omega) \ln\left(1 - e^{-\beta\hbar\omega}\right) = -F(T,V) = -\Omega(T,V)$$

$$P = \frac{\pi^2}{45\hbar^3 c^3}(kT)^4$$

$$P = \frac{\langle E \rangle}{3V} \rightarrow \langle E \rangle = \mathbf{3PV}$$

General expression for relativistic free quantum particles

Density of photons at T:

$$\rho(T) = \frac{\langle N \rangle}{V} = \frac{V}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^2}{e^{\beta\hbar\omega} - 1} = \left(\frac{kT}{\hbar c}\right)^3 \frac{I_2}{\pi^2}, \qquad I_2 = \int_0^\infty dx \frac{x^2}{e^x - 1} \approx 2.404$$

Thermal vibrations in crystals

Harmonic solids: atoms in a crystal held at lattice sites by elastic forces
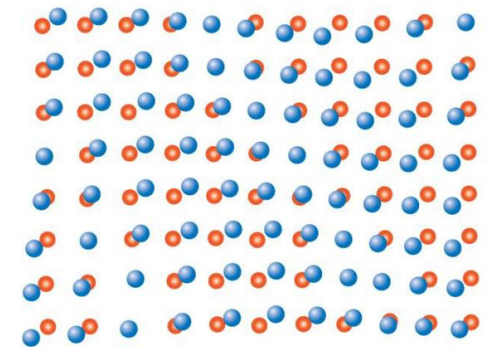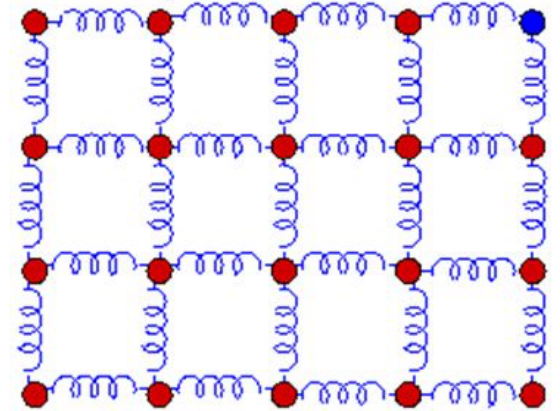
Lattice vibrations: sum of harmonic oscillators

$$H = \frac{1}{2}\sum_{i=1}^{3N}(p_i^2 + \omega_i^2 q_i^2)$$

Virial theorem: $U = 6N\frac{kT}{2} = 3NkT$

Each mode is described as a quantum harmonic oscillator
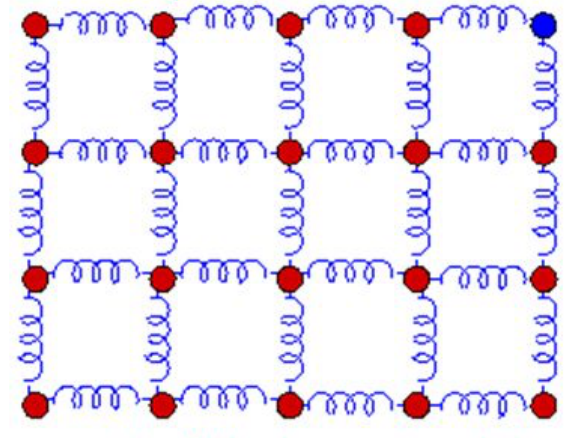$$\epsilon_\omega = \hbar\omega$$

- Normal lattice positions for atoms
- Positions displaced because of vibrations

Phonon gas: Debye model

Atoms vibrate with different frequencies and a linear dispersion

$\omega = kv$, where v is the sound wave in the solid

- Density of elastic modes is analogous to the density of states for photons

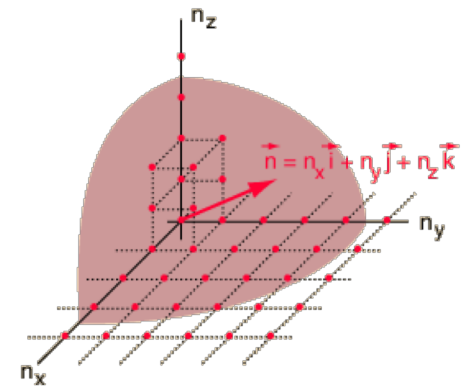(in the long-wavelength approximation, continuum elastic medium)

$$D_\omega(\omega) = 3\frac{V}{2\pi^2}\frac{\omega^2}{v^2}\frac{dk}{d\omega} = 3\frac{V}{2\pi^2}\frac{\omega^2}{v^3}, \qquad for \ 0 \le \omega \le \omega_D$$

