Lecture Notes
On
General Relativity

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Preface

These notes are a transcript of lectures delivered by Øyvind Grøn during the spring of 1997 at the University of Oslo. Two compendia, (Grøn and Flø 1984) and (Ravndal 1978) were provided by Grøn as additional reference material during the lectures.

The present version of this document is an extended and corrected version of a set of Lecture Notes which were typesetted by S. Bard, Andreas O. Jaunsen, Frode Hansen and Ragnvald J. Irgens using LaTeX2ϵ. Svend E. Hjelmeland has made many useful suggestions which have improved the text.

While we hope that these typeset notes are of benefit particularly to students of general relativity and look forward to their comments, we welcome all interested readers and accept all feedback with thanks.

All comment may be sent to the author either by e-mail or snail mail.

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1.1 The force law of gravitation

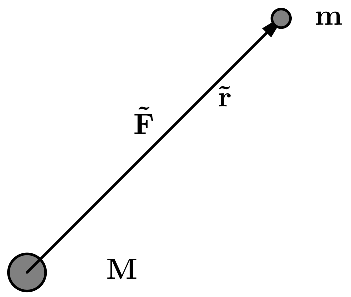


Figure 1.1: Newton's law of universal gravitation states that the force between two masses is attractive, acts along the line joining them and is inversely proportional to the distance separating the masses.

$$\vec{F} = -mG\frac{M}{r^3}\vec{r} = -mG\frac{M}{r^2}\vec{e}_r \tag{1.1}$$

Let V be the potential energy of m (see figure 1.1). Then

$$\vec{F} = -\nabla V(\vec{r}), \quad F_i = -\frac{\partial V}{\partial x_i} \tag{1.2}$$

For a spherical mass distribution: $V(\vec{r}) = -mG\frac{M}{r}$, with zero potential infinitely far from the center of M. Newton's law of gravitation is valid for "small" velocities, i.e. velocities much smaller than the velocity of light and "weak" fields. Weak fields are fields in which the gravitational potential energy of a test particle is very small compared to its rest mass energy. (Note that here one is interested only in the absolute values of the above quantities and not their sign).

$$mG\frac{M}{r} \ll mc^2 \quad \Rightarrow \quad r \gg \frac{GM}{c^2}. \tag{1.3}$$

The **Schwarzschild radius** for an object of mass M is $R_s = \frac{2GM}{c^2}$. Far outside the Schwarzschild radius we have a weak field. To get a feeling for magnitudes consider that $R_s \approxeq 1$ cm for the Earth which is to be compared with $R_E \approxeq 6400$ km. That is, the gravitational field at the Earth's surface can be said to be weak! This explains, in part, the success of the Newtonian theory.

1.2 Newton's law of gravitation in its local form

Let P be a point in the field (see figure 1.2) with position vector $\vec{r} = x^i \vec{e}_i$ and let the gravitating point source be at $\vec{r'} = x^{i'} \vec{e}_{i'}$. Newton's law of gravitation for a continuous distribution of mass is

$$\begin{aligned} \vec{F} &= -mG \int_r^\infty \rho(\vec{r'}) \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} d^3 r' \\ &= -\nabla V(\vec{r}) \end{aligned} \tag{1.4}$$

See figure (1.2) for symbol definitions.

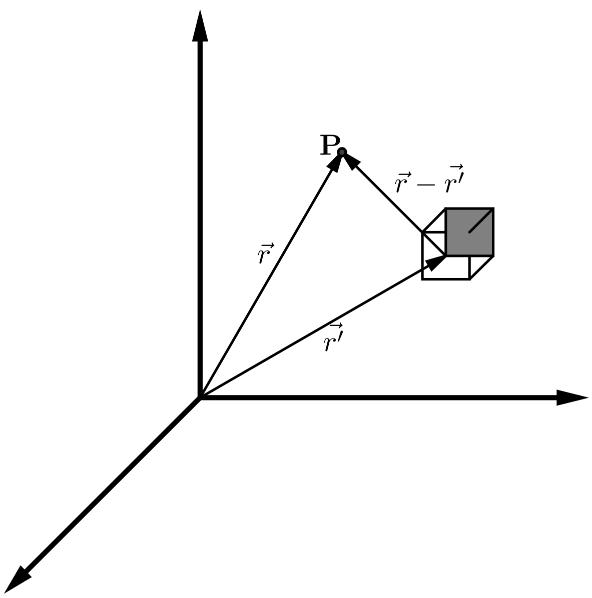


Figure 1.2: Newton's law of gravitation in its local form.

Let's consider equation (1.4) term by term.

$$\begin{aligned}
\nabla \frac{1}{|\vec{r} - \vec{r'}|} &= \vec{e}_i \frac{\partial}{\partial x_i} \frac{1}{\left[(x^j - x^{j'})(x_j - x_{j'})\right]^{1/2}} \\
&= \vec{e}_i \frac{\partial}{\partial x_i} \left[(x^j - x^{j'})(x_j - x_{j'})\right]^{-1/2} \\
&= \vec{e}_i \frac{-1}{2} 2(x_j - x_{j'}) \frac{\partial x^j}{\partial x_i} \left[(x^k - x^{k'})(x_k - x_{k'})\right]^{-3/2} \\
&= -\vec{e}_i \frac{(x^j - x^{j'})\delta^i{}_j}{\left[(x^k - x^{k'})(x_k - x_{k'})\right]^{3/2}} \\
&= -\vec{e}_i \frac{(x^i - x^{i'})}{\left[(x^j - x^{j'})(x_j - x_{j'})\right]^{3/2}} \\
&= -\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3}
\end{aligned} \tag{1.5}$$

Now equations (1.4) and (1.5) together $\Rightarrow$

$$V(\vec{r}) = -mG \int \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} d^3 r' \tag{1.6}$$

Gravitational potential at point P:

$$\begin{aligned}
\phi(\vec{r}) \equiv \frac{V(\vec{r})}{m} &= -G \int \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} d^3 r' \\
\Rightarrow \quad \nabla\phi(\vec{r}) &= G \int \rho(\vec{r'}) \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} d^3 r' \\
\Rightarrow \quad \nabla^2\phi(\vec{r}) &= G \int \rho(\vec{r'}) \nabla \cdot \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} d^3 r'
\end{aligned} \tag{1.7}$$

The above equation simplifies considerably if we calculate the divergence in the integrand.

Note that "∇" operates on $\vec{r}$ only!

$$\begin{aligned}
\nabla \cdot \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} &= \frac{\nabla \cdot \vec{r}}{|\vec{r} - \vec{r'}|^3} + (\vec{r} - \vec{r'}) \cdot \nabla \frac{1}{|\vec{r} - \vec{r'}|^3} \\
&= \frac{3}{|\vec{r} - \vec{r'}|^3} - (\vec{r} - \vec{r'}) \cdot \frac{3(\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^5} \\
&= \frac{3}{|\vec{r} - \vec{r'}|^3} - \frac{3}{|\vec{r} - \vec{r'}|^3} \\
&= 0 \quad \forall \quad \vec{r} \neq \vec{r'}
\end{aligned} \tag{1.8}$$

We conclude that the Newtonian gravitational potential at a point in a gravitational field outside a mass distribution satisfies Laplace's equation

$$\boxed{\nabla^2 \phi = 0} \tag{1.9}$$

`Digression` **1.2.1 (Dirac's delta function)**
The Dirac delta function has the following properties:

1. $\delta(\vec{r}-\vec{r'}) = 0 \quad \forall \quad \vec{r} \neq \vec{r'}$

2. $\int \delta(\vec{r}-\vec{r'})d^3r' = 1$ when $\vec{r} = \vec{r'}$ is contained in the integration domain. The integral is identically zero otherwise.

3. $\int f(\vec{r'})\delta(\vec{r}-\vec{r'})d^3r' = f(\vec{r})$

A calculation of the integral $\int \nabla \cdot \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} d^3r'$ which is valid also in the case where the field point is inside the mass distribution is obtained through the use of Gauss' integral theorem:

$$\int_{\mathrm{v}} \nabla \cdot \vec{A} d^3r' = \oint_{\mathrm{s}} \vec{A} \cdot d\vec{s}, \tag{1.10}$$

where s is the boundary of v ($s = \partial v$ is an area).

Definition 1.2.1 (Solid angle)

$$d\Omega \equiv \frac{ds'_\perp}{|\vec{r}-\vec{r'}|^2} \tag{1.11}$$

where $ds'_\perp$ is the projection of the area ds' normal to the line of sight. $\vec{ds'}_\perp$ is the component vector of $\vec{ds'}$ along the line of sight which is equal to the normal vector of $ds'_\perp$ (see figure (1.3)).

Now, let's apply Gauss' integral theorem.

$$\int_{\mathrm{v}} \nabla \cdot \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} d^3r' = \oint_{\mathrm{s}} \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} \cdot d\vec{s'} = \oint_{\mathrm{s}} \frac{ds'_\perp}{|\vec{r}-\vec{r'}|^2} = \oint_{\mathrm{s}} d\Omega \tag{1.12}$$

So that,

$$\int_{\mathrm{v}} \nabla \cdot \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} d^3r' = \begin{cases} 4\pi & \text{if P is inside the mass distribution,} \\ 0 & \text{if P is outside the mass distribution.} \end{cases} \tag{1.13}$$

The above relation is written concisely in terms of the Dirac delta function:

$$\nabla \cdot \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} = 4\pi\delta(\vec{r}-\vec{r'}) \tag{1.14}$$

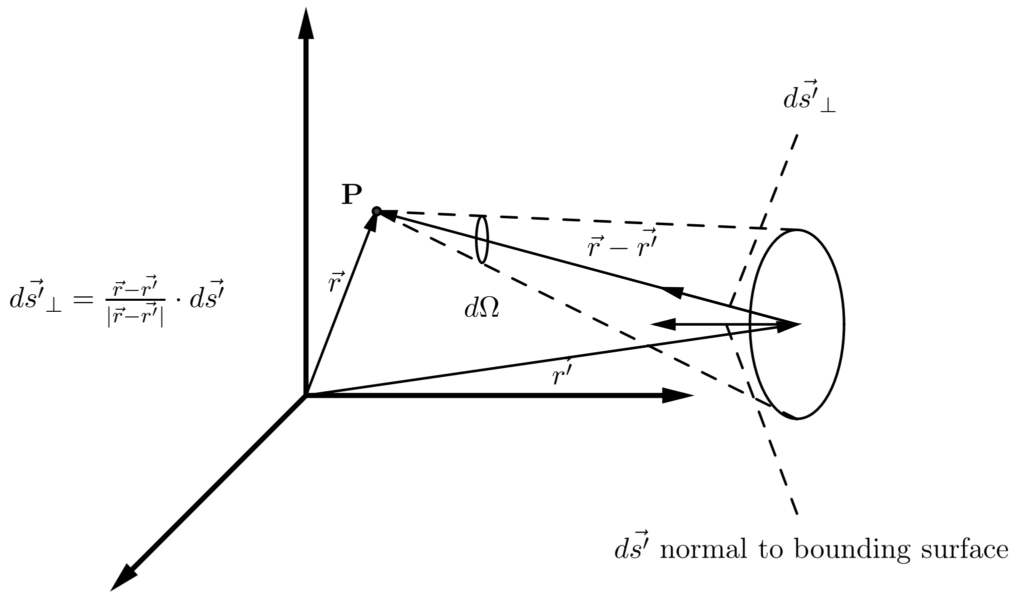


Figure 1.3: The solid angle $d\Omega$ is defined such that the surface of a sphere subtends 4π at the center

We now have

$$
\begin{aligned}
\nabla^2\phi(\vec{r}) &= G\int \rho(\vec{r'})\nabla\cdot\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}d^3r' \\
&= G\int \rho(\vec{r'})4\pi\delta(\vec{r}-\vec{r'})d^3r' \\
&= 4\pi G\rho(\vec{r})
\end{aligned}
\tag{1.15}
$$

Newton's theory of gravitation can now be expressed very succinctly indeed!

1. Mass generates gravitational potential according to

$$\nabla^2\phi = 4\pi G\rho \tag{1.16}$$

2. Gravitational potential generates motion according to

$$\vec{g} = -\nabla\phi \tag{1.17}$$

where $\vec{g}$ is the field strength of the gravitational field.

1.3 Tidal Forces

Tidal force is difference of gravitational force on two neighboring particles in a gravitational field. The tidal force is due to the inhomogeneity of a gravitational field.

In figure 1.4 two points have a separation vector $\vec{\zeta}$. The position vectors of 1 and 2 are $\vec{r}$ and $\vec{r}+\vec{\zeta}$, respectively, where $|\vec{\zeta}| \ll |\vec{r}|$. The gravitational forces on

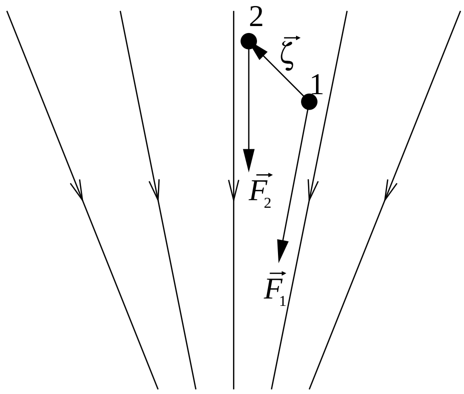


Figure 1.4: Tidal Forces

a mass m at 1 and at 2 are $\vec{F}(\vec{r})$ and $\vec{F}(\vec{r}+\vec{\zeta})$. By means of a Taylor expansion to lowest order in $|\vec{\zeta}|$ we get for the i-component of the tidal force

$$f_i = F_i(\vec{r}+\vec{\zeta}) - F_i(\vec{r}) = \zeta_j \left(\frac{\partial F_i}{\partial x^j}\right)_{\vec{r}}. \tag{1.18}$$

The corresponding vector equation is

$$\vec{f} = (\vec{\zeta}\cdot\nabla)_{\vec{r}}\vec{F}. \tag{1.19}$$

Using that

$$\vec{F} = -m\nabla\phi, \tag{1.20}$$

the tidal force may be expressed in terms of the gravitational potential according to

$$\vec{f} = -m(\vec{\zeta}\cdot\nabla)\nabla\phi. \tag{1.21}$$

It follows that in a local Cartesian coordinate system, the i-coordinate of the relative acceleration of the particles is

$$\frac{d^2\zeta_i}{dt^2} = -\left(\frac{\partial^2\phi}{\partial x^i\partial x^j}\right)_{\vec{r}}\zeta_j. \tag{1.22}$$

Let us look at a few simple examples. In the first one $\vec{\zeta}$ has the same direction as $\vec{g}$. Consider a small Cartesian coordinate system at a distance R from a mass M (see figure 1.5). If we place a particle of mass m at a point $(0,0,+z)$, it will, according to eq. (1.1) be acted upon by a force

$$F_z(+z) = -m\frac{GM}{(R+z)^2} \tag{1.23}$$

while an identical particle at the origin will be acted upon by the force

$$F_z(0) = -m\frac{GM}{R^2}. \tag{1.24}$$

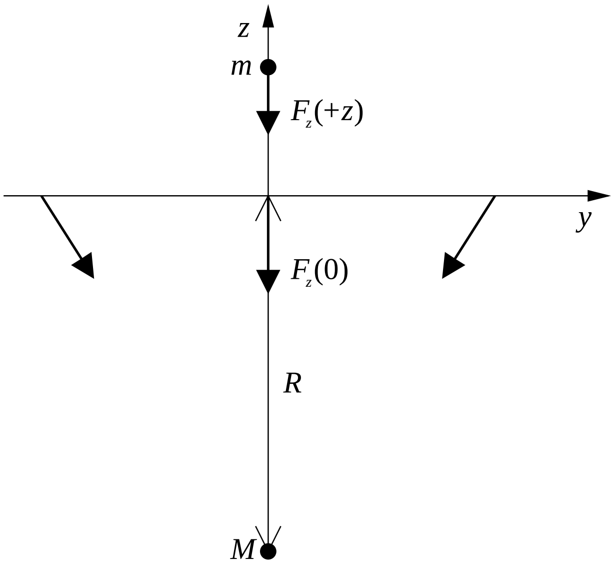


Figure 1.5: A small Cartesian coordinate system at a distance R from a mass M.

If this little coordinate system is falling freely towards M, an observer at the origin will say that the particle at $(0, 0, +z)$ is acted upon by a force

$$f_z = F_z(z) - F_z(0) \approx 2mz\frac{GM}{R^3} \tag{1.25}$$

directed away from the origin, along the positive z-axis. We have assumed $z \ll R$. This is the tidal force.

In the same way particles at the points $(+x, 0, 0)$ and $(0, +y, 0)$ are attracted towards the origin by tidal forces

$$f_x = -mx\frac{GM}{R^3}, \tag{1.26}$$

$$f_y = -my\frac{GM}{R^3}. \tag{1.27}$$

Eqs. (1.25)–(1.27) have among others the following consequence: If an elastic, circular ring is falling freely in the Earth's gravitational field, as shown in figure 1.6, it will be stretched in the vertical direction and compressed in the horizontal direction.

In general, tidal forces cause changes of shape.

The tidal forces from the Sun and the Moon cause flood and ebb on the Earth. Let us consider the effect due to the Moon. We then let M be the mass of the Moon, and choose a coordinate system with origin at the Earth's center. The tidal force per unit mass at a point is the negative gradient of the tidal potential

$$\phi(\vec{r}) = -\frac{GM}{R^3}\left(z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) = -\frac{GM}{2R^3}r^2(3\cos^2\theta - 1), \tag{1.28}$$

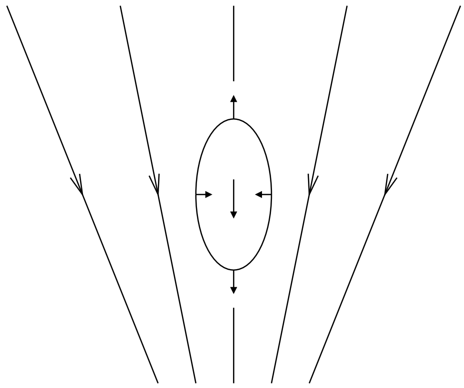


Figure 1.6: An elastic, circular ring falling freely in the Earth's gravitational field

where we have introduced spherical coordinates, $z = r\cos\theta$, $x^2 + y^2 = r^2 \sin^2\theta$, R is the distance between the Earth and the Moon, and the radius r of the spherical coordinate is equal to the radius of the Earth.

The potential at a height h above the surface of the Earth has one term, mgh, due to the attraction of the Earth and one given by eq. (1.28), due to the attraction of the Moon. Thus,

$$\Theta(r) = gh - \frac{GM}{2R^3} r^2 (3\cos^2\theta - 1). \tag{1.29}$$

At equilibrium, the surface of the Earth will be an equipotential surface, given by Θ = constant. The height of the water at flood, $\theta = 0$ or $\theta = \pi$, is therefore

$$h_{\text{flood}} = h_0 + \frac{GM}{gR}\left(\frac{r}{R}\right)^2, \tag{1.30}$$

where h_0 is an unknown constant. The height of the water at ebb ($\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$) is

$$h_{\text{ebb}} = h_0 - \frac{1}{2}\frac{GM}{gR}\left(\frac{r}{R}\right)^2. \tag{1.31}$$

The height difference between flood and ebb is therefore

$$\Delta h = \frac{3}{2}\frac{GM}{gR}\left(\frac{r}{R}\right)^2. \tag{1.32}$$

For a numerical result we need the following values:

$$M_{\text{Moon}} = 7.35 \cdot 10^{25}\text{g}, \qquad g = 9.81\text{m/s}^2, \tag{1.33}$$

$$R = 3.85 \cdot 10^5\text{km}, \qquad r_{\text{Earth}} = 6378\text{km}. \tag{1.34}$$

With these values we find $\Delta h = 53$cm, which is typical of tidal height differences.

1.4 The Principle of Equivalence

Galilei investigated experimentally the motion of freely falling bodies. He found that they moved in the same way, regardless what sort of material they consisted of and what mass they had.

In Newton's theory of gravitation mass appears in two different ways; as gravitational mass, m_G, in the law of gravitation, analogously to charge in Coulomb's law, and as inertial mass, m_I in Newton's 2nd law.

The equation of motion of a freely falling particle in the field of gravity from a spherical body with mass M then takes the form

$$\frac{d^2\vec{r}}{dt^2} = -G\frac{m_G}{m_I}\frac{M}{r^3}\vec{r}. \tag{1.35}$$

The results of Galilei's measurements imply that the quotient between gravitational and inertial mass must be the same for all bodies. With a suitable choice of units, we then obtain

$$m_G = m_I. \tag{1.36}$$

Measurements performed by the Hungarian baron Eötvös around the turn of the century indicated that this equality holds with an accuracy better than 10^{-8}. More recent experiments have given the result $|\frac{m_I}{m_G} - 1| < 9 \cdot 10^{-13}$.

Einstein assumed the exact validity of eq.(1.52). He did not consider this as an accidental coincidence, but rather as an expression of a fundamental principle, called *the principle of equivalence.*

A consequence of this principle is the possibility of removing the effect of a gravitational force by being in free fall. In order to clarify this, Einstein considered a homogeneous gravitational field in which the acceleration of gravity, g, is independent of the position. In a freely falling, non-rotating reference frame in this field, all free particles move according to

$$m_I\frac{d^2\vec{r}}{dt^2} = (m_G - m_I)\vec{g} = 0, \tag{1.37}$$

where eq. (1.36) has been used.

This means that an observer in such a freely falling reference frame will say that the particles around him are not acted upon by forces. They move with constant velocities along straight paths. In other words, such a reference frame is inertial.

Einstein's heuristic reasoning suggests equivalence between inertial frames in regions far from mass distributions, where there are no gravitational fields, and inertial frames falling freely in a gravitational field. This equivalence between all types of inertial frames is so intimately connected with the equivalence between gravitational and inertial mass, that the term "principle of equivalence" is used whether one talks about masses or inertial frames. The equivalence of different types of inertial frames encompasses all types of physical phenomena, not only particles in free fall.

The principle of equivalence has also been formulated in an "opposite" way. An observer at rest in a homogeneous gravitational field, and an observer in an accelerated reference frame in a region far from any mass distributions, will obtain identical results when they perform similar experiments. An inertial field caused by the acceleration of the reference frame, is equivalent to a field of gravity caused by a mass distribution, as far is tidal effects can be ignored.

1.5 The general principle of relativity

The principle of equivalence led Einstein to a generalization of the special principle of relativity. In his general theory of relativity Einstein formulated a general principle of relativity, which says that not only velocities are relative, but accelerations, too.

Consider two formulations of the special principle of relativity.

S1 All laws of Nature are the same (may be formulated in the same way) in all inertial frames.

S2 Every inertial observer can consider himself to be at rest.

These two formulations may be interpreted as different formulations of a single principle. But the generalization of S1 and S2 to the general case, which encompasses accelerated motion and non-inertial frames, leads to two different principles G1 and G2.

G1 The laws of Nature are the same in all reference frames.

G2 Every observer can consider himself to he at rest.

In the literature both G1 and G2 are mentioned as *the general principle of relativity*. But G2 is a stronger principle (i.e. stronger restriction on natural phenomena) than G1. Generally the course of events of a physical process in a certain reference frame, depends upon the laws of physics, the boundary conditions, the motion of the reference frame and the geometry of space-time. The *two* latter properties are described by means of a metrical tensor. By formulating the physical laws in a metric independent way, one obtains that G1 is valid for all types of physical phenomena.

Even if the laws of Nature are the same in all reference frames, the course of events of a physical process will, as mentioned above, depend upon the motion of the reference frame. As to the spreading of light, for example, the law is that light follows null-geodesic curves (see ch. 4). This law implies that the path of a light particle is curved in non-inertial reference frames and straight in inertial frames.

