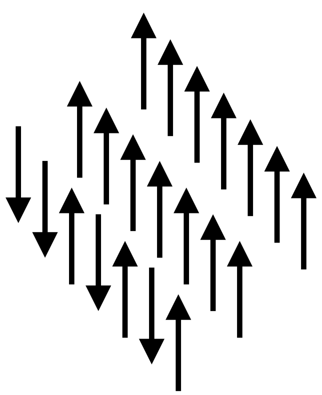
Ising model



$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

exchange parameter **magnetic field**

"spin" $\sigma_i = \pm 1$

nearest neighbors counted once

Model proposed by Wilhelm Lenz as a model of magnetism in 1920.

XYZ-model

$$H = \sum_{\langle ij \rangle} \left(J_x \sigma_i^x \sigma_j^x + J_y \sigma_i^y \sigma_j^y + J_z \sigma_i^z \sigma_j^z \right)$$

$\sigma^x, \sigma^y, \sigma^z$ Pauli matrices

XXZ-model: $J_x = J_y, J_z$

Heisenberg model: $J_x = J_y = J_z$

XY model: $J_x = J_y, J_z = 0$

Ising model: $J_x = J_y = 0, J_z$

Ising model exact solutions

- 1D: Ernst Ising solved the 1D model exactly in 1924, but no phase transition…a disappointing conclusion.

- 2D: Lars Onsager, solved the 2D Ising model exactly in 1944. There is a phase transition! A true “tour de force” in theoretical physics. Very impressive, he even introduced quaternions to solve it.

- 3D: Ising model has not yet been solved exactly.

Ising chain (1D) h=0

$$H = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}$$

Open boundary conditions

New variables:

$$\tau_1 = \sigma_1 \qquad \tau_i = \sigma_{i-1}\sigma_i \qquad H = -J \sum_{i=2}^{N} \tau_i$$

$$Z = \sum_{\sigma} e^{-\beta H} = \sum_{\tau} e^{-\beta H}$$

$$Z = \sum_{\tau_1} \prod_{i=2}^{N} \left(\sum_{\tau_i} e^{\beta J \tau_i} \right) = 2 \left(2 \cosh\left(\beta J\right)\right)^{N-1}$$

Ising chain (1D) h=0

$$Z = 2\left(2\cosh\beta J\right)^{N-1}$$

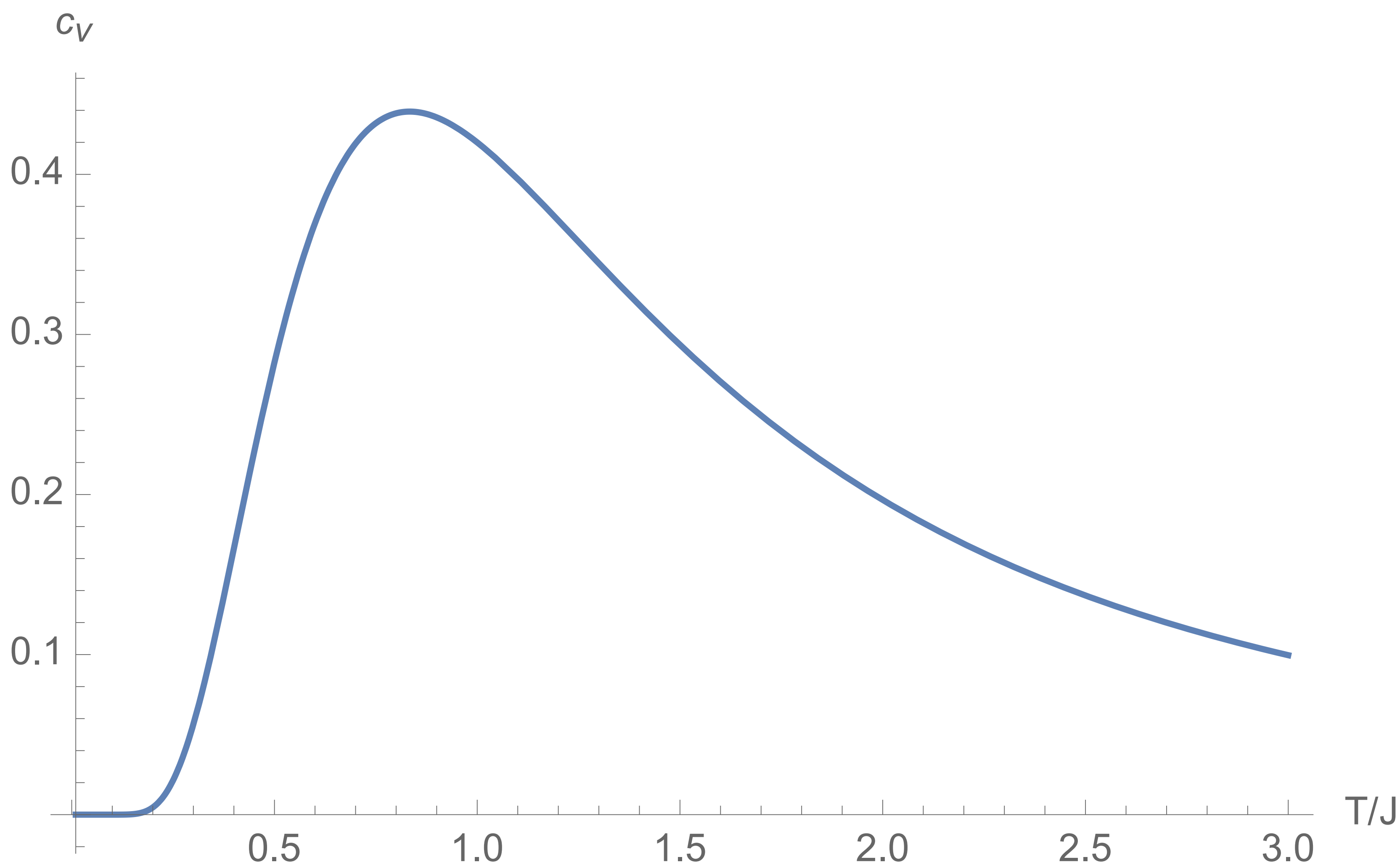
$$-\beta F = \ln 2 + (N-1)\ln\left[2\cosh\left(\beta J\right)\right]$$