- Total number of modes: $3N$ normal modes (in 3D) for N atoms

$$3N = \int_0^{\omega_D} d\omega \, D(\omega) = 3\frac{V}{2\pi^2 v^3}\int_0^{\omega_D} d\omega \, \omega^2$$

Debye frequency $\omega_D = v\left(\frac{6\pi^2 N}{V}\right)^{\frac{1}{3}} \rightarrow \omega_D = \frac{2\pi v}{\lambda_{\min}} = v(6\pi^2\rho)^{\frac{1}{3}}, \quad \rho = \frac{N}{V}$

Debye model: heat capacity of crystals

- Total average energy of phonons

$$U(T,V) = \int_0^{\omega_D} d\omega\, D(\omega) \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} = 3k\frac{V}{2\pi^2 v^3} \int_0^{\omega_D} d\omega\, \omega^2 \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$
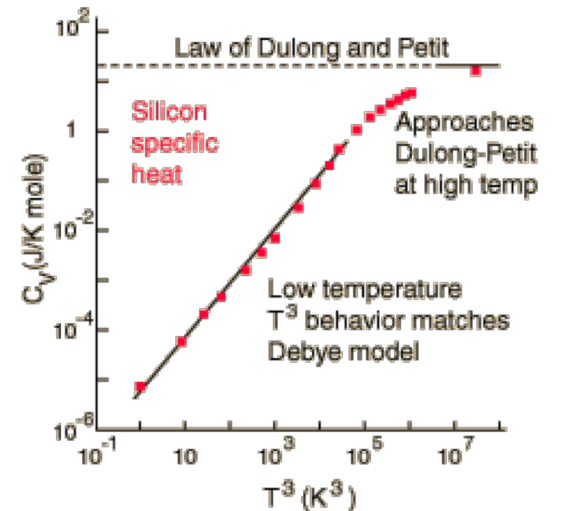
- Heat capacity

$$C_V(T) = \left(\frac{\partial U}{\partial T}\right)_V = 3k\frac{V}{2\pi^2 v^3}\int_0^{\omega_D} d\omega\, \omega^2 \left(\frac{\hbar\omega}{kT}\right)^2 \frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}$$

$$C_V(T) = 9Nk\left(\frac{T}{T_D}\right)^3 F\left(\frac{T_D}{T}\right), \qquad F(y) = \int_0^y dx\, \frac{x^4 e^x}{(e^x-1)^2}$$

$$\boldsymbol{C_V(T) \approx 3Nk,} \qquad \boldsymbol{T \gg T_D}$$

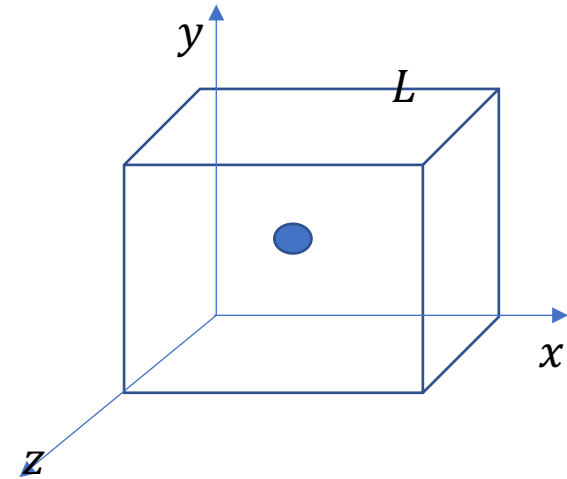$$C_V(T) \approx \frac{12}{5}Nk\pi^4\left(\frac{T}{T_D}\right)^3, \qquad \boldsymbol{T \ll T_D}$$

Ideal Bose gas: Density of states in 3D

- Energy levels for a particle in a box with periodic boundary conditions:

$$\epsilon_n = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 |\vec{n}|^2 = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 n^2, \qquad \vec{n} = (n_x, n_y, n_z), \quad n_{x,y,z} \text{ integers}$$

- Quantum state of the particle is described by its wavefunction $\psi_1 = e^{\frac{2\pi i}{L}\vec{n}\cdot\vec{r}}$, which is determined by $\vec{n}$

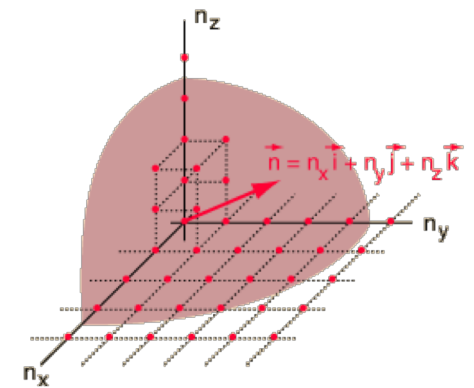Number of available states between modes with n and $n + dn$ *in 3D*

$$D(n)dn = 4\pi n^2 dn$$

$$D_\epsilon(\epsilon)d\epsilon = D(n)dn \rightarrow D_\epsilon(\epsilon) = \frac{V}{\sqrt{2}\pi^2}\frac{m^{3/2}}{\hbar^3}\epsilon^{\frac{1}{2}}$$