The question whether G2 is true in the general theory of relativity has been thoroughly discussed recently, and the answer is not clear yet.

1.6 The covariance principle

The principle of relativity is a physical principle. It is concerned with physical phenomena. This principle motivates the introduction of a formal principle, called the *covariance principle*: The equations of a physical theory shall have the same form in every coordinate system.

This principle is not concerned directly with physical phenomena. The principle may be fulfilled for every theory by writing the equations in a form-invariant i.e. covariant way. This may he done by using tensor (vector) quantities, only, in the mathematical formulation of the theory.

The covariance principle and the equivalence principle may be used to obtain a description of what happens in the presence of gravitation. We then start with the physical laws as formulated in the special theory of relativity. Then the laws are written in a covariant form, by writing them as tensor equations. They are then valid in an arbitrary, accelerated system. But the inertial field ("fictive force") in the accelerated frame is equivalent to a gravitational field. So, starting with in a description referred to an inertial frame, we have obtained a description valid in the presence of a gravitational field.

The tensor equations have in general a coordinate independent form. Yet, such form-invariant, or covariant, equations need not fulfill the principle of relativity.

This is due to the following circumstances. A physical principle, for example the principle of relativity, is concerned with observable relationships. Therefore, when one is going to deduce the observable consequences of an equation, one has to establish relations between the tensor-components of the equation and observable physical quantities. Such relations have to be defined; they are not determined by the covariance principle.

From the tensor equations, that are covariant, and the defined relations between the tensor components and the observable physical quantities, one can deduce equations between physical quantities. The special principle of relativity, for example, demands that the laws which these equations express must be the same with reference to every inertial frame

The relationships between physical quantities and tensors (vectors) are theory dependent. The relative velocity between two bodies, for example, is a vector within Newtonian kinematics. However, in the relativistic kinematics of four-dimensional space-time, an ordinary velocity, which has only three components, is not a vector. Vectors in space-time, so called 4-vectors, have four components. Equations between physical quantities are not covariant in general.

For example, Maxwell's equations in three-vector-form are not invariant under a Galilei transformation. However, if these equations are rewritten in tensor-form, then neither a Galilei transformation nor any other transformation will change the form of the equations.

If all equations of a theory are tensor equations, the theory is said to be given a *manifestly covariant form*. A theory that is written in a manifestly covariant form, will automatically fulfill the covariance principle, but it need not fulfill the principle of relativity.

1.7 Mach's principle

Einstein gave up Newton's idea of an absolute space. According to Einstein all motion is relative. This may sound simple, but it leads to some highly non-trivial and fundamental questions.

Imagine that there are only two particles connected by a spring, in the universe. What will happen if the two particles rotate about each other? Will the spring be stretched due to centrifugal forces? Newton would have confirmed that this is indeed what will happen. However, when there is no longer any absolute space that the particles can rotate relatively to, the answer is not so obvious. If we, as observers, rotate around the particles, and they are at rest, we would not observe any stretching of the spring. But this situation is now kinematically equivalent to the one with rotating particles and observers at rest, which leads to stretching.

Such problems led Mach to the view that all motion is relative. The motion of a particle in an empty universe is not defined. All motion is motion relatively to something else, i.e. relatively to other masses. According to Mach this implies that inertial forces must be due to a particle's acceleration relatively to the great masses of the universe. If there were no such cosmic masses, there would not exist inertial forces, like the centrifugal force. In our example with two particles connected by a string, there would not be any stretching of the spring, if there were no cosmic masses that the particles could rotate relatively to.

Another example may be illustrated by means of a turnabout. If we stay on this, while it rotates, we feel that the centrifugal forces lead us outwards. At the same time we observe that the heavenly bodies rotate. According to Mach identical centrifugal forces should appear if the turnabout is static and the heavenly bodies rotate.

Einstein was strongly influenced by Mach's arguments, which probably had some influence, at least with regards to motivation, on Einstein's construction of his general theory of relativity. Yet, it is clear that general relativity does not fulfill all requirements set by Mach's principle. For example there exist general relativistic, rotating cosmological models, where free particles will tend to rotate relative to the cosmic masses of the model.

However, some Machian effects have been shown to follow from the equations of the general theory of relativity. For example, inside a rotating, massive shell the inertial frames, i.e. the free particles, are dragged on and tend to rotate in the same direction as the shell. This was discovered by Lense and Thirring in 1918 and is therefore called the Lense-Thirring effect. More recent investigations of this effect have, among others, lead to the following result (Brill and Cohen 1966): "A massive shell with radius equal to its Schwarzschild radius has often been used as an idealized model of our universe. Our result shows that in such models local inertial frames near the center cannot rotate relatively to the mass of the universe. In this way our result gives an explanation in accordance with Mach's principle, of the fact that the "fixed stars" is at rest on heaven as observed from an inertial reference frame."

Chapter 2

Vectors, Tensors and Forms

2.1 Vectors

An expression on the form $a^\mu \vec{e}_\mu$, where a^μ, $\mu = 1, 2, ..., n$ are real numbers, is known as a **linear combination** of the vectors $\vec{e}_\mu$.

The vectors $\vec{e}_1, ..., \vec{e}_n$ are said to be linearly independent if there does **not** exist real numbers $a^\mu \neq 0$ such that $a^\mu \vec{e}_\mu = 0$.

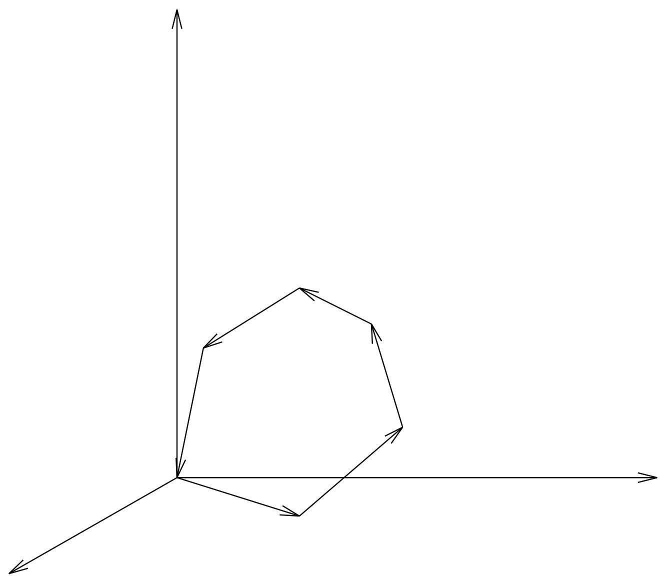


Figure 2.1: Closed polygon (linearly dependent)

Geometrical interpretation: A set of vectors are **linearly independent** if it is **not** possible to construct a closed polygon of the vectors (even by adjusting their lengths).

A set of vectors $\vec{e}_1, \ldots, \vec{e}_n$ are said to be **maximally linearly independent** if $\vec{e}_1, \ldots, \vec{e}_n, \vec{v}$ are linearly dependent for all vectors $\vec{v} \neq \vec{e}_\mu$. We define the **dimension** of a vector-space as the number of vectors in a maximally linearly independent set of vectors of the space. The vectors $\vec{e}_\mu$ in such a set are known as the **basis-vectors** of the space.

$$
\begin{aligned}
\vec{v} + a^\mu \vec{e}_\mu &= 0 \\
&\Downarrow \\
\vec{v} &= -a^\mu \vec{e}_\mu
\end{aligned}
\tag{2.1}
$$

The components of $\vec{v}$ are the numbers v^μ defined by $v^\mu = -a^\mu \Rightarrow \vec{v} = v^\mu \vec{e}_\mu$.

2.1.1 4-vectors

4-vectors are vectors which exist in (4-dimensional) space-time. A 4-vector equation represents 4 independent component equations.

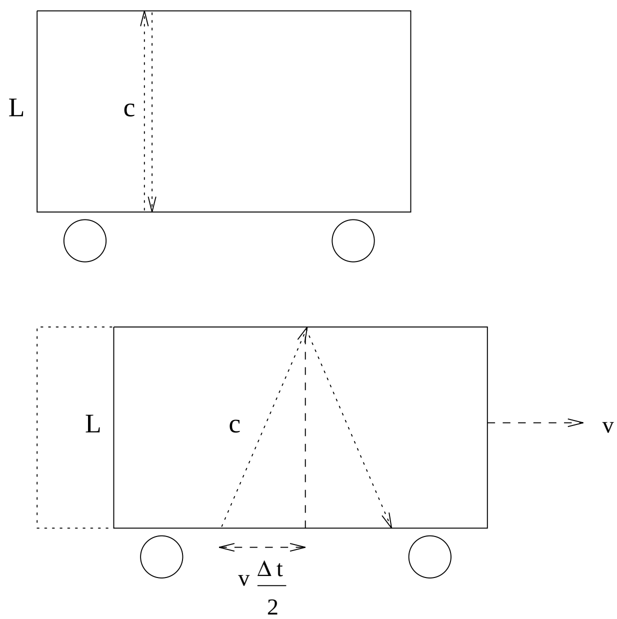


Figure 2.2: Carriage at rest (top) and with velocity $\vec{v}$ (bottom)

`Example` **2.1.1 (Photon clock)**
Carriage at rest:

$$\Delta t_0 = \frac{2L}{c}$$

Carriage with velocity $\vec{v}$:

$$
\begin{aligned}
\Delta t &= \frac{2\sqrt{(v\frac{\Delta t}{2})^2 + L^2}}{c} \\
&\Downarrow \\
c^2\Delta t^2 &= v^2\Delta t^2 + 4L^2 \\
&\Downarrow \\
\Delta t &= \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L/c}{\sqrt{1 - v^2/c^2}} = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} \qquad (2.2)
\end{aligned}
$$

The proper time-interval is denoted by $d\tau$ (above it was denoted Δt_0). The proper time-interval for a particle is measured with a standard clock which follows the particle.

Definition 2.1.1 (4-velocity)

$$\vec{U} = c\frac{dt}{d\tau}\vec{e}_t + \frac{dx}{d\tau}\vec{e}_x + \frac{dy}{d\tau}\vec{e}_y + \frac{dz}{d\tau}\vec{e}_z, \qquad (2.3)$$

where t is the coordinate time, measured with clocks at rest in the reference frame.

$$
\begin{aligned}
\vec{U} &= U^\mu \vec{e}_\mu = \frac{dx^\mu}{d\tau}\vec{e}_\mu, \quad x^\mu = (ct, x, y, z), \quad x^0 \equiv ct \\
\frac{dt}{d\tau} &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma \qquad (2.4)
\end{aligned}
$$

$\vec{U} = \gamma(c, \vec{v})$, where $\vec{v}$ is the common 3-velocity of the particle.

Definition 2.1.2 (4-momentum)

$$\vec{P} = m_0\vec{U}, \qquad (2.5)$$

where m_0 is the rest mass of the particle.

$\vec{P} = (\frac{E}{c}, \vec{p})$, where $\vec{p} = \gamma m_0\vec{v} = m\vec{v}$ and E is the relativistic energy.

The 4-force or Minkowski-force $\vec{F} \equiv \frac{d\vec{P}}{d\tau}$ and the 'common force' $\vec{f} = \frac{d\vec{p}}{dt}$. Then

$$\vec{F} = \gamma(\frac{1}{c}\vec{f} \cdot \vec{v}, \vec{f}) \qquad (2.6)$$

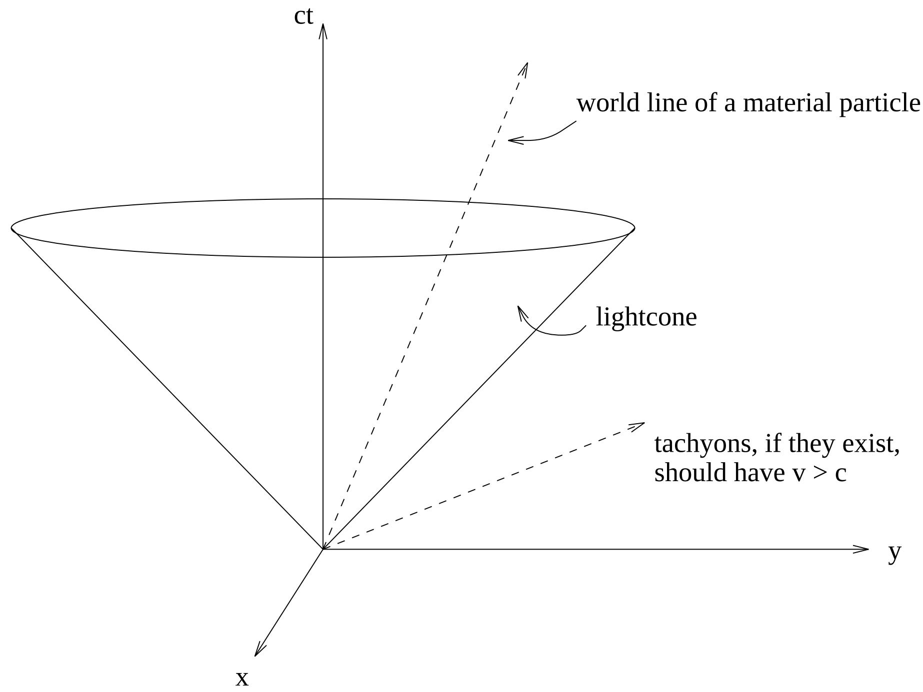


Figure 2.3: World-lines in a Minkowski diagram

Definition 2.1.3 (4-acceleration)

$$\vec{A} = \frac{d\vec{U}}{d\tau} \tag{2.7}$$

The 4-velocity has the scalar value c so that

$$\vec{U} \cdot \vec{U} = -c^2 \tag{2.8}$$

The *4-velocity identity* eq. 2.8 gives $\vec{U} \cdot \vec{A} = 0$, in other words $\vec{A} \perp \vec{U}$ and $\vec{A}$ is space-like.

The line element for Minkowski space-time (flat space-time) with Cartesian coordinates is

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{2.9}$$

In general relativity theory, gravitation is not considered a force. Gravitation is instead described as motion in a curved space-time.

A particle in free fall, is in Newtonian gravitational theory said to be only influenced by the gravitational force. According to general relativity theory the particle is not influenced by any force.

Such a particle has no 4-acceleration. $\vec{A} \neq 0$ implies that the particle is not in free fall. It is then influenced by non-gravitational forces.

One has to distinguish between observed acceleration, ie. common 3-acceleration, and the absolute 4-acceleration.

2.1.2 Tangent vector fields and coordinate vectors

In a curved space position vectors with finite length do not exist. (See figure 2.4).

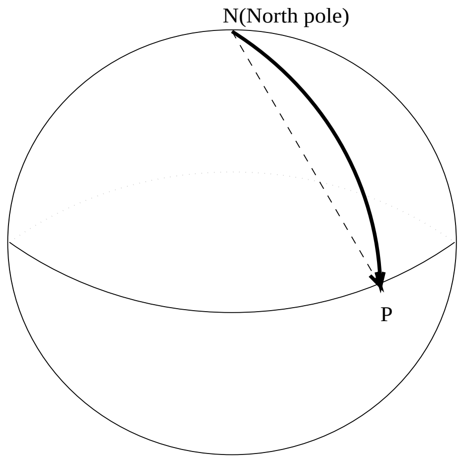


Figure 2.4: In curved space,vectors can only exist in tangent planes.The vectors in the tangent plane of N,do not contain the vector $\overrightarrow{NP}$ (dashed line).

Different points in a curved space have different tangent planes. Finite vectors do only exist in these tangent planes (See figure 2.5). However, infinitesimal position vectors $d\vec{r}$ do exist.

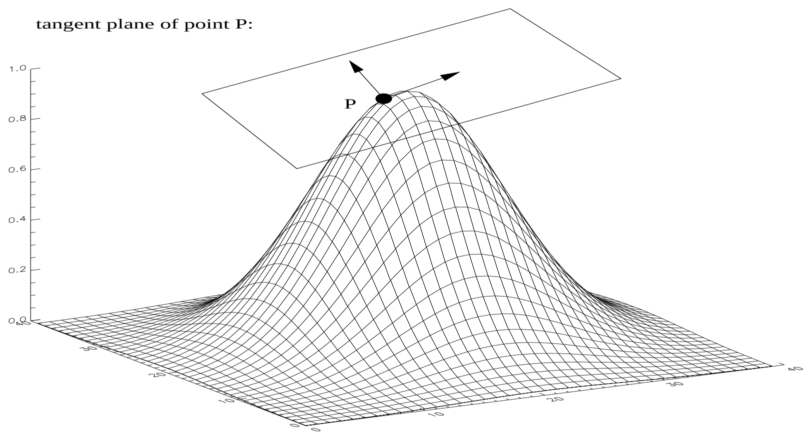


Figure 2.5: In curved space,vectors can only exist in tangent planes

Definition 2.1.4 (Reference frame)
A **reference frame** is defined as a continuum of non-intersecting timelike world lines in spacetime.

We can view a reference frame as a set of reference particles with a specified motion. An *inertial reference frame* is a non-rotating set of free particles.

Definition 2.1.5 (Coordinate system)
A **coordinate system** is a continuum of 4-tuples giving a unique set of coordinates for events in spacetime.

Definition 2.1.6 (Comoving coordinate system)
A **comoving coordinate system** in a frame is a coordinate system where the particles in the reference frame have constant spatial coordinates.

Definition 2.1.7 (Orthonormal basis)
An **orthonormal basis** $\{\vec{e}_{\hat{\mu}}\}$ in spacetime is defined by

$$\begin{aligned} \vec{e}_{\hat{t}} \cdot \vec{e}_{\hat{t}} &= -1 (c = 1) \\ \vec{e}_{\hat{i}} \cdot \vec{e}_{\hat{j}} &= \delta_{\hat{i}\hat{j}} \end{aligned} \tag{2.10}$$

where $\hat{i}$ and $\hat{j}$ are space indices.

Definition 2.1.8 (Coordinate basis vectors.)
Temporary definition of coordinate basis vector:
Assume any coordinate system $\{x^\mu\}$.

$$\vec{e}_\mu \equiv \frac{\partial \vec{r}}{\partial x^\mu} \tag{2.11}$$

A *vector field* is a continuum of vectors in a space, where the components are continuous and differentiable functions of the coordinates. Let $\vec{v}$ be a tangent vector to the curve $\vec{r}(\lambda)$:

$$\vec{v} = \frac{d\vec{r}}{d\lambda} \qquad \text{where} \quad \vec{r} = \vec{r}[x^\mu(\lambda)] \tag{2.12}$$

The chain rule for differentiation yields:

$$\vec{v} = \frac{d\vec{r}}{d\lambda} = \frac{\partial \vec{r}}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{d\lambda}\vec{e_\mu} = v^\mu \vec{e}_\mu \tag{2.13}$$

Thus, the components of the tangent vector field along a curve, parameterised by λ, is given by:

$$v^\mu = \frac{dx^\mu}{d\lambda} \tag{2.14}$$

In the theory of relativity, the invariant parameter is often chosen to be the proper time. Tangent vector to the world line of a material particle:

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{2.15}$$

These are the components of the 4-velocity of the particle!

`Digression` **2.1.1 (Proper time of the photon.)**
Minkowski-space:

$$\begin{aligned} ds^2 &= -c^2 dt^2 + dx^2 \\ &= -c^2 dt^2 \big(1 - \frac{1}{c^2}\big(\frac{dx}{dt}\big)^2\big) \\ &= -\big(1 - \frac{v^2}{c^2}\big)c^2 dt^2 \end{aligned} \tag{2.16}$$

For a photon,$v = c$ so:

$$\lim_{v \to c} ds^2 = 0 \tag{2.17}$$

Thus, the spacetime interval between two points on the world line of a photon, is zero! This also means that the proper time for the photon is zero!! (See example 2.1.2).

`Digression` **2.1.2 (Relationships between spacetime intervals, time and proper time.)**
Physical interpretation of the spacetime interval for a timelike interval:

$$ds^2 = -c^2 d\tau^2 \tag{2.18}$$

where dτ is the proper time interval between two events, measured on a clock moving in a way, such that it is present on both events (figure 2.6).

$$\begin{aligned} -c^2 d\tau^2 &= -c^2\big(1 - \frac{v^2}{c^2}\big)dt^2 \\ \Rightarrow d\tau &= \sqrt{1 - \frac{v^2}{c^2}}dt \end{aligned} \tag{2.19}$$

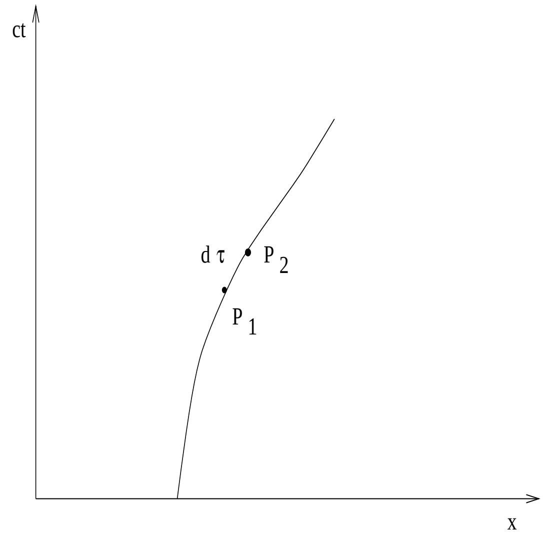


Figure 2.6: P_1 and P_2 are two events in spacetime, separated by a proper time interval $d\tau$.

The time interval between to events in the laboratory, is smaller measured on a moving clock than measured on a stationary one, because the moving clock is ticking slower!

2.1.3 Coordinate transformations

Given two coordinate systems $\{x^\mu\}$ and $\{x^{\mu\prime}\}$.

$$\vec{e}_{\mu'} = \frac{\partial \vec{r}}{\partial x^{\mu'}} \tag{2.20}$$

Suppose there exists a coordinate transformation, such that the primed coordinates are functions of the unprimed, and vice versa. Then we can apply the chain rule:

$$\vec{e}_{\mu'} = \frac{\partial \vec{r}}{\partial x^{\mu'}} = \frac{\partial \vec{r}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial x^{\mu'}} = \vec{e}_{\mu}\frac{\partial x^{\mu}}{\partial x^{\mu'}} \tag{2.21}$$

This is the transformation equation for the basis vectors. $\frac{\partial x^\mu}{\partial x^{\mu'}}$ are elements of the transformation matrix. Indices that are *not* sum-indices are called 'free indices'.

Rule: In *all terms* on each side in an equation, the free indices should behave identically (high or low), **and** there should be exactly the *same* indices in all terms!

Applying this rule, we can now find the inverse transformation

$$\begin{aligned}\vec{e}_\mu &= \vec{e}_{\mu'}\frac{\partial x^{\mu'}}{\partial x^\mu}\\ \vec{v} &= v^{\mu'}\vec{e}_{\mu'} = v^\mu \vec{e}_\mu = v^{\mu'}\vec{e}_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}\end{aligned}$$

So, the transformation rules for the *components* of a vector becomes

$$\boxed{v^\mu = v^{\mu'}\frac{\partial x^\mu}{\partial x^{\mu'}}\,; \qquad v^{\mu'} = v^\mu \frac{\partial x^{\mu'}}{\partial x^\mu}} \tag{2.22}$$

The directional derivative along a curve, parametrised by λ:

$$\frac{d}{d\lambda} = \frac{\partial}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = v^\mu \frac{\partial}{\partial x^\mu} \tag{2.23}$$

where $v^\mu = \frac{dx^\mu}{d\lambda}$ are the components of the tangent vector of the curve. Directional derivative along a coordinate curve:

$$\lambda = x^\nu \qquad \frac{\partial}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu{}_\nu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\nu} \tag{2.24}$$

In the primed system:

$$\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\mu} \tag{2.25}$$

Definition 2.1.9 (Coordinate basis vectors.)
We define the <u>coordinate basis vectors</u> as:

$$\boxed{\vec{e}_\mu = \frac{\partial}{\partial x^\mu}} \tag{2.26}$$

This definition is not based upon the existence of finite position vectors. It applies in curved spaces as well as in flat spaces.