$$U = \frac{\partial\left(\beta F\right)}{\partial\beta} = -(N-1)J\tanh\left(\beta J\right)$$

$$c = \frac{1}{N}\frac{\partial U}{\partial T} = k\beta^2 J^2\left(1-\frac{1}{N}\right)\frac{1}{\cosh^2\left(\beta J\right)}$$

Ising chain (1D) h=0

$$c = \frac{1}{N}\frac{\partial U}{\partial T} = k\beta^2 J^2 \left(1 - \frac{1}{N}\right)\frac{1}{\cosh^2(\beta J)}$$



Transfer matrices
Ising chain

Periodic boundary conditions, write H in a symmetric way:

$$H = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - \frac{h}{2} \sum_{i=1}^{N} (\sigma_i + \sigma_{i+1})$$

$$\begin{aligned}
Z &= \sum_{\{\sigma\}} e^{-\beta\left[-J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - \frac{h}{2}\sum_{i=1}^{N}(\sigma_i+\sigma_{i+1})\right]} \\
&= \sum_{\{\sigma\}} \prod_{i=1}^{N} e^{\left[\beta J \sigma_i \sigma_{i+1} + \frac{\beta h}{2}(\sigma_i + \sigma_{i+1})\right]} \\
&= \sum_{\{\sigma\}} \prod_{i=1}^{N} T_{\sigma_i, \sigma_{i+1}}
\end{aligned}$$

Transfer matrices

$$T_{\sigma_i,\sigma_{i+1}} = e^{\beta J \sigma_i \sigma_{i+1} + \frac{\beta h}{2}(\sigma_i + \sigma_{i+1})}$$

$$T = \begin{pmatrix} e^{\beta J - \beta h} & e^{-\beta J} \\ e^{\beta J} & e^{\beta J + \beta h} \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} - & \sigma_{i+1} \; + \end{array} \\ \sigma_i \begin{array}{c} - \\ + \end{array} & \begin{pmatrix} -- & -+ \\ +- & ++ \end{pmatrix} \end{array}$$

$$Z = \sum_{\{\sigma\}} T_{\sigma_1,\sigma_2} T_{\sigma_2,\sigma_3} \ldots T_{\sigma_N,\sigma_1}$$

Z = Trace (A product of transfer matrices)

$$Z = Tr\left(T^N\right)$$

Transfer matrices

Diagonalize the transfer matrix: $T = R^{-1} D R$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$Z = Tr\left(T^N\right) = Tr\left(R^{-1} D R R^{-1} D R \ldots D R\right)$$

$$= Tr\left(D^N\right) = \lambda_1^N + \lambda_2^N$$

Let $\lambda_1 > \lambda_2$ **then for large- N:**

$$Z = \lambda_1^N \left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right) \approx \lambda_1^N$$