Average number of particles: $\quad \langle N \rangle = \sum_j \frac{1}{e^{\beta(\epsilon_j - \mu)} - 1} = \int d\epsilon D_\epsilon(\epsilon)\frac{1}{e^{\beta(\epsilon - \mu)} - 1}$

Average energy: $\quad U = \sum_j \frac{\epsilon_j}{e^{\beta(\epsilon_j - \mu)} - 1} = \int d\epsilon D_\epsilon(\epsilon)\frac{\epsilon}{e^{\beta(\epsilon - \mu)} - 1}$

Pressure: $\quad PV = -kT\sum_j \log\left(1 - e^{-\beta(\epsilon_j - \mu)}\right) = -kT\int d\epsilon D_\epsilon(\epsilon)\ln\left[1 - e^{-\beta(\epsilon - \mu)}\right]$

Bose-Einstein condensation

At a given temperature T, density ρ and chemical potential μ are related by the following equation

$$\rho(T,\mu) = \frac{1}{V}\frac{1}{\lambda^{-1}-1} + \Lambda^{-3}(T)\frac{2}{\sqrt{\pi}}\int_0^\infty dx \frac{x^{\frac{1}{2}}}{e^x-1}$$

$$\rho(T,\mu) = \rho_0(T,\mu) + \rho_{ex}(T,\mu) = \Lambda^{-3}(T)g_{\frac{3}{2}}(\lambda), \qquad \lambda = e^{\beta\mu}$$

Critical temperature for Bose Einstein condensation:

Determined by the density of the bose gas when $\mu = 0$

$$\boldsymbol{T_c(\rho)} = \frac{h^2}{2\pi mk}\left(\frac{\rho}{\zeta\left(\frac{3}{2}\right)}\right)^{\frac{2}{3}}$$

- $T \leq T_c$: *Density is a mixture of condensed and normal gas*

$$\boldsymbol{\rho = \rho_0(T) + \rho_{ex}(T) = \Lambda^{-3}(T_c)\zeta(3/2)}$$

 **A macroscopic fraction of particles condense into the ground state with the zero momentum**

 - **Chemical potential $\mu = 0$ remains zero for all temperatures below T_c**
 - **Excess density depends only on temperature and actually decreases with decreasing temperature**
 - **Ground state density becomes non-zero and increases with decreasing temperature**

$$\frac{\rho_0}{\rho} = 1 - \frac{\rho_{ex}(T)}{\rho} = 1 - \frac{\Lambda^3(T_c)}{\Lambda^3(T)} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

- $T > T_c$: *Density is that of the normal gas*

$$\boldsymbol{\rho(T,\mu) = \rho_{ex}(T,\mu) = \Lambda^{-3}(T)g_{\frac{3}{2}}(\lambda)}$$

This equation determines the chemical potential $\mu(T,\rho)$
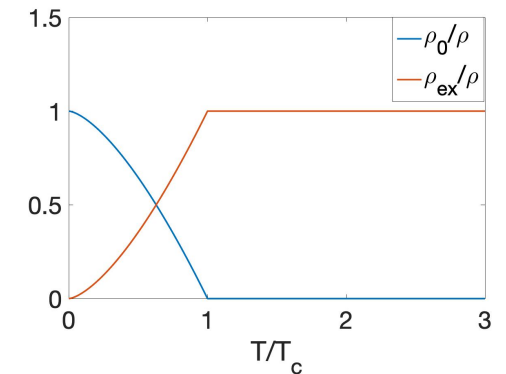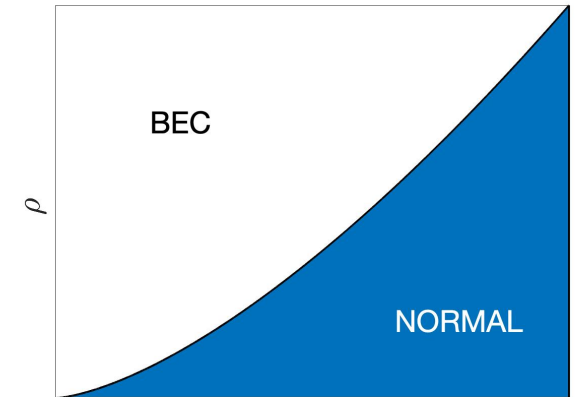
Pressure

$$\frac{P}{kT} = -\frac{1}{V}\log(1-\lambda) - \frac{1}{\Lambda^3(T)}\frac{2}{\sqrt{\pi}}\int_0^\infty dx\, x^{\frac{1}{2}}\log(1-\lambda e^{-x})$$