`Example` **2.1.2 (Coordinate transformation)**
From Figure 2.7 we see that

$$\boxed{x = r\cos\theta,\ y = r\sin\theta} \tag{2.27}$$

Coordinate basis vectors were defined by

$$\vec{e_\mu} \equiv \frac{\partial}{\partial x^\mu} \tag{2.28}$$

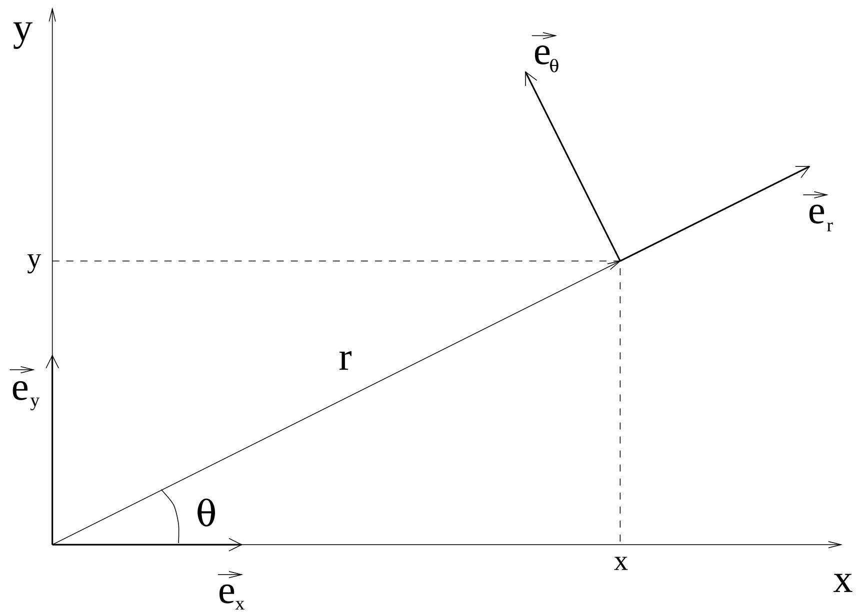


Figure 2.7: Coordinate transformation, flat space.

This means that we have

$$\begin{aligned} \vec{e_x} &= \frac{\partial}{\partial x}\ , \quad \vec{e_y} = \frac{\partial}{\partial y}\ , \quad \vec{e_r} = \frac{\partial}{\partial r}\ , \quad \vec{e_\theta} = \frac{\partial}{\partial \theta} \\ \vec{e_r} &= \frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} \end{aligned} \tag{2.29}$$

Using the chain rule and Equations (2.27) and (2.29) we get

$$\begin{aligned} \vec{e_r} &= \cos\theta \vec{e_x} + \sin\theta \vec{e_y} \\ \vec{e_\theta} &= \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} \\ &= -r\sin\theta \vec{e_x} + r\cos\theta \vec{e_y} \end{aligned} \tag{2.30}$$

But are the vectors in (2.30) also unit vectors?

$$\vec{e_r} \cdot \vec{e_r} = \cos^2\theta + \sin^2\theta = 1 \tag{2.31}$$

So $\vec{e_r}$ is a unit vector, $|\vec{e_r}| = 1$.

$$\vec{e_\theta} \cdot \vec{e_\theta} = r^2(\cos^2\theta + \sin^2\theta) = r^2 \tag{2.32}$$

and we see that $\vec{e_\theta}$ is **not** a unit vector, $|\vec{e_\theta}| = r$. But we have that $\vec{e_r} \cdot \vec{e_\theta} = 0 \Rightarrow \vec{e_r} \perp \vec{e_\theta}$. **Coordinate basis vectors are not generally unit vectors.**

Definition 2.1.10 (Orthonormal basis)
An orthonormal basis is a vector basis consisting of unit vectors that are normal to each other. To show that we are using an orthonormal basis we will use 'hats' over the indices, $\{\vec{e}_{\hat{\mu}}\}$.

Orthonormal basis associated with planar polar coordinates:

$$\vec{e}_{\hat{r}} = \vec{e}_r \ , \quad \vec{e}_{\hat{\theta}} = \frac{1}{r}\vec{e}_\theta \tag{2.33}$$

`Example` **2.1.3 (Relativistic Doppler Effect)**
The Lorentz transformation is known from special relativity and relates the reference frames of two systems where one is moving with a constant velocity v with regard to the other,

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(t - \frac{vx}{c^2}) \end{aligned}$$

According to the vector component transformation (2.22), the 4-momentum for a particle moving in the x-direction, $P^\mu = (\frac{E}{c}, p, 0, 0)$ transforms as

$$\begin{aligned} P^{\mu'} &= \frac{\partial x^{\mu'}}{\partial x^\mu} P^\mu, \\ E' &= \gamma(E - vp). \end{aligned}$$

Using the fact that a photon has energy $E = h\nu$ and momemtum $p = \frac{h\nu}{c}$, where h is Planck's constant and ν is the photon's frequency, we get the equation for the frequency shift known as the relativistic Doppler effect,

$$\nu' = \gamma(\nu - \frac{v}{c}\nu) = \frac{\left(1 - \frac{v}{c}\right)\nu}{\sqrt{\left(1 - \frac{v}{c}\right)\left(1 + \frac{v}{c}\right)}}$$

$$\boxed{\frac{\nu'}{\nu} = \sqrt{\frac{c - v}{c + v}}} \tag{2.34}$$

2.1.4 Structure coefficients

Definition 2.1.11 (Commutators between vectors)
The commutator between two vectors, $\vec{u}$ and $\vec{v}$, is defined as

$$[\vec{u}\ ,\ \vec{v}] \equiv \vec{u}\vec{v} - \vec{v}\vec{u} \tag{2.35}$$

where $\vec{u}\vec{v}$ is defined as

$$\vec{u}\vec{v} \equiv u^\mu \vec{e_\mu}(v^\nu \vec{e_\nu}) = u^\mu \frac{\partial}{\partial x^\mu}(v^\nu \frac{\partial}{\partial x^\nu}) \tag{2.36}$$

We can think of a vector as a linear combination of partial derivatives. We get:

$$\begin{aligned} \vec{u}\vec{v} &= u^\mu \frac{\partial v^\nu}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + u^\mu v^\nu \frac{\partial^2}{\partial x^\mu \partial x^\nu} \\ &= u^\mu \frac{\partial v^\nu}{\partial x^\mu}\vec{e_\nu} + u^\mu v^\nu \frac{\partial^2}{\partial x^\mu \partial x^\nu} \end{aligned} \tag{2.37}$$

Due to the last term, $\vec{u}\vec{v}$ is **not** a vector.

$$\begin{aligned} \vec{v}\vec{u} &= v^\nu \frac{\partial}{\partial x^\nu}(u^\mu \frac{\partial}{\partial x^\mu}) \\ &= v^\nu \frac{\partial u^\mu}{\partial x^\nu}\vec{e_\mu} + v^\nu u^\mu \frac{\partial^2}{\partial x^\nu \partial x^\mu} \\ \\ \vec{u}\vec{v} - \vec{v}\vec{u} &= u^\mu \frac{\partial v^\nu}{\partial x^\mu}\vec{e_\nu} - \underbrace{v^\nu \frac{\partial u^\mu}{\partial x^\nu}\vec{e_\mu}}_{v^\mu \frac{\partial u^\nu}{\partial x^\mu}\vec{e_\nu}} \\ &= (u^\mu \frac{\partial v^\nu}{\partial x^\mu} - v^\mu \frac{\partial u^\nu}{\partial x^\mu})\vec{e_\nu} \end{aligned} \tag{2.38}$$

Here we have used that

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu} = \frac{\partial^2}{\partial x^\nu \partial x^\mu} \tag{2.39}$$

The Einstein comma notation ⇒

$$\vec{u}\vec{v} - \vec{v}\vec{u} = (u^\mu v^\nu{}_{,\mu} - v^\mu u^\nu{}_{,\mu})\vec{e_\nu} \tag{2.40}$$

As we can see, the commutator between two vectors is itself a vector.

Definition 2.1.12 (Structure coefficients $c^\rho{}_{\mu\nu}$)
The structure coefficients $c^\rho{}_{\mu\nu}$ in an arbitrary basis $\{\vec{e_\mu}\}$ are defined by:

$$[\vec{e_\mu}\ ,\ \vec{e_\nu}] \equiv c^\rho{}_{\mu\nu}\vec{e_\rho} \tag{2.41}$$

Structure coefficients in a coordinate basis:

$$\begin{aligned} [\vec{e_\mu}\ ,\ \vec{e_\nu}] &= [\frac{\partial}{\partial x^\mu}\ ,\ \frac{\partial}{\partial x^\nu}] \\ &= \frac{\partial}{\partial x^\mu}(\frac{\partial}{\partial x^\nu}) - \frac{\partial}{\partial x^\nu}(\frac{\partial}{\partial x^\mu}) \\ &= \frac{\partial^2}{\partial x^\mu \partial x^\nu} - \frac{\partial^2}{\partial x^\nu \partial x^\mu} = 0 \end{aligned} \tag{2.42}$$

The commutator between two coordinate basis vectors is zero, so the structure coefficients are zero in coordinate basis.

`Example` **2.1.4 (Structure coefficients in planar polar coordinates)**
We will find the structure coefficients of an orthonormal basis in planar polar coordinates. In (2.33) we found that

$$\vec{e}_{\hat{r}} = \vec{e}_r \ , \quad \vec{e}_{\hat{\theta}} = \frac{1}{r}\vec{e}_\theta \tag{2.43}$$

We will now use this to find the structure coefficients.

$$\begin{aligned}
[\vec{e}_{\hat{r}} \ , \ \vec{e}_{\hat{\theta}}] &= [\frac{\partial}{\partial r} \ , \ \frac{1}{r}\frac{\partial}{\partial \theta}] \\
&= \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial \theta}) - \frac{1}{r}\frac{\partial}{\partial \theta}(\frac{\partial}{\partial r}) \\
&= -\frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial^2}{\partial \theta \partial r} \\
&= -\frac{1}{r^2}\vec{e}_\theta = -\frac{1}{r}\vec{e}_{\hat{\theta}}
\end{aligned} \tag{2.44}$$

To find the structure coefficients in coordinate basis we must use $[\vec{e}_{\hat{r}} \ , \ \vec{e}_{\hat{\theta}}] = -\frac{1}{r}\vec{e}_{\hat{\theta}}$.

$$[\vec{e}_{\hat{\mu}} \ , \ \vec{e}_{\hat{\nu}}] = c^{\hat{\rho}}{}_{\hat{\mu}\hat{\nu}}\vec{e}_{\hat{\rho}} \tag{2.45}$$

Using (2.44) and (2.45) we get

$$c^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} = -\frac{1}{r} \tag{2.46}$$

From the definition of $c^\rho{}_{\mu\nu}$ ($[\vec{u}\,,\,\vec{v}] = -[\vec{v}\,,\,\vec{u}]$) we see that the structure coefficients are anti symmetric in their lower indices:

$$\boxed{c^\rho{}_{\mu\nu} = -c^\rho{}_{\nu\mu}} \tag{2.47}$$

$$c^{\hat{\theta}}{}_{\hat{\theta}\hat{r}} = \frac{1}{r} = -c^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} \tag{2.48}$$

2.2 Tensors

A 1-form-basis $\underline{\omega}^1, \ldots, \underline{\omega}^n$ is defined by:

$$\underline{\omega}^\mu(\vec{e_\nu}) = \delta^\mu{}_\nu \tag{2.49}$$

An **arbitrary** 1-form can be expressed, in terms of its components, as a linear combination of the basis forms:

$$\underline{\alpha} = \alpha_\mu \underline{\omega}^\mu \tag{2.50}$$

where α_μ are the components of $\underline{\alpha}$ in the given basis.
Using eqs.(2.49) and (2.50), we find:

$$\begin{aligned} \underline{\alpha}(\vec{e}_\nu) &= \alpha_\mu \underline{\omega}^\mu(\vec{e}_\nu) = \alpha_\mu \delta^\mu_\nu = \alpha_\nu \\ \underline{\alpha}(\vec{v}) &= \underline{\alpha}(v^\mu \vec{e}_\mu) = v^\mu \underline{\alpha}(\vec{e}_\mu) = v^\mu \alpha_\mu = v^1\alpha_1 + v^2\alpha_2 + \dots \end{aligned} \tag{2.51}$$

We will now look at functions of multiple variables.

Definition 2.2.1 (Multilinear function, tensors)
A multilinear function is a function that is linear in all its arguments and maps one-forms and vectors into real numbers.

- A **covariant tensor** only maps vectors.
- A **contravariant tensor** only maps forms.
- A **mixed tensor** maps both vectors and forms into R.

A tensor of **rank** $\binom{N}{N'}$ maps N one-forms and N' vectors into R. It is usual to say that a tensor is of rank $(N+N')$. A one-form, for example, is a covariant tensor of rank 1:

$$\underline{\alpha}(\vec{v}) = v^\mu \alpha_\mu \tag{2.52}$$

Definition 2.2.2 (Tensor product)
The basis of a tensor R of rank q contains a **tensor product**, $\otimes$. If T and S are two tensors of rank m and n, the tensor product is defined by:

$$T \otimes S(\vec{u_1},\dots,\vec{u_m},\vec{v_1},\dots,\vec{v_n}) \equiv T(\vec{u_1},\dots,\vec{u_m})S(\vec{v_1},\dots,\vec{v_n}) \tag{2.53}$$

where T and S are tensors of rank m and n, respectively. $T \otimes S$ is a tensor of rank $(m+n)$.

Let $R = T \otimes S$. We then have

$$R = R_{\mu_1,\dots,\mu_q} \underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} \otimes \dots \otimes \underline{\omega}^{\mu_q} \tag{2.54}$$

Notice that $S \otimes T \neq T \otimes S$. We get the components of a tensor (R) by using the tensor on the basis vectors:

$$R_{\mu_1,\dots,\mu_q} = R(\vec{e_{\mu_1}},\dots,\vec{e_{\mu_q}}) \tag{2.55}$$

The indices of the components of a contravariant tensor are written as upper indices, and the indices of a covariant tensor as lower indices.

Example 2.2.1 (Example of a tensor)
Let $\vec{u}$ and $\vec{v}$ be two vectors and $\underline{\alpha}$ and $\underline{\beta}$ two 1-forms.

$$\vec{u} = u^{\mu}\vec{e}_{\mu}; \quad \vec{v} = v^{\mu}\vec{e}_{\mu}; \quad \underline{\alpha} = \alpha_{\mu}\underline{\omega}^{\mu}; \quad \underline{\beta} = \beta_{\mu}\underline{\omega}^{\mu} \tag{2.56}$$

From these we can construct tensors of rank 2 through the relation $R = \vec{u} \otimes \vec{v}$ as follows: The components of R are

$$\begin{aligned} R^{\mu_1\mu_2} &= R(\underline{\omega}^{\mu_1}, \underline{\omega}^{\mu_2}) \\ &= \vec{u} \otimes \vec{v}(\underline{\omega}^{\mu_1}, \underline{\omega}^{\mu_2}) \\ &= \vec{u}(\underline{\omega}^{\mu_1})\vec{v}(\underline{\omega}^{\mu_2}) \\ &= u^{\mu}\vec{e}_{\mu}(\underline{\omega}^{\mu_1})v^{\nu}\vec{e}_{\nu}(\underline{\omega}^{\mu_2}) \\ &= u^{\mu}\delta^{\mu_1}_{\mu}v^{\nu}\delta^{\mu_2}_{\nu} \\ &= u^{\mu_1}v^{\mu_2} \end{aligned} \tag{2.57}$$

2.2.1 Transformation of tensor components

We shall not limit our discussion to coordinate transformations. Instead, we will consider arbitrary transformations between bases, $\{\vec{e}_{\mu}\} \longrightarrow \{\vec{e}_{\mu'}\}$. The elements of transformation matrices are denoted by $M^{\mu}_{\ \mu'}$ such that

$$\vec{e}_{\mu'} = \vec{e}_{\mu}M^{\mu}_{\ \mu'} \quad \text{and} \quad \vec{e}_{\mu} = \vec{e}_{\mu'}M^{\mu'}_{\ \mu} \tag{2.58}$$

where $M^{\mu'}_{\ \mu}$ are elements of the inverse transformation matrix. Thus, it follows that

$$M^{\mu}_{\ \mu'}M^{\mu'}_{\ \nu} = \delta^{\mu}_{\nu} \tag{2.59}$$

If the transformation is a coordinate transformation, the elements of the matrix become

$$\boxed{M^{\mu'}_{\ \mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}} \tag{2.60}$$

2.2.2 Transformation of basis 1-forms

$$\begin{aligned} \underline{\omega}^{\mu'} &= M^{\mu'}_{\ \mu}\underline{\omega}^{\mu} \\ \underline{\omega}^{\mu} &= M^{\mu}_{\ \mu'}\underline{\omega}^{\mu'} \end{aligned} \tag{2.61}$$

The components of a tensor of higher rank transform such that every contravariant index (upper) transforms as a basis 1-form and every covariant index (lower) as a basis vector. Also, all elements of the transformation matrix are multiplied with one another.

Example **2.2.2 (A mixed tensor of rank 3)**

$$T^{\alpha'}_{\mu'\nu'} = M^{\alpha'}_{\alpha} M^{\mu}_{\mu'} M^{\nu}_{\nu'} T^{\alpha}_{\mu\nu} \tag{2.62}$$

The components in the **primed basis** are linear combinations of the components in the **unprimed basis**.

Tensor transformation of components means that tensors have a basis **independent** existence. That is, if a tensor has non-vanishing components in a **given basis** then it has non-vanishing components in **all bases**. This means that tensor equations have a basis independent form. **Tensor equations are invariant**. A basis transformation might result in the vanishing of one or more tensor components. Equations in **component** form may differ from one basis to another. But an equation expressed in tensor components can be transformed from one basis to another using the tensor component transformation rules. An equation that is expressed only in terms of tensor components is said to be **covariant**.

2.2.3 The metric tensor

Definition 2.2.3 (The metric tensor)
The scalar product of two vectors $\vec{u}$ and $\vec{v}$ is denoted by g($\vec{u}$,$\vec{v}$) and is defined as a symmetric linear mapping which for each pair of vectors gives a scalar g($\vec{v}$,$\vec{u}$) = g($\vec{u}$,$\vec{v}$).

The value of the scalar product g($\vec{u}$,$\vec{v}$) is given by specifying the scalar products of each pair of basis-vectors in a basis.

g is a symmetric **covariant** tensor of rank 2. This tensor is known as the **metric tensor**. The components of this tensor are

$$\boxed{g(\vec{e}_\mu, \vec{e}_\nu) = g_{\ \mu\nu}} \tag{2.63}$$

$$\vec{u} \cdot \vec{v} = g(\vec{u}, \vec{v}) = g(u^\mu \vec{e}_\mu, v^\nu \vec{e}_\nu) = u^\mu v^\nu g(\vec{e}_\mu, \vec{e}_\nu) = u^\mu u^\nu g_{\ \mu\nu} \tag{2.64}$$

Usual notation:

$$\boxed{\vec{u} \cdot \vec{v} = g_{\ \mu\nu} u^\mu v^\nu} \tag{2.65}$$

The absolute value of a vector:

$$|\vec{v}| = \sqrt{g(\vec{v}, \vec{v})} = \sqrt{|g_{\ \mu\nu} v^\mu v^\nu|} \tag{2.66}$$

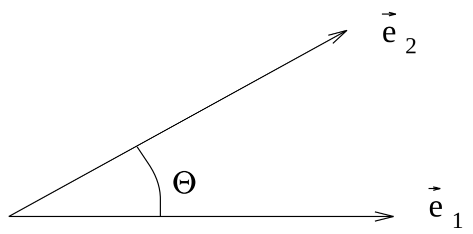


Figure 2.8: Basis-vectors $\vec{e}_1$ and $\vec{e}_2$

`Example` **2.2.3 (Cartesian coordinates in a plane)**

$$\begin{gathered}\vec{e}_x \cdot \vec{e}_x = 1, \qquad \vec{e}_y \cdot \vec{e}_y = 1, \qquad \vec{e}_x \cdot \vec{e}_y = \vec{e}_y \cdot \vec{e}_x = 0 \\ g_{\,xx} = g_{\,yy} = 1, \qquad g_{\,xy} = g_{\,yx} = 0 \\ g_{\,\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{gathered} \tag{2.67}$$

`Example` **2.2.4 (Basis-vectors in plane polar-coordinates)**

$$\vec{e}_r \cdot \vec{e}_r = 1, \quad \vec{e}_\theta \cdot \vec{e}_\theta = r^2, \quad \vec{e}_r \cdot \vec{e}_\theta = 0, \tag{2.68}$$

The metric tensor in plane polar-coordinates:

$$g_{\,\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{2.69}$$

`Example` **2.2.5 (Non-diagonal basis-vectors)**

$$\begin{gathered}\vec{e}_1 \cdot \vec{e}_1 = 1, \qquad \vec{e}_2 \cdot \vec{e}_2 = 1, \qquad \vec{e}_1 \cdot \vec{e}_2 = \cos\theta = \vec{e}_2 \cdot \vec{e}_1 \\ g_{\,\mu\nu} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix} = \begin{pmatrix} g_{\,11} & g_{\,12} \\ g_{\,21} & g_{\,22} \end{pmatrix}\end{gathered} \tag{2.70}$$

Definition 2.2.4 (Contravariant components)
The **contravariant components** $g^{\mu\alpha}$ of the metric tensor are defined as:

$$g^{\mu\alpha} g_{\,\alpha\nu} \equiv \delta^{\mu}{}_{\nu} \quad g^{\mu\nu} = \vec{w}^{\mu} \cdot \vec{w}^{\nu}, \tag{2.71}$$

where $\vec{w}^{\mu}$ is defined by

$$\vec{w}^{\mu} \cdot \vec{w}_{\nu} \equiv \delta^{\mu}{}_{\nu}. \tag{2.72}$$

$g^{\mu\nu}$ is the inverse matrix of $g_{\,\mu\nu}$.

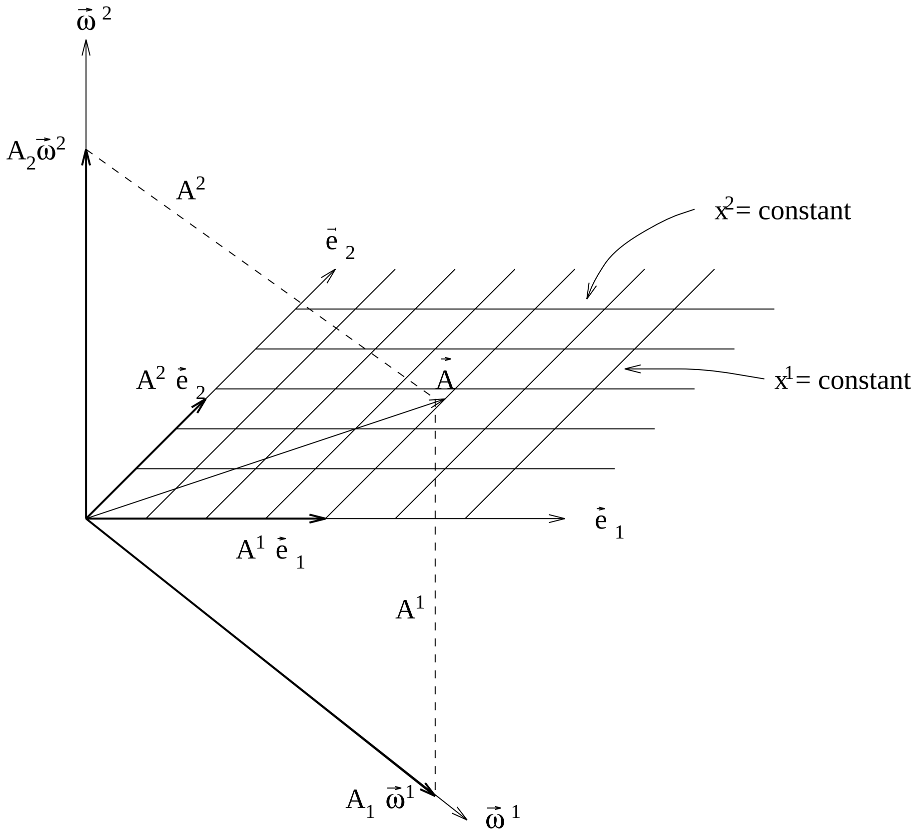


Figure 2.9: The covariant- and contravariant components of a vector

It is possible to define a **mapping** between tensors of different type (eg. covariant on contravariant) using the metric tensor.