Transfer matrices

Ising chain (exercise):

$$\lambda_1 = e^{\beta J}\cosh\beta h + \sqrt{e^{2\beta J}\cosh^2(\beta h) - 2\sinh 2\beta J}$$

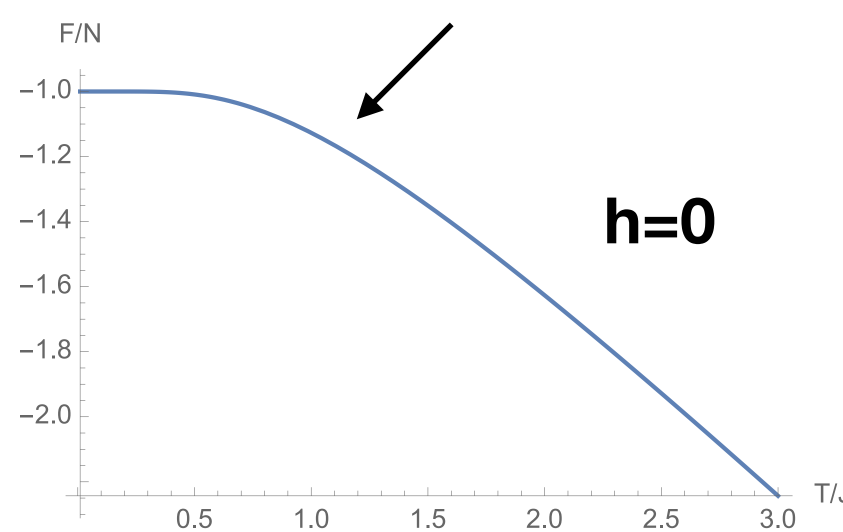
$$\frac{F}{N} = -kT\ln\left(e^{\beta J}\cosh(\beta h) + \sqrt{e^{2\beta J}\cosh^2(\beta h) - 2\sinh(2\beta J)}\right)$$

Exercise: Show that the above reduces to

$$\frac{F}{N} = -kT\ln\left(2\cosh(\beta J)\right)$$

for h=0

Analytic: No phase transitions!