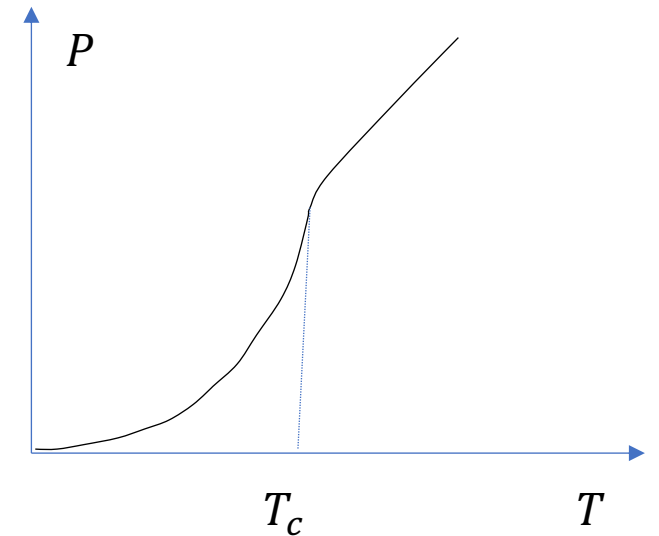$$= \frac{1}{\Lambda^3(T)}g_{5/2}(\lambda)$$

- $T \leq T_c$ $(\lambda = 1)$

$$\frac{P}{kT} = \frac{1}{\Lambda^3(T)}g_{5/2}(1) \rightarrow P(T) \sim T^{\frac{5}{2}}$$

The pressure becomes independent of density below T_c

- $T > T_c$ $(\lambda < 1)$

$$\frac{P(T,\lambda)}{kT} = \frac{1}{\Lambda^3(T)}g_{5/2}(\lambda), \qquad \rho(T,\lambda) = \Lambda^{-3}(T)g_{\frac{3}{2}}(\lambda)$$

Chemical potential and equation of state for $T > T_c$

- *The chemical potential is determined perturbatively from the density equation above T_c.*

$$\rho = \frac{1}{\Lambda^3(T)} g_{\frac{3}{2}}(\lambda) \rightarrow \frac{1}{\Lambda^3(T_c)} \zeta\left(\frac{3}{2}\right) = \frac{1}{\Lambda^3(T)} \sum_{n=1}^{\infty} \frac{\lambda^n}{n^{\frac{3}{2}}} \rightarrow \zeta\left(\frac{3}{2}\right) = \left(\frac{T}{T_c}\right)^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{\lambda^n}{n^{\frac{3}{2}}}$$

Inverting the equation above we can determine the fugacity $\lambda(\rho, T)$:

$$\lambda\left(1 + 2^{-3/2}\lambda + \cdots\right) = \zeta\left(\frac{3}{2}\right)\left(\frac{T}{T_c}\right)^{-3/2},$$
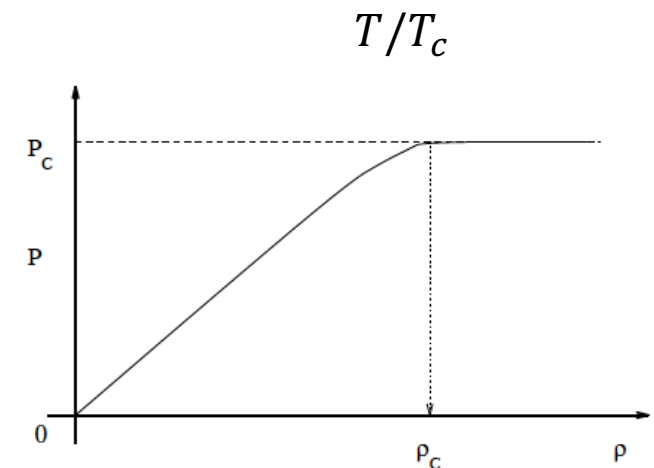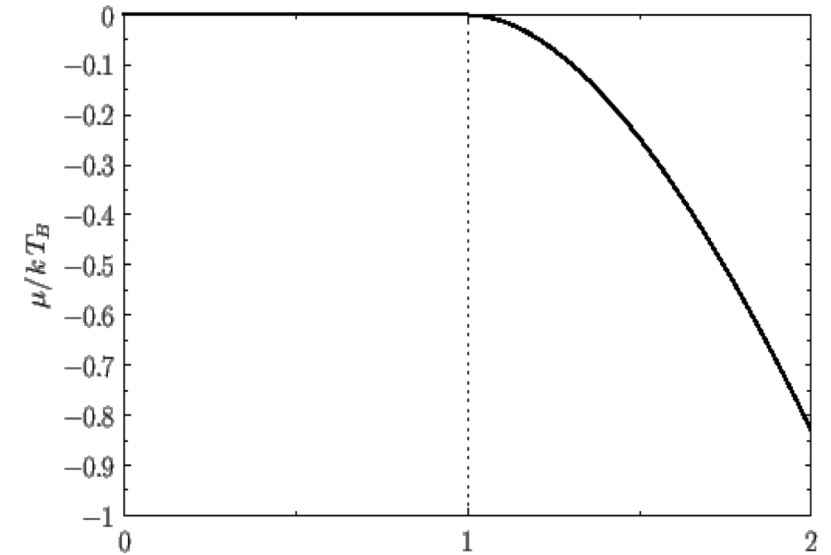
$$\lambda \approx \zeta\left(\frac{3}{2}\right)\left(\frac{T}{T_c}\right)^{-\frac{3}{2}}\left(1 - \zeta\left(\frac{3}{2}\right)\left(\frac{2T}{T_c}\right)^{-\frac{3}{2}}\right) \rightarrow$$