We can for instance map a vector on a 1-form:

$$v_\mu = g(\vec{v}, \vec{e}_\mu) = g(v^\alpha \vec{e}_\alpha, \vec{e}_\mu) = v^\alpha g(\vec{e}_\alpha, \vec{e}_\mu) = v^\alpha g_{\alpha\mu} \tag{2.73}$$

This is known as lowering of an index. Raising of an index becomes :

$$v^\mu = g^{\mu\alpha} v_\alpha \tag{2.74}$$

The mixed components of the metric tensor becomes:

$$g^\mu{}_\nu = g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \tag{2.75}$$

We now define distance along a curve. Let the curve be parameterized by λ (proper-time τ for time-like curves). Let $\vec{v}$ be the tangent vector-field of the curve.

The squared distance ds^2 between the points along the curve is defined as:

$$ds^2 \equiv g(\vec{v}, \vec{v}) d\lambda^2 \tag{2.76}$$

gives

$$ds^2 = g_{\mu\nu} v^\mu v^\nu d\lambda^2. \tag{2.77}$$

The tangent vector has components $v^\mu = \frac{dx^\mu}{d\lambda}$, which gives:

$$\boxed{ds^2 = g_{\mu\nu} dx^\mu dx^\nu} \tag{2.78}$$

The expression ds^2 is known as the **line-element**.

`Example` **2.2.6 (Cartesian coordinates in a plane)**

$$\begin{aligned} g_{xx} = g_{yy} = 1, \qquad g^x_{\ y} = g^y_{\ x} = 0 \\ ds^2 = dx^2 + dy^2 \end{aligned} \tag{2.79}$$

`Example` **2.2.7 (Plane polar coordinates)**

$$\begin{aligned} g_{rr} = 1, \quad g_{\theta\theta} = r^2 \\ ds^2 = dr^2 + r^2 d\theta^2 \end{aligned} \tag{2.80}$$

Cartesian coordinates in the (flat) Minkowski space-time :

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{2.81}$$

In an arbitrary curved space, an orthonormal basis can be adopted in any point. If $\vec{e}_{\hat{t}}$ is tangent vector to the world line of an observer, then $\vec{e}_{\hat{t}} = \vec{u}$ where $\vec{u}$ is the 4-velocity of the observer. In this case, we are using what we call the *comoving orthonormal basis* of the observer. In a such basis, we have the Minkowski-metric:

$$ds^2 = \eta_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} \tag{2.82}$$

2.3 Forms

An **antisymmetric tensor** is a tensor whose sign changes under an arbitrary exchange of two arguments.

$$A(\cdots, \vec{u}, \cdots, \vec{v}, \cdots) = -A(\cdots, \vec{v}, \cdots, \vec{u}, \cdots) \tag{2.83}$$

The components of an antisymmetric tensor change sign under exchange of two indices.

$$A_{\cdots\mu\cdots\nu\cdots} = -A_{\cdots\nu\cdots\mu\cdots} \tag{2.84}$$

Definition 2.3.1 (p-form)
A **p-form** is defined to be an antisymmetric, covariant tensor of rank p.
An antisymmetric tensor product $\wedge$ is defined by:

$$\underline{\omega}^{[\mu_1} \otimes \cdots \otimes \underline{\omega}^{\mu_p]} \wedge \underline{\omega}^{[\nu_1} \otimes \cdots \otimes \underline{\omega}^{\nu_q]} \equiv \frac{(p+q)!}{p!q!} \underline{\omega}^{[\mu_1} \otimes \cdots \otimes \underline{\omega}^{\nu_q]} \tag{2.85}$$

where [ ] denotes antisymmetric combinations defined by:

$$\underline{\omega}^{[\mu_1} \otimes \cdots \otimes \underline{\omega}^{\mu_p]} \equiv \frac{1}{p!} \cdot (\text{the sum of terms with all possible permutations of indices with, "+" for even and "-" for odd permutations}) \tag{2.86}$$

`Example` **2.3.1 (antisymmetric combinations)**

$$\underline{\omega}^{[\mu_1} \otimes \underline{\omega}^{\mu_2]} = \frac{1}{2}(\underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} - \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_1}) \tag{2.87}$$

`Example` **2.3.2 (antisymmetric combinations)**

$$\begin{aligned}
&\underline{\omega}^{[\mu_1} \otimes \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_3]} = \\
&\quad \frac{1}{3!}(\underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_3} + \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} + \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_1} \\
&\quad - \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_3} - \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_1} - \underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_2}) \\
&\quad = \frac{1}{3!}\epsilon_{ijk}(\underline{\omega}^{\mu_i} \otimes \underline{\omega}^{\mu_j} \otimes \underline{\omega}^{\mu_k})
\end{aligned} \tag{2.88}$$

`Example` **2.3.3 (A 2-form in 3-space)**

$$\underline{\alpha} = \alpha_{12}\underline{\omega}^1 \otimes \underline{\omega}^2 + \alpha_{21}\underline{\omega}^2 \otimes \underline{\omega}^1 + \alpha_{13}\underline{\omega}^1 \otimes \underline{\omega}^3 + \alpha_{31}\underline{\omega}^3 \otimes \underline{\omega}^1 + \alpha_{23}\underline{\omega}^2 \otimes \underline{\omega}^3 + \alpha_{32}\underline{\omega}^3 \otimes \underline{\omega}^2 \tag{2.89}$$

Now the antisymmetry of $\underline{\alpha}$ means that

$$+\underline{\alpha}_{21} = -\underline{\alpha}_{12}; \quad +\underline{\alpha}_{31} = -\underline{\alpha}_{13}; \quad +\underline{\alpha}_{32} = -\underline{\alpha}_{23} \tag{2.90}$$

$$\begin{aligned}\underline{\alpha} =&\underline{\alpha}_{12}(\underline{\omega}^1 \otimes \underline{\omega}^2 - \underline{\omega}^2 \otimes \underline{\omega}^1)\\ &+ \underline{\alpha}_{13}(\underline{\omega}^1 \otimes \underline{\omega}^3 - \underline{\omega}^3 \otimes \underline{\omega}^1)\\ &+ \underline{\alpha}_{23}(\underline{\omega}^2 \otimes \underline{\omega}^3 - \underline{\omega}^3 \otimes \underline{\omega}^2)\\ &= \alpha_{|\mu\nu|} 2\underline{\omega}^{[\mu} \otimes \underline{\omega}^{\nu]}\end{aligned} \tag{2.91}$$

where $|\mu\nu|$ means summation only for $\mu < \nu$ (see (Misner, Thorne and Wheeler 1973)). We now use the definition of $\wedge$ with $p = q = 1$. This gives

$$\boxed{\underline{\alpha} = \alpha_{|\mu\nu|}\underline{\omega}^\mu \wedge \underline{\omega}^\nu}$$

We can also write

$\underline{\omega}^\mu \wedge \underline{\omega}^\nu$ is the form basis.

$$\boxed{\underline{\alpha} = \frac{1}{2}\alpha_{\mu\nu}\underline{\omega}^\mu \wedge \underline{\omega}^\nu}$$

A tensor of rank 2 can always be split up into a symmetric and an antisymmetric part. (Note that tensors of higher rank can not be split up in this way.)

$$\begin{aligned}T_{\mu\nu} &= \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) + \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})\\ &= A_{\mu\nu} + S_{\mu\nu}\end{aligned} \tag{2.92}$$

We thus have:

$$\begin{aligned}S_{\mu\nu}A^{\mu\nu} &= \frac{1}{4}(T_{\mu\nu} + T_{\nu\mu})(T^{\mu\nu} - T^{\nu\mu})\\ &= \frac{1}{4}(T_{\mu\nu}T^{\mu\nu} - T_{\mu\nu}T^{\nu\mu} + T_{\nu\mu}T^{\mu\nu} - T_{\nu\mu}T^{\nu\mu})\\ &= 0\end{aligned} \tag{2.93}$$

In general, summation over indices of a symmetric and an antisymmetric quantity vanishes. In a summation $T_{\mu\nu}A^{\mu\nu}$ where $A^{\mu\nu}$ is antisymmetric and $T_{\mu\nu}$ has no symmetry, only the antisymmetric part of $T_{\mu\nu}$ contributes. So that, in

$$\underline{\alpha} = \frac{1}{2}\alpha_{\mu\nu}\underline{\omega}^\mu \wedge \underline{\omega}^\nu \tag{2.94}$$

only the antisymmetric elements $\alpha_{\nu\mu} = -\alpha_{\mu\nu}$, contribute to the summation. These antisymmetric elements are the **form components**

Forms are antisymmetric covariant tensors. Because of this antisymmetry a form with two identical components must be a **null form** (= zero). e.g. $\alpha_{131} = -\alpha_{131} \Rightarrow \alpha_{131} = 0$

In an n-dimensional space all p-forms with $p > n$ are null forms.

Chapter 3

Accelerated Reference Frames

3.1 Rotating reference frames

3.1.1 The spatial metric tensor

Let $\vec{e}_{\hat{0}}$ be the 4-velocity field ($x^0 = ct, c = 1, x^0 = t$) of the reference particles in a reference frame R. We are going to find the metric tensor γ_{ij} in a tangent space orthogonal to $\vec{e}_{\hat{0}}$, expressed by the metric tensor $g_{\mu\nu}$ of spacetime.

In an arbitrary coordinate basis $\{\vec{e}_\mu\}$, $\{\vec{e}_i\}$ is not necessarily orthogonal to $\vec{e}_0$. We choose $\vec{e}_0 \| \vec{e}_{\hat{0}}$. Let $\vec{e}_{\perp i}$ be the component of $\vec{e}_i$ orthogonal to $\vec{e}_0$, that is:$\vec{e}_{\perp i} \cdot \vec{e}_0 = 0$. The metric tensor of space is defined by:

$$\gamma_{ij} = \vec{e}_{\perp i} \cdot \vec{e}_{\perp j}, \gamma_{i0} = 0, \gamma_{00} = 0$$

$$\vec{e}_{\perp i} = \vec{e}_i - \vec{e}_{\| i}$$

$$\vec{e}_{\| i} = \frac{\vec{e}_i \cdot \vec{e}_0}{\vec{e}_0 \cdot \vec{e}_o} \vec{e}_0 = \frac{g_{i0}}{g_{00}} \vec{e}_0$$

$$\begin{aligned}
\gamma_{ij} &= (\vec{e}_i - \vec{e}_{\| i}) \cdot (\vec{e}_j - \vec{e}_{\| j}) \\
&= (\vec{e}_i - \frac{g_{i0}}{g_{00}} \vec{e}_0) \cdot (\vec{e}_j - \frac{g_{j0}}{g_{00}} \vec{e}_0) \\
&= \vec{e}_i \cdot \vec{e}_j - \frac{g_{j0}}{g_{00}} \vec{e}_0 \cdot \vec{e}_i - \frac{g_{i0}}{g_{00}} \vec{e}_0 \cdot \vec{e}_j + \frac{g_{i0} g_{j0}}{g_{00}^2} \vec{e}_0 \cdot \vec{e}_0 \\
&= g_{ij} - \frac{g_{i0} g_{j0}}{g_{00}} - \frac{g_{i0} g_{j0}}{g_{00}} + \frac{g_{i0} g_{j0}}{g_{00}}
\end{aligned}$$

$$\Rightarrow \gamma_{ij} = g_{ij} - \frac{g_{i0} g_{j0}}{g_{00}} \tag{3.1}$$

(Note:$g_{ij} = g_{ji} \Rightarrow \gamma_{ij} = \gamma_{ji}$)
The line element in space:

$$dl^2 = \gamma_{ij} dx^i dx^j = \big(g_{ij} - \frac{g_{i0} g_{j0}}{g_{00}}\big) dx^i dx^j \tag{3.2}$$

gives the distance between simultaneous events in a reference frame where the metric tensor of spacetime in a comoving coordinate system is $g_{\mu\nu}$.

The line element for spacetime can be expressed as:

$$ds^2 = -d\hat{t}^2 + dl^2 \tag{3.3}$$

It follows that $d\hat{t} = 0$ represents the simultaneity defining the spatial line element. The temporal part of the spacetime line element may be expressed as

$$\begin{aligned}
d\hat{t}^2 &= dl^2 - ds^2 = (\gamma_{\mu\nu} - g_{\mu\nu})dx^\mu dx^\nu \\
&= (\gamma_{ij} - g_{ij})dx^i dx^j + 2(\gamma_{i0} - g_{i0})dx^i dx^0 + (\gamma_{00} - g_{00})dx^0 dx^0 \\
&= (g_{ij} - \frac{g_{i0}g_{j0}}{g_{00}} - g_{ij})dx^i dx^j - 2g_{i0}dx^i dx^0 - g_{00}(dx^0)^2 \\
&= -g_{00}\left[(dx^0)^2 + 2\frac{g_{i0}}{g_{00}}dx^0 dx^i + \frac{g_{i0}g_{j0}}{g_{00}^2}dx^i dx^j\right] \\
&= \left[(-g_{00})^{1/2}(dx^0 + \frac{g_{i0}}{g_{00}}dx^i)\right]^2
\end{aligned}$$

So finally we get

$$d\hat{t} = (-g_{00})^{1/2}(dx^0 + \frac{g_{i0}}{g_{00}}dx^i) \tag{3.4}$$

The 3-space orthogonal to the world lines of the reference particles in R, $d\hat{t} = 0$, corresponds to a coordinate time interval $dt = -\frac{g_{i0}}{g_{00}}dx^i$. This is not an exact differential, that is, dt is not integrable. this means that one cannot in general define a 3-space orthogonal to the world lines of the world lines of the reference particles, i.e. a "simultaneity space", in an arbitrary reference frame. We must also conclude that unless g_{i0}/g_{00} is constant, it is not possible to Einstein synchronize clocks around closed curves.

In particular, it is not possible to Einstein-synchronize clocks around a closed curve in a rotating reference frame. If this is attempted, contradictory boundary conditions in the non-rotating lab frame will arise, due to the relativity of simultaneity. (See figure 3.1)

The distance in the laboratory frame between two points is:

$$L_0 = \frac{2\pi r}{n} \tag{3.5}$$

Lorentz transformation from the instantaneous rest frame (x', t') to the laboratory system (x, t):

$$\begin{aligned}
\Delta t &= \gamma(\Delta t' + \frac{v}{c^2}\Delta x'), \quad \gamma = \frac{1}{\sqrt{1 - \frac{r^2\omega^2}{c^2}}} \\
\Delta x &= \gamma(\Delta x' + v\Delta t')
\end{aligned} \tag{3.6}$$

Since we for simultaneous events in the rotating reference frame have $\Delta t' = 0$, and proper distance $\Delta x' = \gamma L_0$, we get in the laboratory frame

$$\Delta t = \gamma^2 \frac{r\omega}{c^2} L_0 = \gamma^2 \frac{r\omega}{c^2} \frac{2\pi r}{n} \tag{3.7}$$

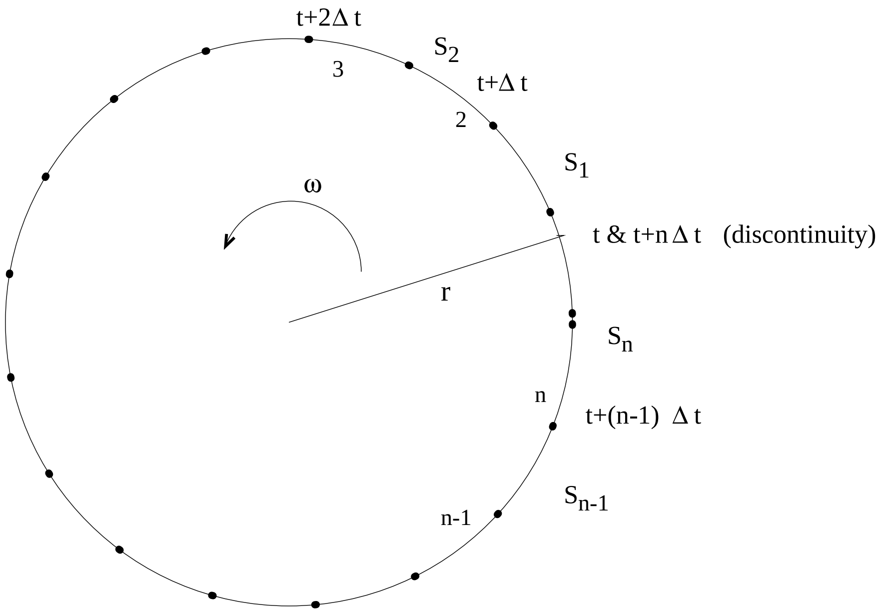


Figure 3.1: Events simultanous in the rotating reference frame. 1 comes before 2, before 3, etc... Note the discontinuity at t.

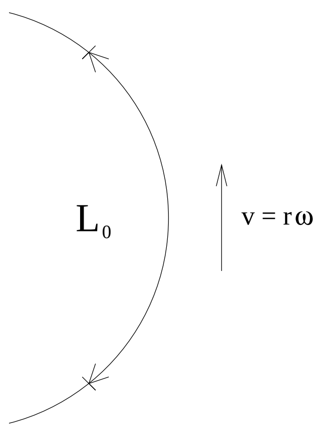


Figure 3.2: The distance between two points on the circumference is L_0.

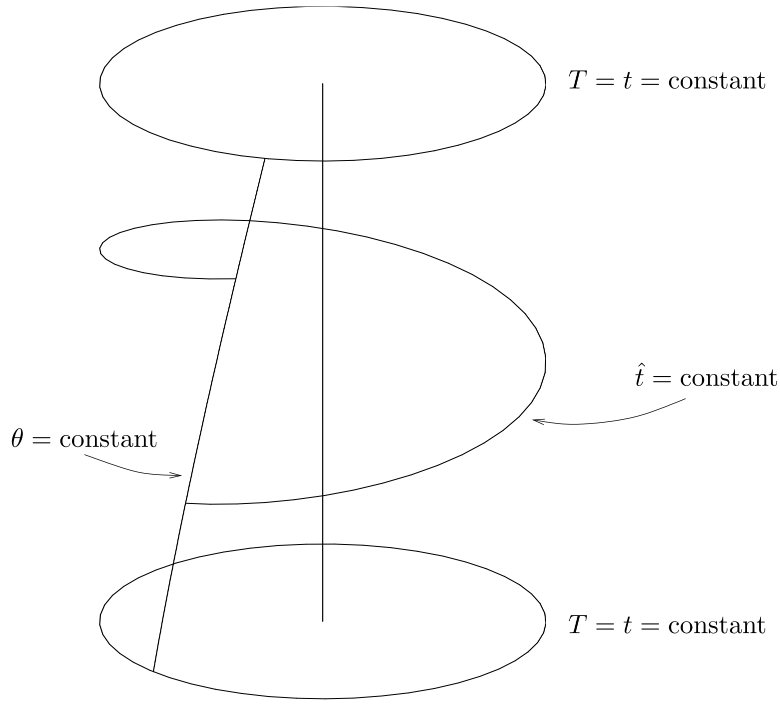


Figure 3.3: Discontinuity in simultaneity.

The fact that $\Delta t' = 0$ and $\Delta t \neq 0$ is an expression of the relativity of simultaneity. Around the circumference this is accumulated to

$$n\Delta t = \gamma^2 \frac{2\pi r^2 \omega}{c^2} \tag{3.8}$$

and we get a discontinuity in simultaneity, as shown in figure 3.3. Let IF be an inertial frame with cylinder coordinates (T, R, Θ, Z). The line element is then given by

$$ds^2 = -dT^2 + dR^2 + R^2 d\Theta^2 + dZ^2 \quad (c = 1) \tag{3.9}$$

In a rotating reference frame, RF, we have cylinder coordinates (t, r, θ, z). We then have the following coordinate transformation :

$$t = T, \quad r = R, \quad \theta = \Theta - \omega T, \quad z = Z \tag{3.10}$$

The line element in the co-moving coordinate system in RF is then

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2(d\theta + \omega dt)^2 + dz^2 \\ &= -(1 - r^2\omega^2)dt^2 + dr^2 + r^2 d\theta^2 + dz^2 + 2r^2\omega d\theta dt \quad (c = 1) \end{aligned} \tag{3.11}$$

The metric tensor have the following components:

$$\begin{aligned} &g_{tt} = -(1 - r^2\omega^2),\ g_{rr} = 1,\ g_{\theta\theta} = r^2,\ g_{zz} = 1 \\ &g_{\theta t} = g_{t\theta} = r^2\omega \end{aligned} \tag{3.12}$$

$dt = 0$ gives

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \tag{3.13}$$

This represents the Euclidean geometry of the 3-space (simultaneity space, $t = T$) in IF.

As applied to the rotating system the spatial line element takes the form

$$\begin{aligned} dl^2 &= (g_{ij} - \frac{g_{i0}g_{j0}}{g_{00}})dx^i dx^j \\ \gamma_{rr} &= g_{rr} = 1,\ \gamma_{zz} = g_{zz} = 1, \\ \gamma_{\theta\theta} &= g_{\theta\theta} - \frac{g_{\theta 0}^2}{g_{00}} \\ &= r^2 - \frac{(r^2\omega)^2}{-(1 - r^2\omega^2)} = \frac{r^2}{1 - r^2\omega^2} \end{aligned}$$

$$\Rightarrow \quad dl^2 = dr^2 + \frac{r^2 d\theta^2}{1 - r^2\omega^2} + dz^2 \tag{3.14}$$

It describes the geometry of a local 3-space orthogonal to the world line of a reference particle in RF. This 3-space cannot be extended to a finite 3-dimensional space in RF since Einstein synchronization is not integrable in RF. From the line element (3.14) it is seen that the geometry of this local simultaneity space in RF in non-Euclidean. The circumference of a circle with radius r is

$$l_\theta = \frac{2\pi r}{\sqrt{1 - r^2\omega^2}} > 2\pi r \tag{3.15}$$

We see that the quotient between circumference and radius $> 2\pi$ which means that the spatial geometry is *hyperbolic.* (For spherical geometry we have $l_\theta < 2\pi r$.)

3.1.2 Angular acceleration in the rotating frame

We will now investigate what happens when we give RF an angular acceleration. Then we use a rotating circle made of standard measuring rods, as shown in Figure 3.4. All points on a circle are accelerated simultaneously in IF (the laboratory system). We let the angular velocity increase from ω to $\omega + d\omega$, measured in IF. Lorentz transformation to an instantaneous rest frame for a point on the circumference then gives an increase in velocity in this system:

$$r d\omega' = \frac{r d\omega}{1 - r^2\omega^2}, \tag{3.16}$$

where we have used that the initial velocity in this frame is zero.