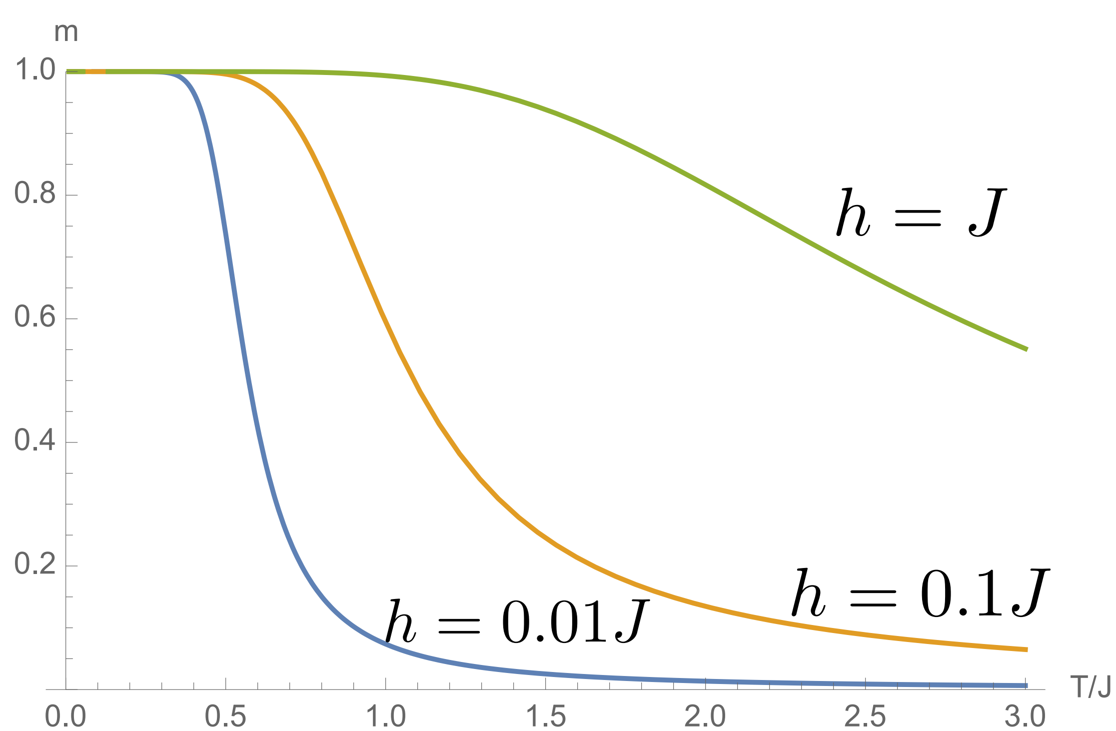


h=0

Magnetization

$$m = \frac{1}{N}\langle \sigma_j \rangle = \frac{1}{\beta N}\frac{\partial}{\partial h}\frac{1}{Z}\sum_{\{\sigma\}} e^{\beta J \sum_i \sigma_i \sigma_{i+1} + \beta h \sum_i \sigma_i}$$

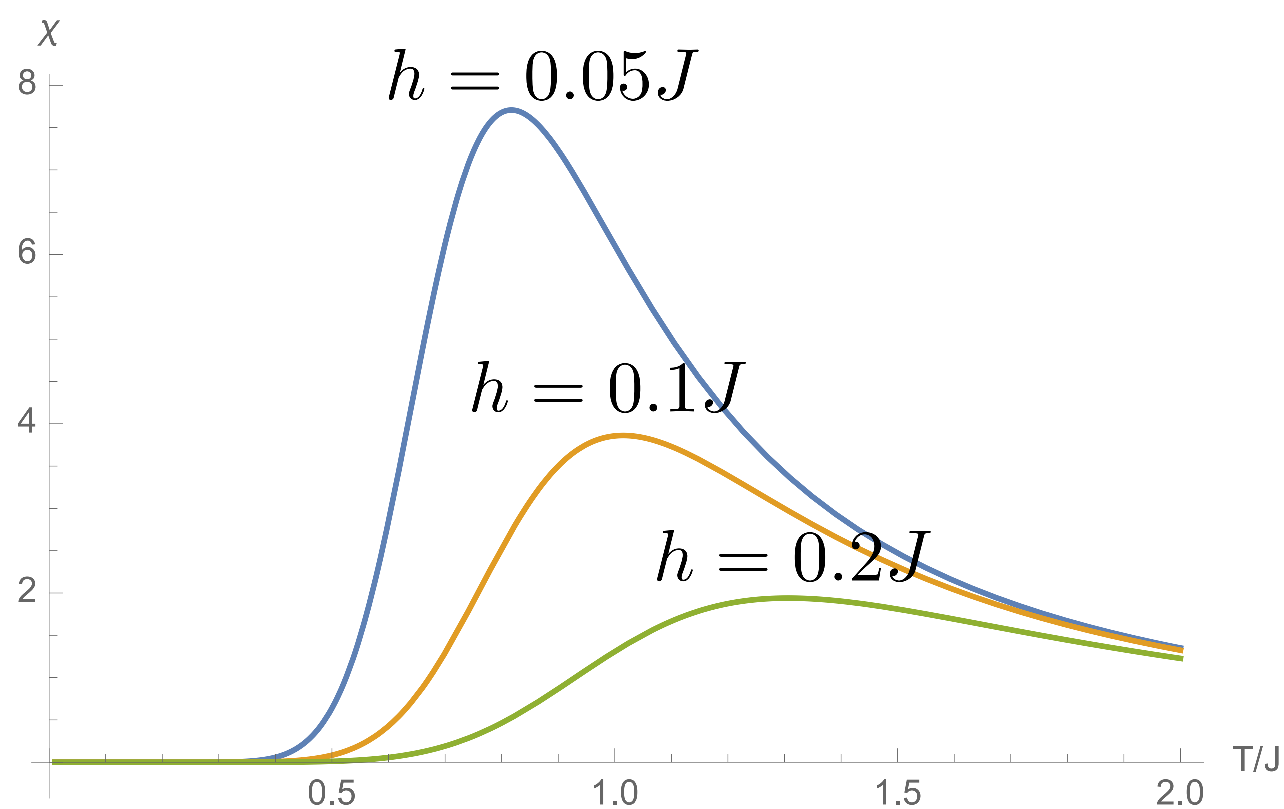
$$= \frac{1}{\beta N}\frac{\partial \ln Z}{\partial h} = -\frac{1}{N}\frac{\partial F}{\partial h}$$



Magnetic susceptibility

Magnetic susceptibility:

$$\chi = \frac{\partial m}{\partial h} \quad = \frac{1}{\beta N} \frac{\partial^2 \ln Z}{\partial^2 h}$$



The peak becomes narrower and moves to lower T as h is reduced. No phase transition.