$$\mu = kT \ln\left[\zeta\left(\frac{3}{2}\right)\left(\frac{T}{T_c}\right)^{-\frac{3}{2}}\left(1 - \zeta\left(\frac{3}{2}\right)\left(\frac{2T}{T_c}\right)^{-\frac{3}{2}}\right)\right]$$

- Equation of state: elimite the chemical potential from $\frac{P}{kT} = \frac{1}{\Lambda^3(T)} g_{5/3}(\lambda)$
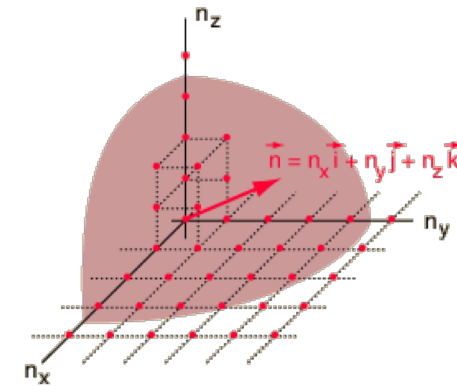
$$\frac{P}{kT} \approx \rho + B_2(T)\rho^2 + \cdots, \quad B_2(T) = -\frac{1}{4\sqrt{2}}\Lambda^3(T) < 0$$

The Bose gas pressure is effectively lowered by statistical attraction forces

Ideal Fermi gases: Density of states in 3D

- $\Psi_1(\boldsymbol{r}) = e^{\frac{2\pi i}{L}\boldsymbol{n}\cdot\boldsymbol{r}}$ 1-particle wave function

- Each fermion (i.e. electron) has a spin moment $= \pm\frac{1}{2}$

- Energy levels a fermion in a box $V = L^3$ with periodic boundary conditions: $\epsilon_n = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 n^2$,

Number of available states between a mode with n between $n \text{ and } n + dn$: $D(n)dn = 2{\times}4\pi n^2 dn$

$$\sum_{\boldsymbol{n}} := 2{\times}\int dn\ 4\pi n^2 = \int dn\ D(n)$$

Density of states corresponding to energy ϵ:

$$D(\epsilon) = D(n)\frac{dn}{d\epsilon} \rightarrow D(\epsilon) = 2\frac{V}{\sqrt{2}\pi^2}\frac{m^{3/2}}{\hbar^3}\epsilon^{\frac{1}{2}}$$

The difference with respect to the density of states of bosons is the spin degeneracy of the energy levels (hence the extra factor of 2).
Different conventions: Sometimes the spin degeneneracy is not included in the density of states, and appears separately on the integral expressions

Pressure and average energy

$$P = kT \frac{\sqrt{2}}{\pi^2} \frac{m^{3/2}}{\hbar^3} \int_0^\infty d\epsilon \ \epsilon^{1/2} \log\left(1 + e^{-\beta(\epsilon - \mu)}\right)$$

$$P = \frac{2}{3} \frac{\sqrt{2}}{\pi^2} \frac{m^{\frac{3}{2}}}{\hbar^3} \int_0^\infty d\epsilon \ \frac{\epsilon^{\frac{3}{2}}}{e^{\beta(\epsilon - \mu)} + 1} = \frac{2}{3} \frac{\langle E \rangle}{V}$$

General expression for non-relativistic quantum ideal gas (independent of $\langle n \rangle (\epsilon)$)

Fermi distribution at T=0 K

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \rightarrow_{T\to 0} \begin{cases} 1, & \epsilon < \mu \\ 0, & \epsilon > \mu \end{cases}$$