The time difference for the accelerations of the front and back ends of the rods (the front end is accelerated first) in the instantaneous rest frame is:

$$\Delta t' = \frac{r\omega L_0}{\sqrt{1 - r^2\omega^2}} \tag{3.17}$$

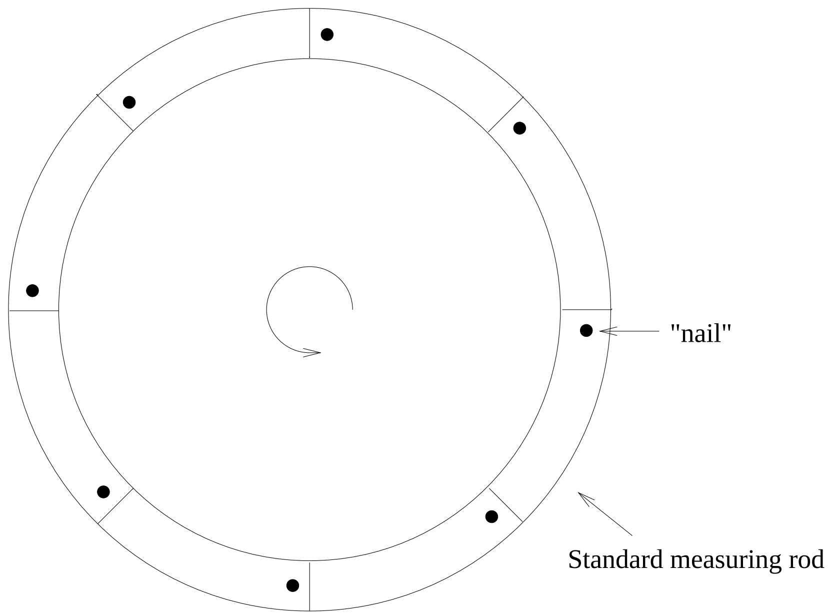


Figure 3.4: The standard measuring rods are fastened with nails in one end. We will see what then happens when we have an angular acceleration.

where L_0 is the distance between points on the circumference when at rest (= the length of the rods when at rest), $L_0 = \frac{2\pi r}{n}$. In IF all points on the circumference are accelerated simultaneously. In RF, however, this is not the case. Here the distance between points on the circumference will increase, see Figure 3.5. The rest distance increases by

$$dL' = rd\omega' \Delta t' = \frac{r^2 \omega L_0 d\omega}{(1 - r^2\omega^2)^{3/2}}. \tag{3.18}$$

The increase of the distance during the acceleration (in an instantaneous rest frame) is

$$L' = r^2 L_0 \int_0^{\omega} \frac{\omega d\omega}{(1 - r^2\omega^2)^{3/2}} = (\frac{1}{\sqrt{1 - r^2\omega^2}} - 1)L_0. \tag{3.19}$$

Hence, after the acceleration there is a proper distance L' between the rods. In

Figure 3.5: In RF two points on the circumference are accelerated at different times. Thus the distance between them is increased.

the laboratory system (IF) the distance between the rods is

$$L = \sqrt{1 - r^2\omega^2} L' = \sqrt{1 - r^2\omega^2} \left(\frac{1}{\sqrt{1 - r^2\omega^2}} - 1 \right) L_0 = L_0 - L_0 \sqrt{1 - r^2\omega^2}, \tag{3.20}$$

where L_0 is the rest length of the rods and $L_0\sqrt{1 - r^2\omega^2}$ is their Lorentz contracted length. We now have the situation shown in Figure 3.6.

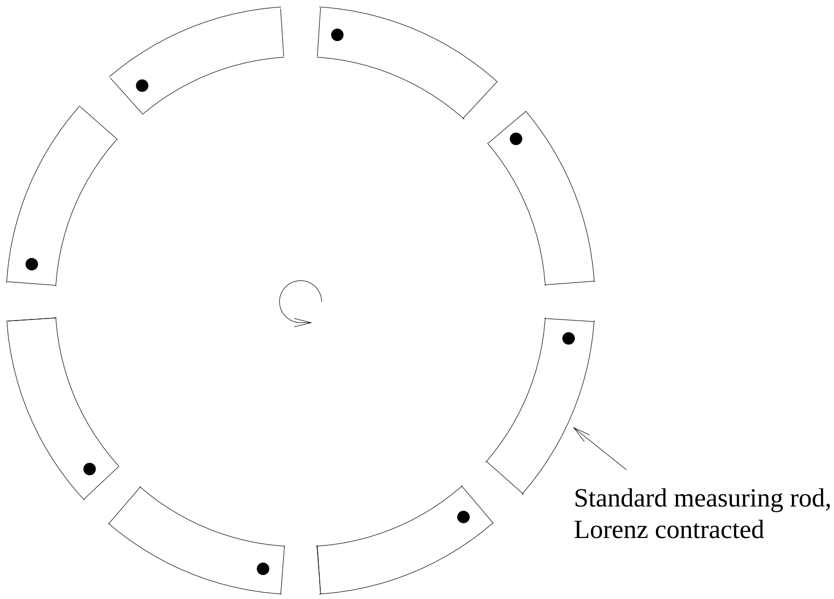


Figure 3.6: The standard measuring rods have been Lorentz contracted.

Thus, there is room for more standard rods around the periphery the faster the disk rotates. This means that the measured length of the periphery (number of standard rods) gets larger with increasing angular velocity.

3.1.3 Gravitational time dilation

$$ds^2 = -(1 - \frac{r^2\omega^2}{c^2})c^2 dt^2 + dr^2 + r^2 d\theta^2 + dz^2 + 2r^2\omega d\theta dt \tag{3.21}$$

We now look at *standard clocks* with constant r and z.

$$ds^2 = c^2 dt^2 [-(1 - \frac{r^2\omega^2}{c^2}) + \frac{r^2}{c^2}(\frac{d\theta}{dt})^2 + 2\frac{r^2\omega}{c^2}\frac{d\theta}{dt}] \tag{3.22}$$

Let $\frac{d\theta}{dt} \equiv \dot{\theta}$ be the angular velocity of the clock in RF. The proper time interval measured by the clock is then

$$ds^2 = -c^2 d\tau^2 \tag{3.23}$$

From this we see that

$$\boxed{d\tau = dt\sqrt{1 - \frac{r^2\omega^2}{c^2} - \frac{r^2\dot{\theta}^2}{c^2} - 2\frac{r^2\omega\dot{\theta}}{c^2}}} \tag{3.24}$$

A non-moving standard clock in RF: $\dot{\theta} = 0 \Rightarrow$

$$d\tau = dt\sqrt{1 - \frac{r^2\omega^2}{c^2}} \tag{3.25}$$

Seen from IF, the non-rotating laboratory system, (3.25) represents the **velocity dependent time dilation** from the special theory of relativity.

But how is (3.25) interpreted in RF? The clock does not move relative to an observer in this system, hence what happens can not bee interpreted as a velocity dependent phenomenon. According to Einstein, the fact that standard clocks slow down the farther away from the axis of rotation they are, is due to a **gravitational effect**.

We will now find the gravitational potential at a distance r from the axis. The sentripetal acceleration is $v^2/r,\ v = r\omega$ so:

$$\Phi = -\int_0^r g(r)dr = -\int_0^r r\omega^2 dr = -\frac{1}{2}r^2\omega^2$$

We then get:

$$d\tau = dt\sqrt{1 - \frac{r^2\omega^2}{c^2}} = dt\sqrt{1 + \frac{2\Phi}{c^2}} \tag{3.26}$$

In RF the position dependent time dilation is interpreted as a **gravitational time dilation***: Time flows slower further down in a gravitational field.*

3.1.4 Path of photons emitted from axes in the rotating reference frame (RF)

We start with description in the inertial frame (IF). In IF photon paths are radial. Consider a photon path with $\Theta = 0,\ R = T$ with light source at $R = 0$. Transforming to RF:

$$\begin{aligned} &t = T, \quad r = R, \quad \theta = \Theta - \omega T \\ \Rightarrow \quad &r = t, \quad \theta = -\omega t \end{aligned} \tag{3.27}$$

The orbit equation is thus $\theta = -\omega r$ which is the equation for an Archimedean spiral. The time used by a photon out to distance r from axis is $t = \frac{r}{c}$.

3.1.5 The Sagnac effect

IF description:

Here the velocity of light is isotropic, but the emitter/receiver moves due to the disc's rotation as shown in Figure 3.7. Photons are emitted/received in/from opposite directions. Let t_1 be the travel time of photons which move *with* the rotation.

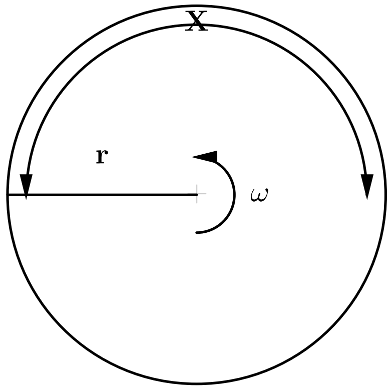


Figure 3.7: The Sagnac effect demonstrates the **anisotropy** of the speed of light when measured in a rotating reference frame.

Then

$$\begin{aligned} &\Rightarrow \quad 2\pi r + r\omega t_1 = ct_1 \\ &\Rightarrow \quad t_1 = \frac{2\pi r}{c - r\omega} \end{aligned} \tag{3.28}$$

Let t_2 be the travel time for photons moving against the rotation of the disc. The difference in travel time is

A is the area enclosed by the photon path or orbit.

$$\begin{aligned} \Delta t = t_1 - t_2 &= 2\pi r \left(\frac{1}{c - r\omega} - \frac{1}{c + r\omega} \right) \\ &= \frac{2\pi r 2r\omega}{c^2 - r^2\omega^2} \\ &= \gamma^2 \frac{4A\omega}{c^2} \end{aligned} \tag{3.29}$$

RF description:

$ds^2 = 0$ along the world line of a photon

$$ds^2 = -\left(1 - \frac{r^2\omega^2}{c^2}\right) c^2 dt^2 + r^2 d\theta^2 + 2r^2\omega d\theta dt$$

$$\begin{aligned} \text{let} \quad \dot{\theta} &= \frac{d\theta}{dt} \\ r^2\dot{\theta}^2 + 2r^2\omega\dot{\theta} - (c^2 - r^2\omega^2) &= 0 \\ \dot{\theta} &= \frac{-r^2\omega \pm \sqrt{(r^4\omega^2 + r^2c^2 - r^4\omega^2)}}{r^2} \end{aligned}$$

$$\begin{aligned} \dot{\theta} &= -\omega \pm \frac{rc}{r^2} \\ &= -\omega \pm \frac{c}{r} \end{aligned} \tag{3.30}$$

The speed of light: $v_{\pm} = r\dot{\theta} = -r\omega \pm c$. We see that in the rotating frame RF, the measured (coordinate) velocity of light is NOT isotropic. The difference in the travel time of the two beams is

$$\begin{aligned} \Delta t &= \frac{2\pi r}{c - r\omega} - \frac{2\pi r}{c + r\omega} \\ &= \gamma^2 \frac{4A\omega}{c^2} \end{aligned} \tag{3.31}$$

(See Phil. Mag. series 6, vol. 8 (1904) for Michelson's article)

3.2 Hyperbolically accelerated reference frames

Consider a particle moving along a straight line with velocity u and acceleration $a = \frac{du}{dT}$. Rest acceleration is $\hat{a}$.

$$\Rightarrow \quad a = \left(1 - u^2/c^2\right)^{3/2} \hat{a}. \tag{3.32}$$

Assume that the particle has constant rest acceleration $\hat{a} = g$. That is

$$\frac{du}{dT} = \left(1 - u^2/c^2\right)^{3/2} g. \tag{3.33}$$

Which on integration with $u(0) = 0$ gives

$$\begin{aligned} u &= \frac{gT}{\left(1 + \frac{g^2T^2}{c^2}\right)^{1/2}} = \frac{dX}{dT} \\ \Rightarrow \quad X &= \frac{c^2}{g}\left(1 + \frac{g^2T^2}{c^2}\right)^{1/2} + k \end{aligned}$$

$$\Rightarrow \quad \frac{c^4}{g^2} = (X - k)^2 - c^2T^2 \tag{3.34}$$

In its final form the above equation describes a hyperbola in the Minkowski diagram as shown in figure(3.8).

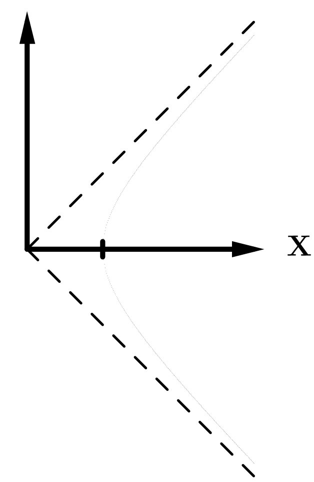


Figure 3.8: Hyperbolically accelerated reference frames are so called because the loci of particle trajectories in space-time are hyperbolae.

The proper time interval as measured by a clock which follows the particle:

$$d\tau = \left(1 - \frac{u^2}{c^2}\right)^{1/2} dT \tag{3.35}$$

Substitution for $u(T)$ and integration with $\tau(0) = 0$ gives

$$\begin{aligned} \tau &= \frac{c}{g}\text{arcsinh}\left(\frac{gT}{c}\right) \\ \text{or} \quad T &= \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right) \\ \text{and} \quad X &= \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right) + k \end{aligned} \tag{3.36}$$

We now use this particle as the origin of space in an hyperbolically accelerated reference frame.

Definition 3.2.1 (Born-stiff motion)
Born-stiff motion of a system is motion such that every element of the system has constant rest length. We demand that our accelerated reference frame is Born-stiff.

Let the inertial frame have coordinates (T, X, Y, Z) and the accelerated frame have coordinates (t, x, y, z). We now denote the X-coordinate of the "origin particle" by X_0.

$$1 + \frac{gX_0}{c^2} = \cosh\frac{g\tau_0}{c} \tag{3.37}$$

where τ_0 is the proper time for this particle and k is set to $\frac{-c^2}{g}$. (These are Møller coordinates. Setting $k = 0$ gives Rindler coordinates).

Let us denote the accelerated frame by Σ. The coordinate time at an arbitrary point in Σ is defined by $t = \tau_0$. That is coordinate clocks in Σ run identically with the standard clock at the "origin particle". Let $\vec{X}_0$ be the position 4-vector of the "origin particle". Decomposed in the laboratory frame, this becomes

$$\vec{X}_0 = \left\{ \frac{c^2}{g} \sinh \frac{gt}{c}, \frac{c^2}{g} \left(\cosh \frac{gt}{c} - 1 \right), 0, 0 \right\} \tag{3.38}$$

P is chosen such that P and P_0 are simultaneous in the accelerated frame Σ. The distance (see figure(3.9)) vector from P_0 to P, decomposed into an orthonormal comoving basis of the "origin particle" is $\hat{X} = (0, \hat{x}, \hat{y}, \hat{z})$ where $\hat{x}, \hat{y}$ and $\hat{z}$ are physical distances measured simultaneously in Σ. The space coordinates in Σ are defined by

$$x \equiv \hat{x}, \quad y \equiv \hat{y}, \quad z \equiv \hat{z}. \tag{3.39}$$

The position vector of P is $\vec{X} = \vec{X}_0 + \hat{\vec{X}}$. The relationship between basis vectors in IF and the comoving orthonormal basis is given by a Lorentz transformation in the x-direction.

$$\begin{aligned} \vec{e}_{\hat{\mu}} &= \vec{e}_{\mu} \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \\ &= (\vec{e}_T, \vec{e}_X, \vec{e}_Y, \vec{e}_Z,) \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{3.40}$$

where θ is the **rapidity** defined by

$$\tanh\theta \equiv \frac{U_0}{c} \tag{3.41}$$

U_0 being the velocity of the "origin particle".

$$\begin{aligned} U_0 &= \frac{dX_0}{dT_0} = c \tanh \frac{gt}{c} \\ \therefore \theta &= \frac{gt}{c} \end{aligned} \tag{3.42}$$

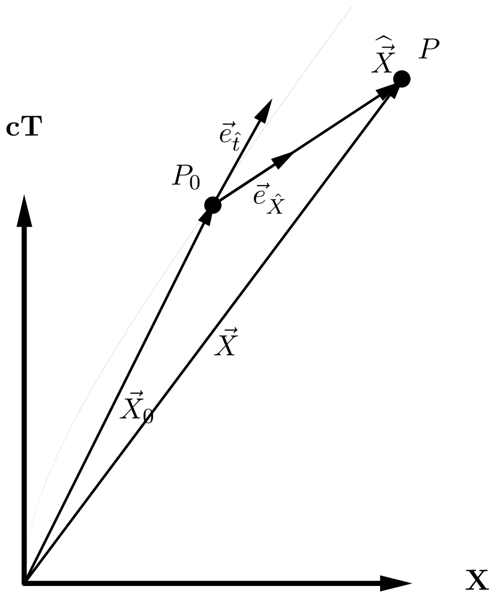


Figure 3.9: Simultaneity in hyperbolically accelerated reference frames. The vector $\widehat{\vec{X}}$ lies along the "simultaneity line" which makes the same angle with the X-axis as does $\vec{e}_{\hat{t}}$ with the cT-axis.

So the basis vectors can be written as follows

$$\begin{aligned}
\vec{e}_{\hat{t}} &= \vec{e}_T \cosh\frac{gt}{c} + \vec{e}_X \sinh\frac{gt}{c} \\
\vec{e}_{\hat{x}} &= \vec{e}_T \sinh\frac{gt}{c} + \vec{e}_X \cosh\frac{gt}{c} \\
\vec{e}_{\hat{y}} &= \vec{e}_Y \\
\vec{e}_{\hat{z}} &= \vec{e}_Z
\end{aligned} \tag{3.43}$$

The equation $\vec{X} = \vec{X}_0 + \hat{\vec{X}}$ can now be decomposed in IF:

$$\begin{aligned}
&cT\vec{e}_T + X\vec{e}_X + Y\vec{e}_Y + Z\vec{e}_Z = \\
&\quad \frac{c}{g}\sinh\frac{gt}{c}\vec{e}_T + \frac{c^2}{g}\left(\cosh\frac{gt}{c} - 1\right)\vec{e}_X + \frac{x}{c}\sinh\frac{gt}{c}\vec{e}_T + x\cosh\frac{gt}{c}\vec{e}_X + y\vec{e}_Y + z\vec{e}_Z
\end{aligned} \tag{3.44}$$

This then, gives the coordinate transformations

$$\begin{aligned}
T &= \frac{c}{g}\sinh\frac{gt}{c} + \frac{x}{c}\sinh\frac{gt}{c} \\
X &= \frac{c^2}{g}\left(\cosh\frac{gt}{c} - 1\right) + x\cosh\frac{gt}{c} \\
Y &= y \\
Z &= z \\
\Rightarrow \quad \frac{gT}{c} &= \left(1 + \frac{gx}{c^2}\right)\sinh\frac{gt}{c} \\
1 + \frac{gX}{c^2} &= \left(1 + \frac{gx}{c^2}\right)\cosh\frac{gt}{c}
\end{aligned}$$

Now dividing the last two of the above equations we get

$$\frac{gT}{c} = \left(1 + \frac{gX}{c^2}\right)\tanh\frac{gt}{c} \tag{3.45}$$

showing that the coordinate curves t = constant are straight lines in the T,X-frame passing through the point T = 0, X = $-\frac{c^2}{g}$. Using the identity $\cosh^2\theta - \sinh^2\theta = 1$ we get

$$\left(1 + \frac{gX}{c^2}\right)^2 - \left(\frac{gT}{c}\right)^2 = \left(1 + \frac{gx}{c^2}\right)^2 \tag{3.46}$$

showing that the coordinate curves x = constant are hypebolae in the T,X-diagram.

The line element (the metric) gives :

ds^2 is an invariant quantity

$$\begin{aligned}
ds^2 &= -c^2dT^2 + dX^2 + dY^2 + dZ^2 \\
&= -(1 + \frac{gx}{c^2})^2c^2dt^2 + dx^2 + dy^2 + dz^2 \qquad (3.47)
\end{aligned}$$

Note: When the metric is diagonal the unit vectors are orthogonal.

Clocks as rest in the accelerated system:

$$dx = dy = dz = 0, \quad ds^2 = -c^2d\tau^2$$

$$\Downarrow$$

$$-c^2d\tau^2 = -(1 + \frac{gx}{c^2})^2c^2dt^2$$

$$\Downarrow$$

$$\boxed{d\tau = (1 + \frac{gx}{c^2})dt} \tag{3.48}$$

Here $d\tau$ is the **proper time** and dt the **coordinate time**.

An observer in the accelerated system Σ experiences a gravitational field in the negative x-direction. When $x < 0$ then $d\tau < dt$. The coordinate clocks

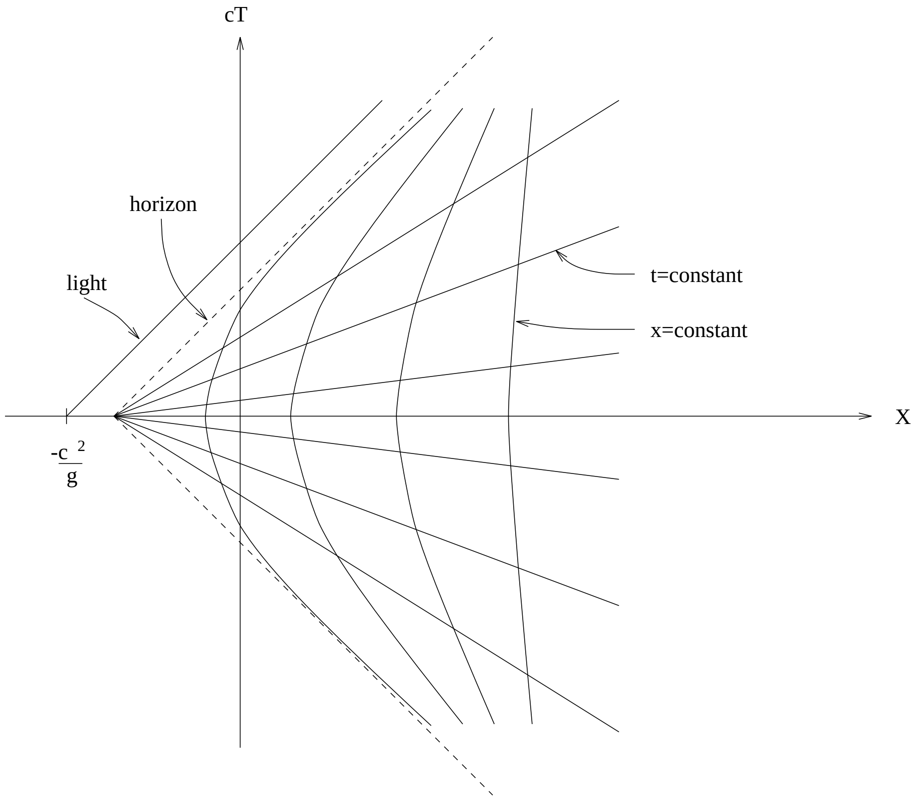


Figure 3.10: The hyperbolically accelerated reference system

tick equally fast independently of their position. This implies that time passes slower further down in a gravitational field.

Consider a standard clock moving in the x-direction with velocity $v = dx/dt$. Then

$$\begin{aligned} -c^2 d\tau^2 &= -\left(1 + \frac{gx}{c^2}\right)^2 c^2 dt^2 + dx^2 \\ &= -\left[\left(1 + \frac{gx}{c^2}\right)^2 - \frac{v^2}{c^2}\right] c^2 dt^2 \end{aligned} \tag{3.49}$$

Hence

$$d\tau = \sqrt{\left(1 + \frac{gx}{c^2}\right)^2 - \frac{v^2}{c^2}} dt \tag{3.50}$$

This expresses the combined effect of the gravitational- and the kinematic time dilation.

Chapter 4

Covariant Differentiation

4.1 Differentiation of forms

We must have a method of differentiation that maintains the anti symmetry, thus making sure that what we end up with after differentiation is still a form.

4.1.1 Exterior differentiation

The exterior derivative of a 0-form, i.e. a scalar function, f, is given by:

$$\underline{d}f = \frac{\partial f}{\partial x^\mu}\,\underline{\omega}^\mu = f_{,\mu}\underline{\omega}^\mu \tag{4.1}$$

where $\underline{\omega}^\mu$ are coordinate basis forms:

$$\underline{\omega}^\mu(\frac{\partial}{\partial x^\nu}) = \delta^\mu_{\ \nu} \tag{4.2}$$

We then (in general) get:

$$\underline{\omega}^\mu = \delta^\mu_{\ \nu}\underline{\omega}^\nu = \frac{\partial x^\mu}{\partial x^\nu}\underline{\omega}^\nu = \underline{d}x^\mu \tag{4.3}$$

In coordinate basis we can always write the basis forms as exterior derivatives of the coordinates. The differential dx^μ is given by

$$\underline{d}x^\mu(d\vec{r}) = dx^\mu \tag{4.4}$$

where $d\vec{r}$ is an infinitesimal position vector. $\underline{d}x^\mu$ are *not* infinitesimal quantities. In coordinate basis the exterior derivative of a p-form

$$\underline{\alpha} = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}\underline{d}x^{\mu_1}\wedge\cdots\wedge\underline{d}x^{\mu_p} \tag{4.5}$$

will have the following component form:

$$\underline{d}\,\underline{\alpha} = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p,\mu_0}\underline{d}x^{\mu_0}\wedge\underline{d}x^{\mu_1}\wedge\cdots\wedge\underline{d}x^{\mu_p} \tag{4.6}$$

where $,\mu_0 \equiv \frac{\partial}{\partial x^{\mu_0}}$. **The exterior derivative of a p-form is a $(p+1)$-form.**

Consider the exterior derivative of a p-form $\underline{\alpha}$.

$$\underline{d\alpha} = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p,\mu_0}\underline{d}x^{\mu_0} \wedge \cdots \wedge \underline{d}x^{\mu_p}. \tag{4.7}$$

Let $(\underline{d\alpha})_{\mu_0\cdots\mu_p}$ be the form components of $\underline{d\alpha}$. They must, by definition, be antisymmetric under an arbitrary interchange of indices.

$$\begin{aligned} \underline{d\alpha} &= \frac{1}{(p+1)!}(\underline{d\alpha})_{\mu_0\cdots\mu_p}\underline{d}x^{\mu_0} \wedge \cdots \wedge \underline{d}x^{\mu_p} \\ \text{which, by (4.7)} \Rightarrow \qquad &= \frac{1}{p!}\alpha_{[\alpha_{\mu_1}\cdots\mu_p,\mu_0]}\underline{d}x^{\mu_0} \wedge \cdots \wedge \underline{d}x^{\mu_p} \end{aligned}$$

$$\boxed{\therefore (\underline{d\alpha})_{\mu_0\cdots\mu_p} = (p+1)\alpha_{[\mu_1\cdots\mu_p,\mu_0]}} \tag{4.8}$$

The form equation $\underline{d\alpha} = 0$ in component form is

$$\alpha_{[\mu_1\cdots\mu_p,\mu_0]} = 0 \tag{4.9}$$

`Example` **4.1.1 (Outer product of 1-forms in 3-space)**

$$\begin{aligned} \underline{\alpha} &= \alpha_i\underline{d}x^i \qquad x^i = (x,y,z) \\ \underline{d\alpha} &= \alpha_{i,j}\underline{d}x^j \wedge \underline{d}x^i \end{aligned} \tag{4.10}$$

Also, assume that $\underline{d\alpha} = 0$. The corresponding component equation is

$$\begin{aligned} &\alpha_{[i,j]} = 0 \qquad \Rightarrow \qquad \alpha_{i,j} - \alpha_{j,i} = 0 \\ \Rightarrow \frac{\partial \alpha_x}{\partial y} - \frac{\partial \alpha_y}{\partial x} = 0, \quad &\frac{\partial \alpha_x}{\partial z} - \frac{\partial \alpha_z}{\partial x} = 0, \quad \frac{\partial \alpha_y}{\partial z} - \frac{\partial \alpha_z}{\partial y} = 0 \end{aligned} \tag{4.11}$$

which corresponds to

$$\nabla \times \vec{\alpha} = 0 \tag{4.12}$$

The outer product of an outer product!

$$\begin{aligned} d^2\underline{\alpha} &\equiv \underline{d}(\underline{d\alpha}) \\ d^2\underline{\alpha} &= \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p,\nu_1\nu_2}\underline{d}x^{\nu_2} \wedge \underline{d}x^{\nu_1} \wedge \cdots \wedge \underline{d}x^{\mu_p} \end{aligned} \tag{4.13}$$

$$\boxed{,_{\nu_1\nu_2} \equiv \frac{\partial^2}{\partial x^{\nu_1}\partial x^{\nu_2}}} \tag{4.14}$$

Since

$$_{,\nu_1\nu_2} \equiv \frac{\partial^2}{\partial x^{\nu_1}\partial x^{\nu_2}} =_{,\nu_2\nu_1} \equiv \frac{\partial^2}{\partial x^{\nu_2}\partial x^{\nu_1}} \tag{4.15}$$

summation over ν_1 and ν_2 which are symmetric in $\alpha_{\mu_1\cdots\mu_p,\nu_1\nu_2}$ and antisymmetric in the basis we get **Poincaré's lemma** (valid only for **scalar fields**)

$$\boxed{\underline{d}^2\underline{\alpha} = 0} \tag{4.16}$$

This corresponds to the vector equation

$$\nabla\cdot(\nabla\times\vec{A}) = 0 \tag{4.17}$$

Let $\underline{\alpha}$ be a p-form and $\underline{\beta}$ be a q-form. Then

$$\underline{d}(\underline{\alpha}\wedge\underline{\beta}) = \underline{d\alpha}\wedge\underline{\beta} + (-1)^p\underline{\alpha}\wedge\underline{d\beta} \tag{4.18}$$

4.1.2 Covariant derivative

The general theory of relativity contains a **covariance principle** which states that all equations expressing laws of nature must have the same form irrespective of the coordinate system in which they are derived. This is achieved by writing all equations in terms of tensors. Let us see if the partial derivative of vector components transform as tensor components. Given a vector $\vec{A} = A^\mu\vec{e}_\mu = A^{\mu'}\vec{e}_{\mu'}$ with the transformation of basis given by

$$\frac{\partial}{\partial x^{\nu\prime}} = \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial}{\partial x^\nu} \tag{4.19}$$

So that

$$\begin{aligned} A^{\mu'}_{\ ,\nu'} &\equiv \frac{\partial}{\partial x^{\nu\prime}}\left(A^{\mu'}\right) \\ &= \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial}{\partial x^\nu}\left(A^{\mu'}\right) \\ &= \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial}{\partial x^\nu}\left(\frac{\partial x^{\mu\prime}}{\partial x^\mu}A^\mu\right) \\ &= \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial x^{\mu\prime}}{\partial x^\mu}A^\mu_{\ ,\nu} + \frac{\partial x^\nu}{\partial x^{\nu\prime}}A^\mu\frac{\partial^2 x^{\mu'}}{\partial x^\nu\partial x^\mu} \end{aligned} \tag{4.20}$$

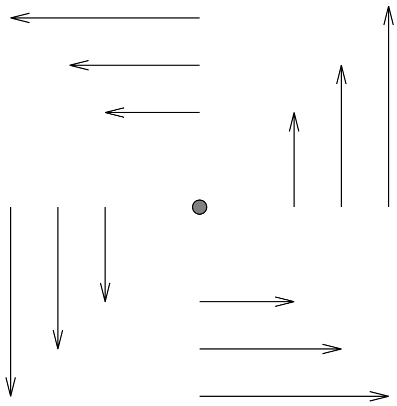
The first term corresponds to a tensorial transformation. The existence of the last term shows that $A^\mu_{,\nu}$ does not, in general, transform as the components of a tensor. Note that $A^\mu_{,\nu}$ will transform as a tensor under linear transformations such as the Lorentz transformations.

The partial derivative must be generalized such as to ensure that when it is applied to tensor components it produces tensor components.

`Example` **4.1.2 (The derivative of a vector field with rotation)**
We have a vector field:

$$\vec{A} = kr\vec{e}_\theta$$

The chain rule for derivation gives:



$$\frac{d}{d\tau} = \frac{\partial}{\partial x^\nu} \cdot \frac{dx^\nu}{d\tau} = u^\nu \frac{\partial}{\partial x^\nu}$$

$$\begin{aligned} \frac{d\vec{A}}{d\tau} &= u^\nu \left(A^\mu \vec{e}_\mu\right)_{,\nu} \\ &= u^\nu \left({A^\mu}_{,\nu} \vec{e}_\mu + A^\mu \vec{e}_{\mu,\nu}\right) \end{aligned}$$

The change of the vector field with a displacement along a coordinate-curve is expressed by:

$$\frac{\partial \vec{A}}{\partial x^\nu} = \vec{A}_{,\nu} = {A^\mu}_{,\nu} \vec{e}_\mu + A^\mu \vec{e}_{\mu,\nu}$$

The change in $\vec{A}$ with the displacement in the θ-direction is:

$$\frac{\partial \vec{A}}{\partial \theta} = {A^\mu}_{,\theta} \vec{e}_\mu + A^\mu \vec{e}_{\mu,\theta}$$

For our vector field, with $A^r = 0$, we get

$$\frac{\partial \vec{A}}{\partial \theta} = \underbrace{{A^\theta}_{,\theta}}_{=0} \vec{e}_\theta + A^\theta \vec{e}_{\theta,\theta}$$

and since ${A^\theta}_{,\theta} = 0$ because $A^\theta = kr$ we end up with

$$\frac{\partial \vec{A}}{\partial \theta} = A^\theta \vec{e}_{\theta,\theta} = kr\vec{e}_{\theta,\theta}$$

We now need to calculate the derivative of $\vec{e}_\theta$. We have:

$$x = r\cos\theta \qquad y = r\sin\theta$$

Using $\vec{e}_\mu = \frac{\partial}{\partial x^\mu}$ we can write:

$$\begin{aligned}\vec{e}_\theta &= \frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} \\ &= -r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y\end{aligned}$$

$$\vec{e}_r = \frac{\partial}{\partial r} = \cos\theta\vec{e}_x + \sin\theta\vec{e}_y$$

Gives:

$$\begin{aligned}\vec{e}_{\theta,\theta} &= -r\cos\theta\vec{e}_x - r\sin\theta\vec{e}_y \\ &= -r(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) = -r\vec{e}_r\end{aligned}$$

This gives us finally:

$$\frac{\partial\vec{A}}{\partial\theta} = -kr^2\vec{e}_r$$

Thus $\frac{\partial\vec{A}}{\partial\theta} \neq 0$ even if $\vec{A} = A^\theta\vec{e}_\theta$ and $A^\theta{}_{,\theta} = 0$.

4.2 The Christoffel Symbols

The covariant derivative was introduced by Christoffel to be able to differentiate tensor fields. It is defined in coordinate basis by generalizing the partially derivative $A^\mu{}_{,\nu}$ to a derivative written as $A^\mu{}_{;\nu}$ and which transforms tensorially,

$$A^{\mu'}{}_{;\nu'} \equiv \frac{\partial x^{\mu'}}{\partial x^\mu}\cdot\frac{\partial x^\nu}{\partial x^{\nu'}}A^\mu{}_{;\nu}. \tag{4.21}$$

The covariant derivative of the contravariant vector components are written as:

$$A^\mu{}_{;\nu} \equiv A^\mu{}_{,\nu} + A^\alpha\Gamma^\mu_{\alpha\nu} \tag{4.22}$$

This equation defines the Christoffel symbols $\Gamma^\mu_{\alpha\nu}$, which are also called the "connection coefficients in coordinate basis". From the transformation formulae for the two first terms follows that the Christoffel symbols transform as:

$$\Gamma^{\alpha'}_{\mu'\nu'} = \frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\Gamma^\alpha_{\mu\nu} + \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x^{\mu'}\partial x^{\nu'}} \tag{4.23}$$

The Christoffel symbols do not transform as tensor components. It is possible to cancel <u>all</u> Christoffel symbols by transforming into a locally Cartesian coordinate system which is co-moving in a locally non-rotating reference frame in free fall. Such coordinates are known as **Gaussian coordinates**.

In general relativity theory an inertial frame is defined as a non-rotating frame in free fall. The Christoffel symbols are 0 (zero) in a locally Cartesian coordinate system which is co-moving in a local inertial frame. Local Gaussian coordinates are indicated with a bar over the indices, giving

$$\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = 0 \tag{4.24}$$

A transformation from local Gaussian coordinates to any coordinates leads to:

$$\Gamma^{\alpha'}_{\mu'\nu'} = \frac{\partial x^{\alpha'}}{\partial x^{\bar{\alpha}}} \frac{\partial^2 x^{\bar{\alpha}}}{\partial x^{\mu'} \partial x^{\nu'}} \tag{4.25}$$

This equation shows that the Christoffel symbols are symmetric in the two lower indices, ie.

$$\Gamma^{\alpha'}_{\mu'\nu'} = \Gamma^{\alpha'}_{\nu'\mu'} \tag{4.26}$$

`Example` **4.2.1 (The Christoffel symbols in plane polar coordinates)**

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\frac{y}{x}$$

$$\begin{aligned}
&\frac{\partial x}{\partial r} = \cos\theta, && \frac{\partial x}{\partial \theta} = -r\sin\theta && \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, && \frac{\partial r}{\partial y} = \sin\theta \\
&\frac{\partial y}{\partial r} = \sin\theta, && \frac{\partial y}{\partial \theta} = r\cos\theta && \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, && \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}
\end{aligned}$$

$$\begin{aligned}
\Gamma^{r}{}_{\theta\theta} &= \frac{\partial r}{\partial x}\frac{\partial^2 x}{\partial \theta^2} + \frac{\partial r}{\partial y}\frac{\partial^2 y}{\partial \theta^2} \\
&= \cos\theta(-r\cos\theta) + \sin\theta(-r\sin\theta) \\
&= -r(\cos\theta^2 + \sin\theta^2) = -r
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} &= \frac{\partial \theta}{\partial x}\frac{\partial^2 x}{\partial \theta \partial r} + \frac{\partial \theta}{\partial y}\frac{\partial^2 y}{\partial \theta \partial r} \\
&= -\frac{\sin\theta}{r}(-\sin\theta) + \frac{\cos\theta}{r}(\cos\theta) \\
&= \frac{1}{r}
\end{aligned}$$

The geometrical interpretation of the covariant derivative was given by Levi-Civita.

Consider a curve S in any (eg. curved) space. It is parameterized by λ, ie. $x^\mu = x^\mu(\lambda)$. λ is invariant and chosen to be the curve length.

The tangent vector field of the curve is $\vec{u} = (dx^\mu/d\lambda)\vec{e}_\mu$. The curve passes through a vector field $\vec{A}$. The covariant directional derivative of the vector field along the curve is defined as:

$$\nabla_{\vec{u}}\vec{A} = \frac{d\vec{A}}{d\lambda} \equiv A^\mu_{;\nu}\frac{dx^\nu}{d\lambda}\vec{e}_\mu = A^\mu_{;\nu}u^\nu\vec{e}_\mu \tag{4.27}$$

The vectors in the vector field are said to be connected by parallel transport along the curve if

$$A^\mu_{;\nu}u^\nu = 0$$

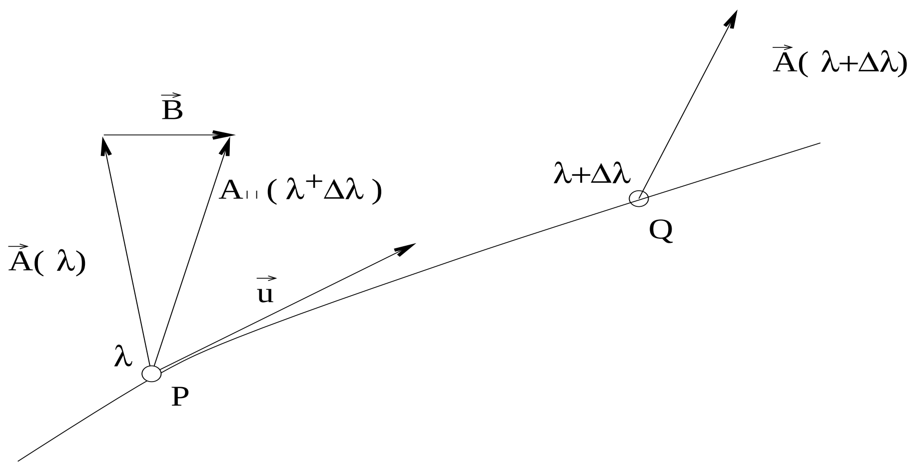


Figure 4.1: Parallel transport from P to Q. The vector $\vec{B} = A^\mu_{;\nu}u^\nu\Delta\lambda\vec{e}_\mu$

$$\vec{u} = \frac{dx^\mu}{d\lambda}\vec{e}_\mu \tag{4.28}$$

According to the geometrical interpretation of Levi-Civita, the covariant directional derivative is:

$$\nabla_{\vec{u}}\vec{A} = A^\mu_{;\nu}u^\nu\vec{e}_\mu = \lim_{\Delta\lambda\to 0}\frac{\vec{A}_\parallel(\lambda+\Delta\lambda) - \vec{A}(\lambda)}{\Delta\lambda} \tag{4.29}$$

where $\vec{A}_\parallel(\lambda+\Delta\lambda)$ means the vector $\vec{A}$ parallel transported from Q to P.

4.3 Geodesic curves

Definition 4.3.1 (Geodesic curves)
A geodesic curve is defined in such a way that,the vectors of the tangent vector field of the curve is connected by parallell transport.

This definition says that geodesic curves are 'as straight as possible'.
If vectors in a vector field $\vec{A}(\lambda)$ are connected by parallell transport by a displacement along a vector $\vec{u}$, we have $A^{\mu}_{;\nu}u^{\nu} = 0$. For geodesic curves, we then have:

$$\boxed{u^{\mu}_{;\nu}u^{\nu} = 0} \tag{4.30}$$

which is the *geodesic equation.*

$$(u^{\mu}_{,\nu} + \Gamma^{\mu}_{\alpha\nu}u^{\alpha})u^{\nu} = 0 \tag{4.31}$$

Then we are using that $\frac{d}{d\lambda} \equiv \frac{dx^{\nu}}{d\lambda}\frac{\partial}{\partial x^{\nu}} = u^{\nu}\frac{\partial}{\partial x^{\nu}}$:

$$\frac{du^{\mu}}{d\lambda} = u^{\nu}\frac{\partial u^{\mu}}{\partial x^{\nu}} = u^{\nu}u^{\mu}_{,\nu} \tag{4.32}$$

The geodesic equation can also be written as:

$$\frac{du^{\mu}}{d\lambda} + \Gamma^{\mu}_{\alpha\nu}u^{\alpha}u^{\nu} = 0 \tag{4.33}$$

Usual notation: $\dot{} = \frac{d}{d\lambda}$

$$u^{\mu} = \frac{dx^{\mu}}{d\lambda} = \dot{x}^{\mu} \tag{4.34}$$

$$\boxed{\ddot{x}^{\mu} + \Gamma^{\mu}_{\alpha\nu}\dot{x}^{\alpha}\dot{x}^{\nu} = 0} \tag{4.35}$$

By comparing eq.4.35 with the equation of motion(4.53) for a free particle (which we deduced from the Lagrangian equations) , we find the equations to be identical. **Conclusion:Free particles follow geodesic curves in spacetime.**

`Example` **4.3.1 (vertical motion of free particle in hyperb. acc. ref. frame)**
Inserting the Christoffel symbols $\Gamma^{x}_{tt} = (1 + \frac{gx}{c^2})g$ from example 4.5.3 into the geodesic equation for a vertical geodesic curve in a hyperbolically accelerated reference frame, we get:

$$\ddot{x} + (1 + \frac{gx}{c^2})g\dot{t}^2 = 0$$

4.4 The covariant Euler-Lagrange equations

Geodesic curves can also be defined as curves with an extremal distance between two points. Let a particle have a world-line (in space-time) between two points (events) P_1 and P_2. Let the curves be described by an invariant parameter λ (proper time τ is used for particles with a rest mass).

The Lagrange-function is a function of coordinates and their derivatives,

$$L = L(x^\mu, \dot{x}^\mu), \qquad \dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}. \tag{4.36}$$

(Note: if $\lambda = \tau$ then $\dot{x}^\mu$ are the 4-velocity components)

The action-integral is $S = \int L(x^\mu, \dot{x}^\mu) d\lambda$. The principle of extremal action (Hamiltons-principle): The world-line of a particle is determined by the condition that S shall be extremal for all infinitesimal variations of curves which keep P_1 and P_2 rigid, ie.

$$\delta \int_{\lambda_1}^{\lambda_2} L(x^\mu, \dot{x}^\mu) d\lambda = 0, \tag{4.37}$$

where λ_1 and λ_2 are the parameter-values at P_1 and P_2. For all the variations

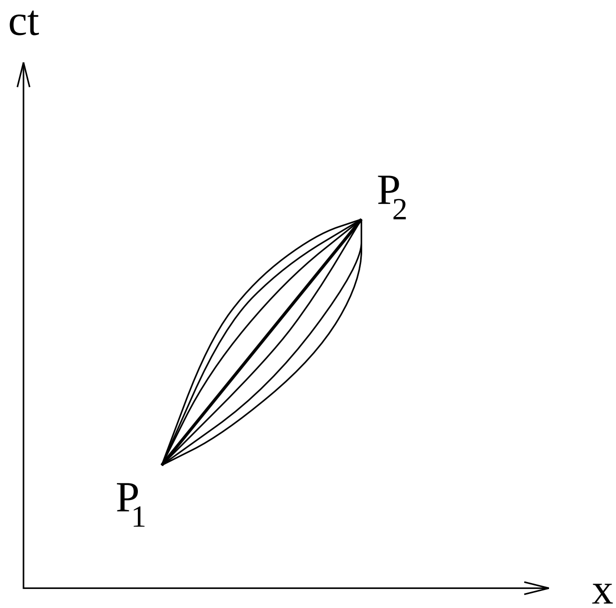


Figure 4.2: Different world-lines connecting P_1 and P_2 in a Minkowski diagram

the following condition applies:

$$\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0 \tag{4.38}$$

We write Eq. (4.37) as

$$\delta \int_{\lambda_1}^{\lambda_2} L d\lambda = \int_{\lambda_1}^{\lambda_2} \left[\frac{\partial L}{\partial x^\mu} \delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu} \delta \dot{x}^\mu \right] d\lambda \tag{4.39}$$

Partial integration of the last term

$$\int_{\lambda_1}^{\lambda_2} \frac{\partial L}{\partial \dot{x}^\mu} \delta \dot{x}^\mu d\lambda = \left[\frac{\partial L}{\partial \dot{x}^\mu} \delta x^\mu \right]_{\lambda_1}^{\lambda_2} - \int_{\lambda_1}^{\lambda_2} \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) \delta x^\mu d\lambda \tag{4.40}$$

Due to the conditions $\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0$ the first term becomes zero. Then we have :

$$\delta S = \int_{\lambda_1}^{\lambda_2} \left[\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) \right] \delta x^\mu d\lambda \tag{4.41}$$

The world-line the particle follows is determined by the condition $\delta S = 0$ for any variation δx^μ. Hence, the world-line of the particle must be given by

$$\boxed{\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) = 0} \tag{4.42}$$

These are the covariant **Euler-Lagrange** equations.