$\epsilon_F \equiv \mu$ Fermi energy level below which all states are occupied

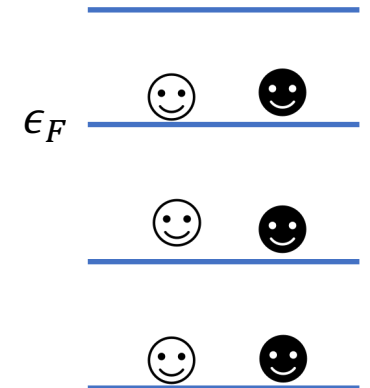Determined by the gas density $\epsilon_F(\rho) = \frac{\hbar^2}{2m}(3\pi^2\rho)^{\frac{2}{3}}$

$$\rho(\epsilon_F) = \frac{1}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon_F^{\frac{3}{2}}$$

Energy

$$\frac{\langle E \rangle_0}{V} = \frac{\sqrt{2}}{\pi^2}\frac{m^{3/2}}{\hbar^3}\int_0^{\epsilon_F} d\epsilon\, \epsilon^{\frac{3}{2}} = \frac{1}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon_F^{\frac{5}{2}}$$

Exclusion pressure

$$P_0 = \frac{\hbar^2}{15\pi^2 m^2}(3\pi^2\rho)^{\frac{5}{3}}$$

f(ε)

1

0   ε_F   ε

ϵ_F
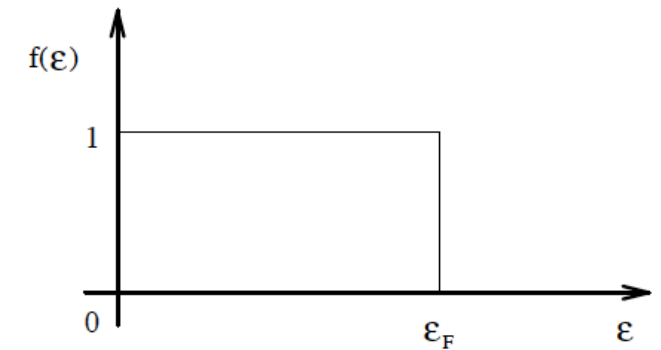
Denenerate ideal Fermi gas $T < T_F$

The Fermi temperature is most often much larges than the gas temperature

Therefore, even though the fermi gas is at finite temperature, it behaves as if it was a near zero temperature when $T \ll T_F$

$$\rho = \Lambda^{-3}(T)\frac{4}{\sqrt{\pi}}\frac{m^{\frac{3}{2}}}{\hbar^3}F\left(\frac{1}{2}\right), \qquad \frac{\langle E \rangle}{V} = \frac{\sqrt{2}}{\pi^2}\frac{m^{\frac{3}{2}}}{\hbar^3}F\left(\frac{3}{2}\right)$$
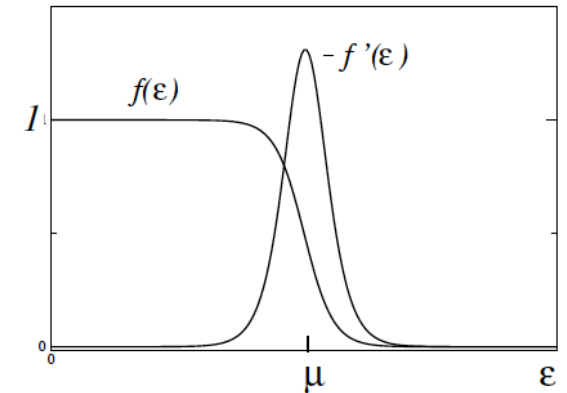
$$F(a) = \int_0^\infty d\epsilon\, \epsilon^a f(\epsilon) = \frac{\beta}{a+1}\int_0^\infty d\epsilon\, \frac{\epsilon^{a+1}e^{\beta(\epsilon-\mu)}}{(e^{\beta(\epsilon-\mu)}+1)^2}$$

Sommerfeld expansion: $\frac{kT}{\mu} \ll 1$:
$\qquad F(a) = \frac{\mu^{a+1}}{a+1}\left(1 + \frac{\pi^2}{6}a(a+1)\left(\frac{kT}{\mu}\right)^2 + \cdots\right)$

$$\rho = \frac{1}{3\pi^2}\frac{(2m)^{\frac{3}{2}}}{\hbar^3}\mu^{3/2}\left(1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right)$$

$$\frac{\langle E \rangle}{V} = \frac{1}{5\pi^2}\frac{(2m)^{\frac{3}{2}}}{\hbar^3}\mu^{5/2}\left(1 + \frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right)$$

Denenerate ideal Fermi gas: chemical potential μ

Sommerfeld expansion: $\dfrac{kT}{\mu} \ll 1$

$$\rho = \frac{1}{3\pi^2} \frac{(2m)^{\frac{3}{2}}}{\hbar^3} \mu^{3/2} \left(1 + \frac{\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 + \cdots \right)$$

Using that $\rho = \dfrac{1}{3\pi^2} \left(\dfrac{2m}{\hbar^2} \right)^{3/2} \epsilon_F^{\frac{3}{2}}$

$$\epsilon_F^{\frac{3}{2}} = \mu^{3/2} \left(1 + \frac{\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 + \cdots \right)$$

$$\mu = \epsilon_F \left(1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 + \cdots \right)$$
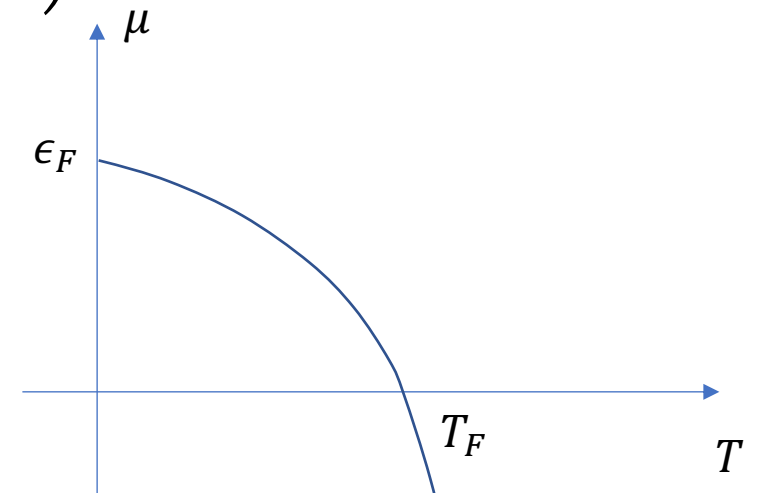
Denenerate ideal Fermi gas: average energy

Sommerfeld expansion: $\dfrac{kT}{\mu} \ll 1$

$$\mu = \epsilon_F \left(1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 + \cdots \right)$$

$$\frac{\langle E \rangle}{V} = \frac{1}{5\pi^2} \frac{(2m)^{\frac{3}{2}}}{\hbar^3} \mu^{5/2} \left(1 + \frac{5\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 + \cdots \right)$$

$$= \frac{1}{5\pi^2} \frac{(2m)^{\frac{3}{2}}}{\hbar^3} \epsilon_F^{5/2} \left(1 + \frac{5\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 + \cdots \right)$$

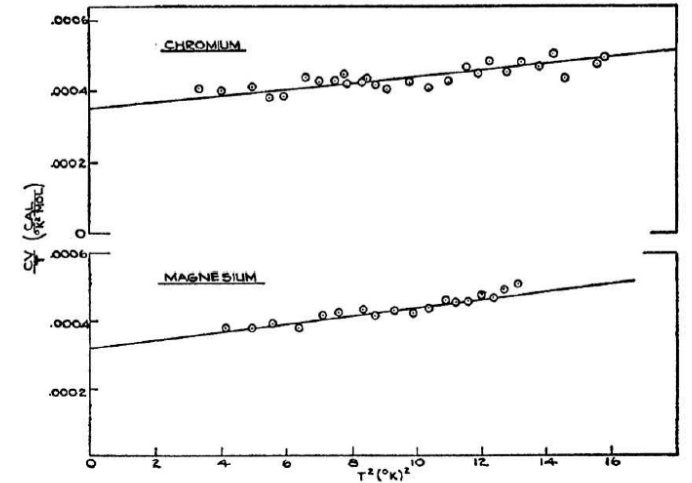$$C_V = \frac{1}{2} \langle N \rangle \pi^2 k \frac{T}{T_F}$$

Figure 5.20: The specific heat C_V/T of a solid as function of T^2 has an intercept determined by the electrons and a slope determined by the phonons. This specific heat is measured for Chromium and Magnesium by *S. A. Friedberg, I. Estermann, and J. E. Goldman* 1951.

Denenerate ideal Fermi gas: pressure

Sommerfeld expansion: $\dfrac{kT}{\mu} \ll 1$

$$\frac{\langle E \rangle}{V} = \frac{1}{5\pi^2} \frac{(2m)^{\frac{3}{2}}}{\hbar^3} \epsilon_F^{5/2} \left(1 + \frac{5\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 + \cdots \right) = \frac{3}{2} PV$$

$$\mathrm{P} = P_0 \left(1 + \frac{5\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 + \cdots \right)$$