The canonical momentum p_μ conjugated to a coordinate x^μ is defined as

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} \tag{4.43}$$

The Lagrange-equations can now be written as

$$\boxed{\frac{dp_\mu}{d\lambda} = \frac{\partial L}{\partial x^\mu} \quad \text{or} \quad \dot{p}_\mu = \frac{\partial L}{\partial x^\mu}} \; . \tag{4.44}$$

A coordinate which the Lagrange-function does not depend on is known as a **cyclic coordinate**. Hence, $\frac{\partial L}{\partial x^\mu} = 0$ for a cyclic coordinate. From this follows:

The canonical momentum conjugated to a cyclic coordinate is a **constant of motion**

ie. $p_\mu = C$ (constant) if x^μ is cyclic.

A free particle in space-time (curved space-time includes gravitation) has the Lagrange function

$$L = \frac{1}{2} \vec{u} \cdot \vec{u} = \frac{1}{2} \dot{x}_\mu \dot{x}^\mu = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{4.45}$$

An integral of the Lagrange-equations is obtained readily from the 4-velocity identity:

$$\begin{cases} \dot{x}_\mu \dot{x}^\mu = -c^2 & \text{for a particle with rest-mass} \\ \dot{x}_\mu \dot{x}^\mu = 0 & \text{for light} \end{cases} \tag{4.46}$$

The line-element is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu d\lambda^2 = 2L d\lambda^2 \; . \tag{4.47}$$

Thus the Lagrange function of a free particle is obtained from the line-element.

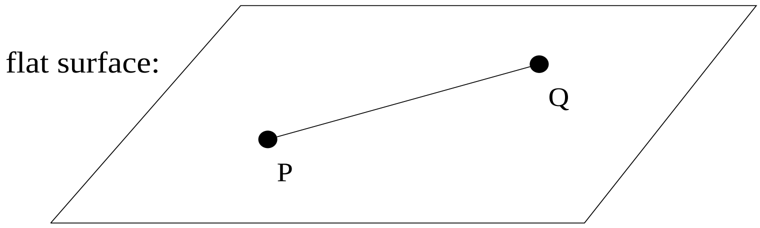


Figure 4.3: On a flat surface, the geodesic curve is the minimal distance between P and Q

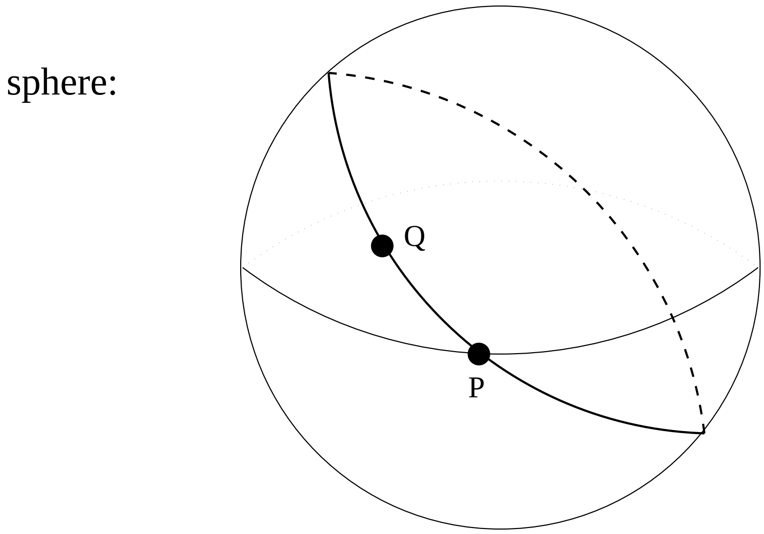


Figure 4.4: On a sphere, the geodesic curves are great circles.

4.5 Application of the Lagrangian formalism to free particles

To describe the motion of a free particle, we start by setting up the line element of the space-time in the chosen coordinate system. There are coordinates on which the metric does not depend. For example, given axial symmetry we may choose the angle θ which is a cyclic coordinate here and the conjugate (covariant) impulse P_θ is a constant of the motion (the orbital spin of the particle). If, in addition, the metric is time independent (**stationary metric**) then t is also cyclic and p_t is a constant of the motion (the mechanical energy of the particle).

A **static metric** is time-independent and unchanged under time reversal (i.e. $t \to -t$). A stationary metric changed under time reversal. Examples of static metrics are Minkowski and hyperbolically accelerated frames. The rotating cylindrical coordinate system is stationary.

4.5.1 Equation of motion from Lagrange's equation

The Lagrange function for a free particle is:

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \tag{4.48}$$

where $g_{\mu\nu} = g_{\mu\nu}(x^{\lambda})$. And the Lagrange equations are

$$\begin{aligned} \frac{\partial L}{\partial x^{\beta}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^{\beta}}\right) &= 0, \\ \frac{\partial L}{\partial \dot{x}^{\beta}} &= g_{\beta\nu} \dot{x}^{\nu}, \\ \frac{\partial L}{\partial x^{\beta}} &= \frac{1}{2} g_{\mu\nu,\beta} \dot{x}^{\mu} \dot{x}^{\nu}. \end{aligned} \tag{4.49}$$

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^{\beta}}\right) \equiv \left(\frac{\partial L}{\partial \dot{x}^{\beta}}\right)^{\bullet} &= \dot{g}_{\beta\nu} \dot{x}^{\nu} + g_{\beta\nu} \ddot{x}^{\nu} \\ &= g_{\beta\nu,\mu} \dot{x}^{\mu} \dot{x}^{\nu} + g_{\beta\nu} \ddot{x}^{\nu} \ . \end{aligned} \tag{4.50}$$

Now, (4.50) and (4.49) together give:

$$\frac{1}{2} g_{\mu\nu,\beta} \dot{x}^{\mu} \dot{x}^{\nu} - g_{\beta\nu,\mu} \dot{x}^{\mu} \dot{x}^{\nu} - g_{\beta\nu} \ddot{x}^{\nu} = 0. \tag{4.51}$$

The second term on the left hand side of (4.51) may be rewritten making use of the fact that $\dot{x}^{\mu} \dot{x}^{\nu}$ is symmetric in $\mu\nu$, as as follows

$$\begin{aligned} g_{\beta\nu,\mu} \dot{x}^{\mu} \dot{x}^{\nu} &= \frac{1}{2}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu}) \dot{x}^{\mu} \dot{x}^{\nu} \\ \Rightarrow\ g_{\beta\nu} \ddot{x}^{\nu} &+ \frac{1}{2}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \dot{x}^{\mu} \dot{x}^{\nu} = 0. \end{aligned} \tag{4.52}$$

Finally, since we are free to multiply (4.52) through by $g^{\alpha\beta}$, we can isolate $\ddot{x}^{\alpha}$ to get the equation of motion in a particularly elegant and simple form:

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = 0 \tag{4.53}$$

where the **Christoffel symbols** $\Gamma^{\alpha}_{\mu\nu}$ in (4.53) are defined by

$$\Gamma^{\alpha}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}). \tag{4.54}$$

Equation(4.53) describes a **geodesic** curve .

4.5.2 Geodesic world lines in spacetime

Consider two timelike curves between two events in spacetime. In fig.4.5 they are drawn in a Minkowski diagram which refers to an inertial reference frame.

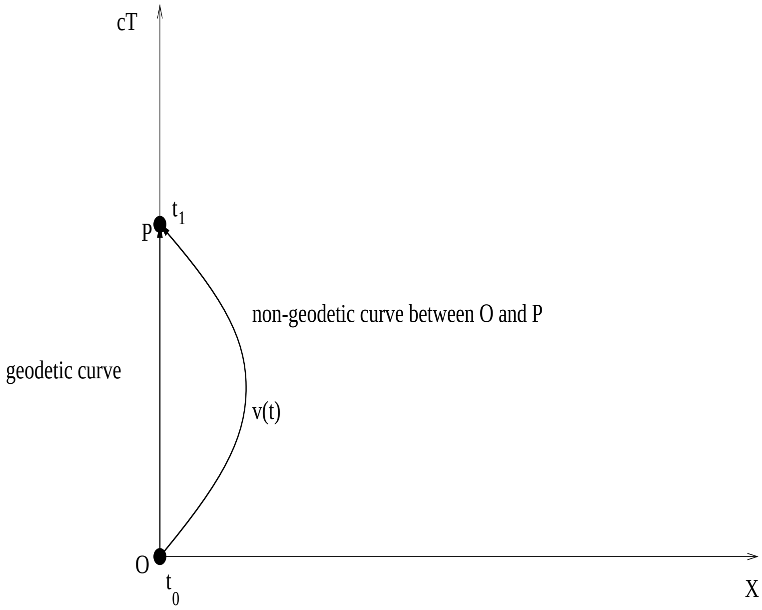


Figure 4.5: Timelike curves in spacetime.

The general interpretation of the line-element for a time-like interval is; The spacetime distance between O and P (See figure 4.5) equals the proper time interval between two events O and P measured on a clock moving in a such way, that it is present both at O and P.

$$\boxed{ds^2 = -c^2 d\tau^2} \tag{4.55}$$

which gives

$$\tau_{0-1} = \int_{T_0}^{T_1} \sqrt{1 - \frac{v^2(T)}{c^2}} dT \tag{4.56}$$

We can see that τ_{0-1} is maximal along the geodesic curve with $v(T) = 0$. Time-like geodesic curves in spacetime have maximal distance between two points.

`Example` **4.5.1 (How geodesics in spacetime can give parabolas in space)**
A geodesic curve between two events O and P has maximal proper time. Consider the last expression in Section 3.2 of the propertime interval of a particle with position x and velocity v in a gravitational field with acceleration of gravity g.

$$d\tau = dt\sqrt{\left(1 + \frac{gx}{c^2}\right)^2 - \frac{v^2}{c^2}}$$

This expression shows that the proper time of the particle proceeds faster the higher up in the field the particle is, and it proceeds slower the faster the particle moves. Consider figure 4.6. The path a free particle follows between the events O and P is a compromize between moving as slowly as possible in space, in order to keep the velocity dependent time dilation small, and moving through regions high up in the gravitational field, in order to prevent the slow proceeding of proper time far down. However if the particle moves too high up, its velocity becomes so large that it proceeds slower again. The compromise between kinematic and gravitational time dilation which gives maximal proper time between O and P is obtained for the thick curve in fig. 4.6. This is the curve followed by a free particle between the events O and P.

We shall now deduce the mathematical expression of what has been said above. Timelike geodesic curves are curves with maximal proper time, i.e.

$$\tau = \int_0^{\tau_1} \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\tau$$

is maximal for a geodesic curve. However the action

$$J = -2\int_0^{\tau_1} L d\tau = -\int_0^{\tau_1} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu d\tau$$

is maximal for the same curves and this gives an easier calculation.

In the case of a vertical curve in a hyperbolically accelerated reference frame the Lagrangian is

$$L = \frac{1}{2}\left(-\left(1+\frac{gx}{c^2}\right)^2 \dot{t}^2 + \frac{\dot{x}^2}{c^2}\right) \tag{4.57}$$

Using the Euler-Lagrange equations now gives

$$\ddot{x} + (1+\frac{gx}{c^2})g\dot{t}^2 = 0$$

which is the equation of the geodesic curve in example 4.3.1.

Since spacetime is flat, the equation represents straight lines in spacetime. The projection of such curves into the three space of arbitrary inertial frames gives straight paths in 3-space, in accordance with Newton's 1st law. However projecting it into an accelerated frame where the particle also has a horizontal motion, and taking the Newtonian limit, one finds the parabolic path of projectile motion.

`Example` **4.5.2 (Spatial geodesics described in the reference frame of a rotating disc.)**
In Figure 4.7, we see a rotating disc. We can see two geodesic curves between P_1 and P_2. The dashed line is the geodesic for the non-rotating disc. The other curve is a geodesic for the 3-space of a rotating reference frame. We can see that the geodesic is curved inward when the disc is rotating. The curve has to curve **inward** since the measuring rods are longer there (because of Lorentz-contraction). Thus, the minimum distance between P_1 and P_2 will be achieved by an inwardly bent curve.

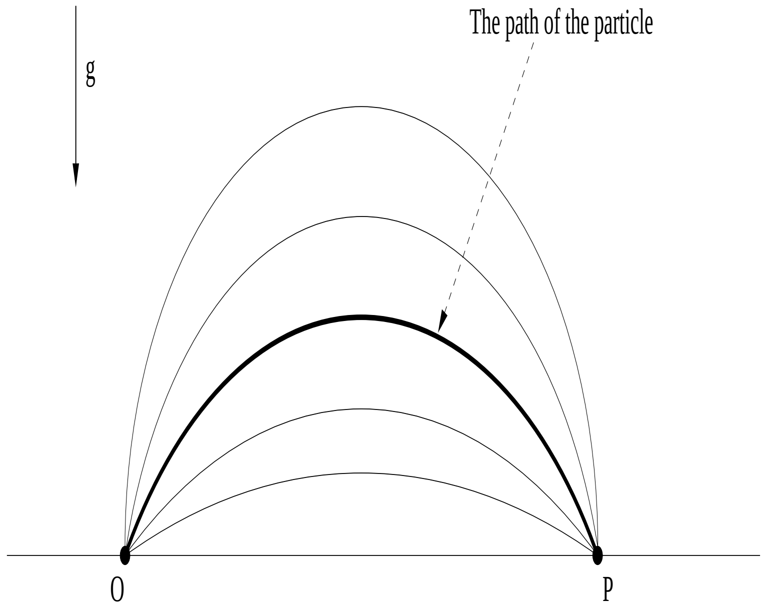


Figure 4.6: The particle moves between two events O and P at fixed points in time. The path chosen by the particle between O and P is such that the proper time taken by the particle betweem these two events is as large as possible. Thus the goal of the particle is to follow a path such that its comoving standard clocks goes as fast as possible. If the particle follows the horizontal line between O and P it goes as slowly as possible and the kinematical time dilation is as small as possible. Then the slowing down of its comoving standard clocks due to the kinematical time dilation is as small as possible, but the particle is far down in the gravitational field and its proper time goes slowly for that reason. Paths further up leads to a greater rate of proper time. But above the curve drawn as a thick line, the kinematical time dilation will dominate, and the proper time proceeds more slowly.

We will show this mathematically, using the Lagrangian equations. The line element for the space $dt = dz = 0$ of the rotating reference frame is

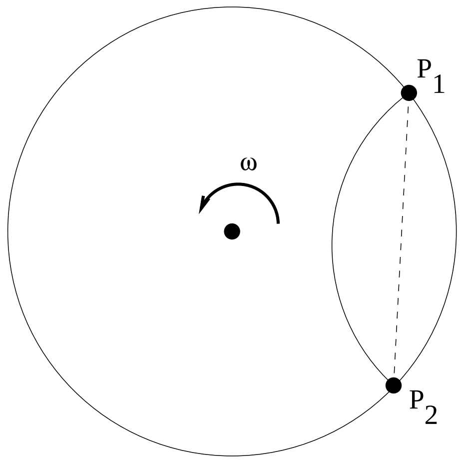


Figure 4.7: Geodesic curves on a non-rotating (dashed line) and rotating (solid line) disc.

$$dl^2 = dr^2 + \frac{r^2 d\theta^2}{1 - \frac{r^2\omega^2}{c^2}}$$

Lagrangian function:

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{r^2\dot{\theta}^2}{1 - \frac{r^2\omega^2}{c^2}}$$

We will also use the identity:

$$\dot{r}^2 + \frac{r^2\dot{\theta}^2}{1 - \frac{r^2\omega^2}{c^2}} = 1 \tag{4.58}$$

(We got this from using $\vec{u} \cdot \vec{u} = 1$) We see that θ is cyclic ($\frac{\partial L}{\partial \theta} = 0$), implying:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{r^2\dot{\theta}}{1 - \frac{r^2\omega^2}{c^2}} = constant$$

This gives:

$$\dot{\theta} = \big(1 - \frac{r^2\omega^2}{c^2}\big)\frac{p_\theta}{r^2} = \frac{p_\theta}{r^2} - \frac{\omega^2 p_\theta}{c^2} \tag{4.59}$$

Inserting 4.59 into 4.58:

$$\dot{r}^2 = 1 + \frac{\omega^2 p_\theta^2}{c^2} - \frac{p_\theta^2}{r^2} \tag{4.60}$$

This gives us the equation of the geodesic curve between P_1 and P_2:

$$\frac{\dot{r}}{\dot{\theta}} = \pm\frac{dr}{d\theta} = \frac{r^2\sqrt{1 + \frac{\omega^2 p_\theta^2}{c^2} - \frac{p_\theta^2}{r^2}}}{p_\theta\big(1 - \frac{r^2\omega^2}{c^2}\big)} \tag{4.61}$$

Boundary conditions:

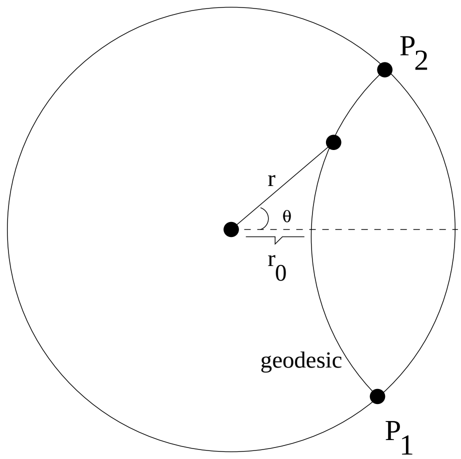


Figure 4.8: Geodesic curves on a rotating disc,coordinates

$$\dot{r} = 0, r = r_0, \text{ for } \theta = 0$$

Inserting this into 4.60 gives:

$$\frac{p_\theta}{r_0} = \sqrt{1 + \frac{p_\theta^2 \omega^2}{c^2}} \tag{4.62}$$

Rearranging 4.61,using 4.62 gives:

$$\frac{dr}{r\sqrt{r^2 - r_0^2}} - \frac{\omega^2}{c^2}\frac{rdr}{\sqrt{r^2 - r_0^2}} = \frac{d\theta}{r_0}$$

Integrating this yields:

$$\theta = \pm \frac{r_0 \omega^2}{c^2}\sqrt{r^2 - r_0^2} \mp \arccos\frac{r_0}{r}$$

`Example` **4.5.3 (Christoffel symbols in a hyperbolically accelerated reference frame)**
The Christoffel symbols were defined in Equation (4.53).

$$\Gamma^\alpha_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta}(g_{\ \beta\mu,\nu} + g_{\ \beta\nu,\mu} - g_{\ \mu\nu,\beta}).$$

In this example

$$g_{tt} = -\left(1 + \frac{gx}{c^2}\right)^2 c^2, \qquad g_{xx} = g_{yy} = g_{zz} = 1$$

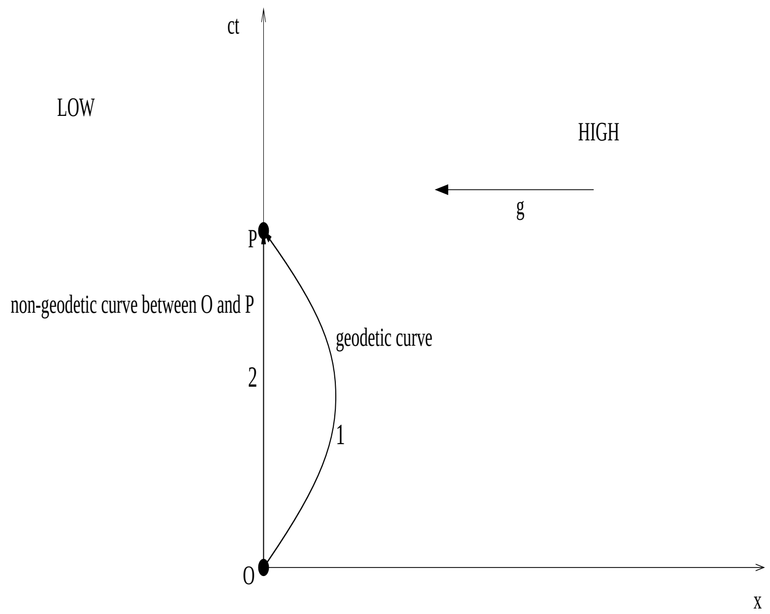


Figure 4.9: Vertical throw in the accelerated referenceframe.

and only the term $\frac{\partial g_{tt}}{\partial x}$ contributes to $\Gamma^{\alpha}_{\mu\nu}$. Thus the only non-vanishing Christoffel symbols are

$$\begin{aligned}
\Gamma^t{}_{xt} = \Gamma^t{}_{tx} &= \frac{1}{2} g^{tt}\left(\frac{\partial g_{tt}}{\partial x}\right) \\
&= \frac{1}{2g_{tt}}\frac{\partial g_{tt}}{\partial x} \\
&= \frac{2\left(1+\frac{gx}{c^2}\right)g}{2\left(1+\frac{gx}{c^2}\right)^2 c^2} \\
&= \frac{1}{\left(1+\frac{gx}{c^2}\right)}\frac{g}{c^2} \\
\Gamma^x_{tt} &= -\frac{1}{2} g^{xx}\left(\frac{\partial g_{tt}}{\partial x}\right) \\
&= -\frac{1}{2}\left\{-2\left(1+\frac{gx}{c^2}\right)\frac{g}{c^2}c^2\right\} \\
&= \left(1+\frac{gx}{c^2}\right)g
\end{aligned}$$

`Example` **4.5.4 (Vertical projectile motion in a hyperbolically accelerated reference frame)**

$$ds^2 = -\left(1+\frac{gx}{c^2}\right)^2 c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{4.63}$$

Vertical motion implies that $dy = dz = 0$ and the Lagrange function becomes

$$\begin{aligned} L &= \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \\ &= -\frac{1}{2}\left(1+\frac{gx}{c^2}\right)^2 c^2\dot{t}^2 + \frac{1}{2}\dot{x}^2 \end{aligned}$$

where the dots imply differentiation w.r.t the particle's proper time, τ. And the initial conditions are:

$$\begin{aligned} x(0) = 0, \quad \dot{x}(0) &= (u^0, u^x, 0, 0) \\ &= \gamma(c, v, 0, 0), \\ \text{where,} \quad \gamma = \left(1 - v^2/c^2\right)^{-1/2}. \end{aligned}$$

What is the maximum height, h reached by the particle?
Newtonian description: $\frac{1}{2}mv^2 = mgh \Rightarrow h = \frac{v^2}{2g}$.
Relativistic description: t is a cyclic coordinate $\Rightarrow$ $x^0 = ct$ is cyclic and $p_0 = constant$.

$$p_0 = \frac{\partial L}{\partial \dot{x}^0} = \frac{1}{c}\frac{\partial L}{\partial \dot{t}} = -c\left(1+\frac{gx}{c^2}\right)^2\dot{t} \tag{4.64}$$

Now the 4-velocity identity is

$$\vec{u}\cdot\vec{u} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -c^2 \tag{4.65}$$

so

$$-\frac{1}{2}\left(1+\frac{gx}{c^2}\right)^2 c^2\dot{t}^2 + \frac{1}{2}\dot{x}^2 = -\frac{1}{2}c^2 \tag{4.66}$$

and given that the maximum height h is reached when $\dot{x} = 0$ we get

$$\left(1+\frac{gh}{c^2}\right)^2 \dot{t}^2_{x=h} = 1. \tag{4.67}$$

Now, since p_0 is a constant of the motion, it preserves its initial value throughout the flight (i.e. $p_0 = -c\dot{t}(0) = -\gamma c$) and particularly at $x = h$,

$$(4.64) \Rightarrow p_0 = -\gamma c = -c\left(1+\frac{gh}{c^2}\right)^2 \dot{t}_{x=h} \tag{4.68}$$

Finally, dividing equation (4.67) by equation (4.68) and substituting back in equation (4.67) gives

$$h = \frac{c^2}{g}(\gamma - 1) \tag{4.69}$$

In the Newtonian limit (4.69) becomes

$$h = \frac{c^2}{g}\left(\frac{1}{(1-v^2/c^2)^{1/2}} - 1\right) \approx \frac{c^2}{g}\left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) \Rightarrow h \approx \frac{v^2}{2g}$$

`Example` **4.5.5 (The twin "paradox")**
Eva travels to Alpha Centauri, 4 light years from the Earth, with a velocity $v = 0.8c$ ($\gamma = 1/0.6$). The trip takes 5 years out and 5 years back. This means that Eli, who remains at Earth is 10 years older when she meets Eva at the end of her journey. Eva, on the other hand, is $10(1 - v^2/c^2)^{1/2} = 10(0.6) = 6$ years older.