High temperature limit (classical ideal gas): $T > T_F$

Pressure equation of state:

$$P = \frac{1}{3\pi^2} \frac{(2m)^{3/2}}{\hbar^3} \int_0^\infty d\epsilon \frac{\epsilon^{\frac{3}{2}}}{\lambda^{-1} e^{\beta\epsilon} + 1}, \qquad \lambda = e^{\beta\mu} < 1$$

$$\frac{P}{kT} = \Lambda^{-3}(T) \frac{8}{3\sqrt{\pi}} \int_0^\infty dx \frac{x^{\frac{3}{2}}}{\lambda^{-1} e^x + 1}, \qquad x = \beta\epsilon$$

Similarily, we can write the density equation with a dimensionless integral form

$$\rho = \frac{\sqrt{2}}{\pi^2} \frac{m^{3/2}}{\hbar^3} \int_0^\infty d\epsilon \frac{\epsilon^{\frac{1}{2}}}{\lambda^{-1} e^{\beta\epsilon} + 1}$$

$$\rho(T, \lambda) = \Lambda^{-3}(T) \frac{4}{\sqrt{\pi}} \int_0^\infty dx \frac{x^{\frac{1}{2}}}{\lambda^{-1} e^x + 1}$$

Density and chemical potential: $T > T_F$

The density equation determines the chemical potential as a function of temperature and density

$$\rho(T, \lambda) = \Lambda^{-3}(T) \frac{4}{\sqrt{\pi}} \int_0^\infty dx \frac{x^{\frac{1}{2}}}{\lambda^{-1} e^x + 1}, \qquad \lambda = e^{\beta \mu} < 1$$

Taylor expand the integrand with respect to the fugacity as the expansion parameter $\lambda < 1$

$$\rho(T, \lambda) = \Lambda^{-3}(T) \frac{4}{\sqrt{\pi}} \int_0^\infty d\epsilon \, \lambda x^{\frac{1}{2}} e^{-x} (1 - \lambda e^{-x} + \cdots)$$

$$\rho = \Lambda^{-3}(T) \frac{4}{\sqrt{\pi}} \lambda \left[\int_0^\infty dx \, x^{\frac{1}{2}} e^{-x} - \lambda \int_0^\infty dx \, x^{\frac{1}{2}} e^{-2x} + \cdots \right]$$

$$\rho = \Lambda^{-3}(T) \frac{4}{\sqrt{\pi}} \lambda \left[\frac{\sqrt{\pi}}{2} - 2^{-\frac{5}{2}} \sqrt{\pi} \lambda + \cdots \right]$$

$$\rho = 2\Lambda^{-3}(T) \lambda \left[1 - 2^{-3/2} \lambda + \cdots \right]$$

Invert the series expansion to find the fugacity in terms of density:

$$\lambda(T) = \frac{\Lambda^3(T)\rho}{2} \left[1 + 2^{-5/2} \Lambda^3 \rho - \cdots \right]$$

High temperature limit of pressure: $T > T_F$

In the high T limit, the pressure can be written as a virial expansion with respect to density dependency.

Expand the pressure in powers of fugacity:

$$\frac{P}{kT} = \Lambda^{-3}(T)\frac{8}{3\sqrt{\pi}}\lambda\left[\int_0^\infty dx\, x^{\frac{3}{2}}e^{-x} - \lambda\int_0^\infty dx\, x^{\frac{3}{2}}e^{-2x} + \cdots\right]$$

$$\frac{P}{kT} = \Lambda^{-3}(T)\frac{8}{3\sqrt{\pi}}\lambda\left[\frac{3\sqrt{\pi}}{4} - \lambda\frac{3\sqrt{\pi}}{16\sqrt{2}} + \cdots\right]$$

$$\frac{P}{kT} = 2\Lambda^{-3}(T)\lambda\left[1 - 2^{-\frac{5}{2}}\lambda + \cdots\right]$$

Inserting the dependence of fugacity on density and keeping only the first two terms:

$\lambda(T) = \frac{\Lambda^3(T)\rho}{2}\left[1 + 2^{-5/2}\Lambda^3\rho - \cdots\right]$

$$\frac{P}{kT} = \rho\left(1 + 2^{-\frac{7}{2}}\Lambda^3(T)\rho\right), \quad B_2(T) = 2^{-\frac{7}{2}}\Lambda^3(T) > 0$$

The positive second virial coefficient means that pressure is larger than the ideal gas pressure due to statistical repelling forces

Equation of state for quantum gases: *high T*

$$P_{\text{fermions}} \approx kT\rho \left(1 + 2^{-\frac{7}{2}}\Lambda^3 \rho\right)$$

$$P_{\text{bosons}} \approx kT\rho \left(1 - 2^{-\frac{5}{2}}\Lambda^3 \rho\right)$$

Nonzero second virial coeff. $B_2(T) \neq 0$

Bosons: $B_2(T) < 0$ *statistical attraction*

Fermions: $B_2(T) > 0$ *statistical repulsion*