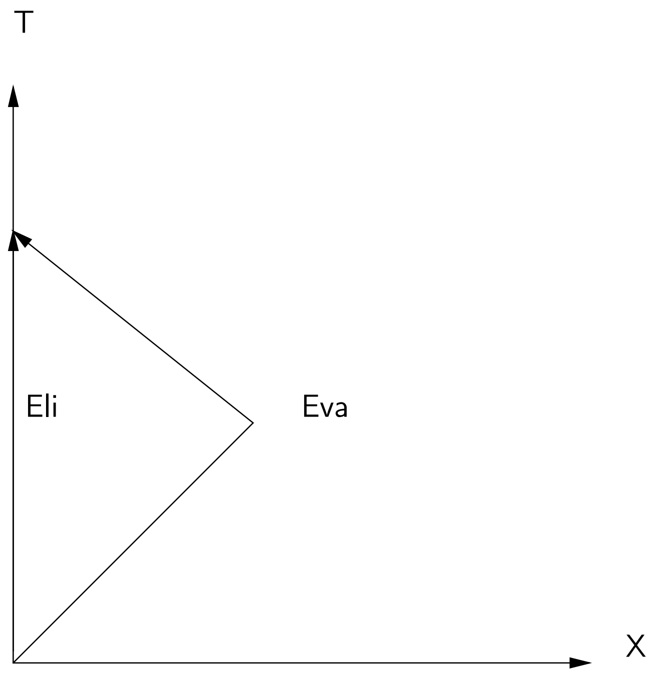


Figure 4.10: The twins Eli and Eva each travel between two fixed events in space-time

According to the theory of relativity, Eva can consider herself as being stationary and Eli as the one whom undertakes the long journey. In this picture it seems that Eva and Eli must be 10 and 6 years older respectively upon their return.

Let us accept the principle of general relativity as applied to accelerated reference frames and review the twin "paradox" in this light.

Eva's description of the trip when she sees herself as stationary is as follows.

Eva perceives a Lorentz contracted distance between the Earth and Alpha Centauri, namely, 4 light years $\times 1/\gamma = 2.4\text{light years}$. The Earth and Eli travel with $v = 0.8c$. Her travel time in one direction is then $\frac{2.4\text{light years}}{0.8c} = 3$ years. So the round trip takes 6 years according to Eva. That is Eva is 6 years older when they meet again. This is in accordance with the result arrived at by Eli. According to

Eva, Eli ages by only 6 years $\times 1/\gamma = 3.6$ years during the round trip, not 10 years as Eli found.

On turning about Eva experiences a force which reduces her velocity and accelerates her towards the Earth and Eli. This means that she experiences a gravitational force directed **away** from the Earth. Eli is higher up in this gravitational field and ages **faster** than Eva, because of the gravitational time dilation. We assume that Eva has constant proper acceleration and is stationary in a hyperbolically accelerated frame as she turns about.

The canonical momentum p_t for Eli is then(see Equation (4.64))

$$p_t = -\left(1+\frac{gx}{c^2}\right)^2 c\dot{t}$$

Inserting this into the 4-velocity identity gives

$$p_t^2 - c^2\left(1+\frac{gx}{c^2}\right)^2 = \left(1+\frac{gx}{c^2}\right)^2 \dot{x}^2, \tag{4.70}$$

or

$$d\tau = \frac{1+\frac{gx}{c^2}}{\sqrt{p_t^2 - c^2\left(1+\frac{gx}{c^2}\right)^2}} dx$$

Now, since $\dot{x}=0 \quad \text{for} \quad x = x_2$ (x_2 is Eli's turning point according to Eva), we have that

$$p_t = c\left(1+\frac{gx_2}{c^2}\right)$$

Let x_1 be Eli's position according to Eva just as Eva begins to notice the gravitational field. That is when Eli begins to slow down in Eva's frame.

Integration from x_1 to x_2 and inserting the value of p_t gives

$$\tau_{1-2} = \frac{c}{g}\sqrt{\left(1+\frac{gx_2}{c^2}\right)^2 - \left(1+\frac{gx_1}{c^2}\right)^2}$$
$$\Rightarrow \lim_{g\to\infty} \tau_{1-2} = \frac{1}{c}\sqrt{x_2^2 - x_1^2}.$$

Now setting $x_2 = 4$ and $x_1 = 2.4$ light years respectively we get

$$\lim_{g\to\infty} \tau_{1-2} = 3.2 \quad \text{years}$$

Eli's aging as she turns about is, according to Eva,

$$\Delta\tau_{Eli} = 2\lim_{g\to\infty} \tau_{1-2} = 6.4 \quad \text{years.}$$

So Eli's has aged by a total of $\tau_{Eli} = 3.6 + 6.4 = 10$ years, according to Eva, which is just what Eli herself found.

4.5.3 Gravitational Doppler effect

This concerns the frequency shift of light traversing up or down in a gravitational field. The 4-momentum of a particle with relativistic energy E and spatial velocity $\vec{w}$ (3-velocity) is given by:

$$\vec{P} = E(1, \vec{w}) \qquad (c = 1) \tag{4.71}$$

Let $\vec{U}$ be the 4-velocity of an observer. In a co-moving orthonormal basis of the observer we have $\vec{U} = (1, 0, 0, 0)$. This gives

$$\vec{U} \cdot \vec{P} = -\hat{E} \tag{4.72}$$

The energy of a particle with 4-momentum $\vec{P}$ measured by an observer with 4-velocity $\vec{U}$ is

$$\hat{E} = -\vec{U} \cdot \vec{P} \tag{4.73}$$

Let $E_S = -(\vec{U} \cdot \vec{P})_S$ and $E_a = -(\vec{U} \cdot \vec{P})_a$ be the energy of a photon, measured locally by observers in rest in the transmitter and receiver positions, respectively. This gives[1]

$$\frac{E_S}{(\vec{U} \cdot \vec{P})_S} = \frac{E_a}{(\vec{U} \cdot \vec{P})_a} \tag{4.74}$$

Let the angular frequency of the light, measured by the transmitter and receiver, be w_s and w_a, respectively. We then have

$$w_s = \frac{E_S}{h}, \qquad w_a = \frac{E_a}{h}, \tag{4.75}$$

which gives:

$$w_a = \frac{(\vec{U} \cdot \vec{P})_a}{(\vec{U} \cdot \vec{P})_s} w_s \tag{4.76}$$

For an observer at rest in a time-independent orthogonal metric we have

$$\vec{U} \cdot \vec{P} = U^t P_t = \frac{dt}{d\tau} P_t \tag{4.77}$$

where P_t is a constant of motion (since t is a cyclic coordinate) for photons and hence has the same value in transmitter and receiver positions. The line-element is

$$ds^2 = g_{tt} dt^2 + g_{ii} (dx^i)^2 \tag{4.78}$$

Using the physical interpretation (4.55) of the line-element for a time like interval, we obtain for the proper time of an observer at rest

$$d\tau^2 = -g_{tt} dt^2 \Rightarrow d\tau = \sqrt{-g_{tt}} dt \tag{4.79}$$

[1] $\vec{A} \cdot \vec{B} = A_0 B^0 + A_1 B^1 + ... = g_{00} A^0 B^0 + g_{11} A^1 B^1 + ...$, an orthonormal basis gives $\vec{A} \cdot \vec{B} = -A^0 B^0 + A^1 B^1 + ...$

Hence

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{-g_{tt}}}, \tag{4.80}$$

which gives

$$\vec{U} \cdot \vec{P} = \frac{1}{\sqrt{-g_{tt}}} P_t. \tag{4.81}$$

Inserting this into the expression for angular frequency (4.76) gives

$$w_a = \sqrt{\frac{(g_{tt})_s}{(g_{tt})_a}} w_s \tag{4.82}$$

Note: we have assumed an orthogonal and time-independent metric, i.e. $P_{t_1} = P_{t_2}$. Inserting the metric of a hyperbolically accelerated reference system with

$$g_{tt} = -(1 + \frac{gx}{c^2})^2 \tag{4.83}$$

gives

$$w_a = \frac{1 + \frac{gx_s}{c^2}}{1 + \frac{gx_a}{c^2}} w_s, \tag{4.84}$$

or

$$\frac{w_a - w_s}{w_s} = \frac{1 + \frac{gx_s}{c^2}}{1 + \frac{gx_a}{c^2}} - 1 = \frac{\frac{g}{c^2}(x_s - x_a)}{1 + \frac{gx_a}{c^2}} \approx \frac{g}{c^2} H, \tag{4.85}$$

where $H = x_s - x_a$ is the difference in height between transmitter and receiver.

`Example` **4.5.6 (Measurements of gravitational Doppler effects (Pound and Rebka 1960))**

$$H \approx 20m, \qquad g = 10m/s^2$$

gives

$$\frac{\Delta w}{w} = \frac{200}{9 \times 10^{16}} = 2.2 \times 10^{-15}.$$

This effect was measured by Pound and Rebka in 1960.

4.6 The Koszul connection

The covariant directional derivative of a scalar field f in the direction of a vector $\vec{u}$ is defined as:

$$\nabla_{\vec{u}} f \equiv \vec{u}(f) \tag{4.86}$$

Here the vector $\vec{u}$ should be taken as a differensial operator. (In coordinate basis, $\vec{u} = u^\mu \frac{\partial}{\partial x^\mu}$)
The directional derivative along a basis vector $\vec{e}_\nu$ is written as:

$$\nabla_\nu \equiv \nabla_{\vec{e}_\nu} \tag{4.87}$$

Hence $\nabla_\mu(\quad) = \nabla_{\vec{e}_\mu}(\quad) = \vec{e}_\mu(\quad)$

Definition 4.6.1 (Koszul's connection coeffecients in an arbitrary basis)
In an aribitrary basis the Koszul connection coefficients are defined by

$$\boxed{\nabla_\nu \vec{e}_\mu \equiv \Gamma^\alpha_{\ \mu\nu} \vec{e}_\alpha} \tag{4.88}$$

which may also be written $\vec{e}_\nu(\vec{e}_\mu) = \Gamma^\alpha_{\ \mu\nu}\vec{e}_\alpha$. In coordinate basis , $\Gamma^\alpha_{\ \mu\nu}$ is reduced to Christoffel symbols and one often writes $\vec{e}_{\mu,\nu} = \Gamma^\alpha_{\ \mu\nu}\vec{e}_\alpha$. In an arbitrary basis , $\Gamma^\alpha_{\ \mu\nu}$ has no symmetry.

`Example` **4.6.1 (The connection coefficients in a rotating reference frame.)**
Coordinate transformation: (T, R, Θ are coordinates in the non-rotating reference frame, t, r, θ in the rotating.) Corresponding Cartesian coordinates:X, Y and x, y.

$$t = T, r = R, \theta = \Theta - \omega T$$
$$X = R\cos\Theta, Y = R\sin\Theta$$
$$X = r\cos(\theta + \omega t), Y = r\sin(\theta + \omega t)$$

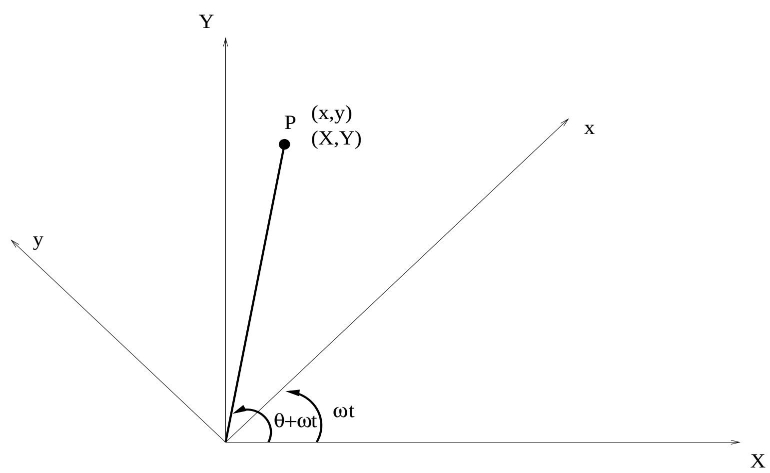


Figure 4.11: The non-rotating coordinate system (X,Y) and the rotating system (x,y),rotating with angular velocity ω

$$\vec{e}_t = \frac{\partial}{\partial t} = \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial t}\frac{\partial}{\partial Y} + \frac{\partial T}{\partial t}\frac{\partial}{\partial T}$$

gives:

$$\begin{aligned}
\vec{e}_t &= -r\omega\sin(\theta+\omega t)\vec{e}_X + r\omega\cos(\theta+\omega t)\vec{e}_Y + \vec{e}_T \\
\vec{e}_r &= \frac{\partial X}{\partial r}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial r}\frac{\partial}{\partial Y} \\
&= \cos(\theta+\omega t)\vec{e}_X + \sin(\theta+\omega t)\vec{e}_Y \\
\vec{e}_\theta &= \frac{\partial X}{\partial \theta}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial \theta}\frac{\partial}{\partial Y} \\
&= -r\sin(\theta+\omega t)\vec{e}_X + r\cos(\theta+\omega t)\vec{e}_Y
\end{aligned}$$

We are going to find the Christoffel symbols, which involves differentiation of basis vectors. This coordinate transformation makes this easy, since $\vec{e}_X, \vec{e}_Y, \vec{e}_T$ are constant. Differentiation gives

$$\nabla_t \vec{e}_t = -r\omega^2\cos(\theta+\omega t)\vec{e}_X - r\omega^2\sin(\theta+\omega t)\vec{e}_Y \tag{4.89}$$

The connection coefficients are (see eq. 4.88)

$$\nabla_\nu \vec{e}_\mu \equiv \Gamma^\alpha_{\ \mu\nu}\vec{e}_\alpha \tag{4.90}$$

So, to calculate $\Gamma^\alpha_{\ \mu\nu}$, the right hand side of eq.4.89 has to by expressed by the basis that we are differentiating.
By inspection, the right hand side is $-r\omega^2\vec{e}_r$.
That is $\nabla_t\vec{e}_t = -r\omega^2\vec{e}_r$ giving $\Gamma^r_{\ tt} = -r\omega^2$.
The other nonzero Christoffel symbols are

$$\begin{aligned}
\Gamma^\theta_{\ rt} = \Gamma^\theta_{\ tr} = \frac{\omega}{r}, \Gamma^\theta_{\ \theta r} = \Gamma^\theta_{\ r\theta} = \frac{1}{r} \\
\Gamma^r_{\ \theta t} = \Gamma^r_{\ t\theta} = r\omega, \Gamma^r_{\ \theta\theta} = -r
\end{aligned}$$

`Example` **4.6.2 (Acceleration in a non-rotating reference frame (Newton))**

$$\ddot{\vec{r}} = \dot{\vec{v}} = (\dot{v}^i + \Gamma^i_{\ jk}v^j v^k)\vec{e}_i,$$

where $\dot{} \equiv \frac{d}{dt}$. i, j, and k are space indices. Inserting the Christoffel symbols for plane polar coordinates (see example 4.2.1), gives:

$$\vec{a}_{inert} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta})\vec{e}_\theta$$

`Example` **4.6.3 (The acceleration of a particle, relative to the rotating reference frame)**
Inserting the Christoffel symbols from example 4.6.1:

$$\begin{aligned}
\vec{a}_{rot} &= (\ddot{r} - r\dot{\theta}^2 - r\omega^2 + \Gamma^r_{\ \theta t}\dot{\theta}\dot{t} + \Gamma^r_{\ t\theta}\dot{t}\dot{\theta})\vec{e}_r + (\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} + \Gamma^\theta_{\ rt}\dot{r}\dot{t} + \Gamma^\theta_{\ tr}\dot{t}\dot{r})\vec{e}_\theta \\
&= (\ddot{r} - r\dot{\theta}^2 - r\omega^2 - 2r\omega\dot{\theta})\vec{e}_{\hat{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} + 2\dot{r}\omega)\vec{e}_{\hat{\theta}} \\
&= \vec{a}_{inert} - (r\omega^2 - 2r\omega\dot{\theta})\vec{e}_{\hat{r}} + 2\dot{r}\omega\vec{e}_{\hat{\theta}}
\end{aligned}$$

The angular velocity of the reference frame, is $\vec{\omega} = \omega\vec{e}_z$. We also introduce $\vec{r} = r\vec{e}_r$. The velocity relative to the rotating reference frame is then:

$$\dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r$$

Furthermore

$$\dot{\vec{e}}_r = \frac{d\vec{e}_r}{dt} = \frac{\partial\vec{e}_r}{\partial x^i}\frac{dx^i}{dt} = v^i\vec{e}_{r,i}$$

Using definition 4.6.1 in a coordinate basis, this may be written

$$\dot{\vec{e}}_r = v^i\Gamma^j{}_{ri}\vec{e}_j$$

Using the expressions of the Christoffel symbols in example 4.6.1, we get

$$\dot{\vec{e}}_r = v^\theta\Gamma^\theta{}_{r\theta}\vec{e}_\theta = \dot{\theta}\frac{1}{r}\vec{e}_\theta = \dot{\theta}\vec{e}_{\hat{\theta}}$$

Hence

$$\vec{v} = \dot{\vec{r}} = \dot{r}\vec{e}_{\hat{r}} + r\dot{\theta}\vec{e}_{\hat{\theta}}$$

Inserting this into the expression for the acceleration, gives:

$$\boxed{\ddot{\vec{r}}_{rot} = \ddot{\vec{r}}_{inert} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2\vec{\omega}\times\vec{v}}$$

We can see that the centrifugal acceleration (the term in the middle) and the coriolis acceleration (last term) is contained in the expression for the covariant derivative.

4.7 Connection coefficients $\Gamma^\alpha_{\mu\nu}$ and structure coefficients $c^\alpha_{\mu\nu}$ in a Riemannian (torsion free) space

The commutator of two vectors, $\vec{u}$ and $\vec{v}$, expressed by covariant directional derivatives is given by:

$$[\vec{u},\vec{v}] = \nabla_{\vec{u}}\vec{v} - \nabla_{\vec{v}}\vec{u} \tag{4.91}$$

Let $\vec{u} = \vec{e_\mu}$ and $\vec{v} = \vec{e_\nu}$. We then have:

$$[\vec{e_\mu},\vec{e_\nu}] = \nabla_\mu\vec{e_\nu} - \nabla_\nu\vec{e_\mu}. \tag{4.92}$$

Using the definitions of the connection and structure coefficients we get:

$$c^\alpha_{\ \mu\nu}\vec{e_\alpha} = (\Gamma^\alpha_{\ \nu\mu} - \Gamma^\alpha_{\ \mu\nu})\vec{e_\alpha} \tag{4.93}$$

Thus in a torsion free space

$$c^\alpha_{\ \mu\nu} = \Gamma^\alpha_{\ \nu\mu} - \Gamma^\alpha_{\ \mu\nu} \tag{4.94}$$

In **coordinate basis** we have

$$\vec{e_\mu} = \frac{\partial}{\partial x^\mu}\ , \quad \vec{e_\nu} = \frac{\partial}{\partial x^\nu} \tag{4.95}$$

And therefore:

$$
\begin{aligned}
[\vec{e_\mu}, \vec{e_\nu}] &= [\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}] \\
&= \frac{\partial}{\partial x^\mu}(\frac{\partial}{\partial x^\nu}) - \frac{\partial}{\partial x^\nu}(\frac{\partial}{\partial x^\mu}) \\
&= \frac{\partial^2}{\partial x^\mu \partial x^\nu} - \frac{\partial^2}{\partial x^\nu \partial x^\mu} = 0
\end{aligned}
\tag{4.96}
$$

Equation (4.96) shows that $c^\alpha{}_{\mu\nu} = 0$, and that the connection coefficients in Equation (4.94) therefore are symmetrical in a coordinate basis:

$$
\Gamma^\alpha{}_{\nu\mu} = \Gamma^\alpha{}_{\mu\nu} \tag{4.97}
$$

4.8 Covariant differentiation of vectors, forms and tensors

4.8.1 Covariant differentiation of a vector in an arbitrary basis

$$
\begin{aligned}
\nabla_\nu \vec{A} &= \nabla_\nu (A^\mu \vec{e_\mu}) \\
&= \nabla_\nu A^\mu \vec{e_\mu} + A^\alpha \nabla_\nu \vec{e_\alpha}
\end{aligned}
\tag{4.98}
$$

$$
\nabla_\nu A^\mu = \vec{e_\nu}(A^\mu) \ , \quad \vec{e_\nu} = M^\mu{}_\nu \frac{\partial}{\partial x^\mu} \ , \tag{4.99}
$$

where $M^\mu{}_\nu$ are the elements of a transformation matrix between a coordinate basis $\{\frac{\partial}{\partial x^\mu}\}$ and an arbitrary basis $\{\vec{e_\nu}\}$. (If $\vec{e_\nu}$ had been a coordinate basis vector, we would have gotten $\vec{e_\nu}(A^\mu) = \frac{\partial}{\partial x^\nu}(A^\mu) = A^\mu{}_{,\nu}$).

$$
\nabla_\nu \vec{A} = [\vec{e_\nu}(A^\mu) + A^\alpha \Gamma^\mu{}_{\alpha\nu}]\vec{e_\mu} \tag{4.100}
$$

Definition 4.8.1 (Covariant derivative of a vector)
The covariant derivative of a vector in an arbitrary basis is defined by:

$$
\nabla_\nu \vec{A} \equiv A^\mu{}_{;\nu} \vec{e}_\mu \tag{4.101}
$$

So:

$$
\begin{aligned}
A^\mu_{;\nu} &= \vec{e}_\nu(A^\mu) + A^\alpha \Gamma^\mu{}_{\alpha\nu} \\
\textit{where } \nabla_\nu \vec{e}_\alpha &\equiv \Gamma^\mu{}_{\alpha\nu} \vec{e}_\mu
\end{aligned}
\tag{4.102}
$$

4.8.2 Covariant differentiation of forms

Definition 4.8.2 (Covariant directional derivative of a one-form field)
Given a vector field $\vec{A}$ and a one-form field $\underline{\alpha}$, the covariant directional derivative of $\underline{\alpha}$ in the direction of the vector $\vec{u}$ is defined by:

$$
(\nabla_{\vec{u}} \underline{\alpha})(\vec{A}\,) \equiv \nabla_{\vec{u}} [\underbrace{\underline{\alpha}(\vec{A}\,)}_{\alpha_\mu A^\mu}] - \underline{\alpha}(\nabla_{\vec{u}} \vec{A}) \tag{4.103}
$$

Let $\underline{\alpha} = \underline{\omega}^\mu$ (basis form), $\underline{\omega}^\mu(\vec{e_\nu}) \equiv \delta^\mu{}_\nu$ and let $\vec{A} = \vec{e_\nu}$ and $\vec{u} = \vec{e_\lambda}$. We then have:

$$(\nabla_\lambda \underline{\omega}^\mu)(\vec{e_\nu}) = \nabla_\lambda[\underbrace{\underline{\omega}^\mu(\vec{e_\nu})}_{\delta^\mu{}_\nu}] - \underline{\omega}^\mu(\nabla_\lambda \vec{e_\nu}) \tag{4.104}$$

The covariant directional derivative ∇_λ of a constant scalar field is zero, $\nabla_\lambda \delta^\mu{}_\nu = 0$. We therefore get:

$$\begin{aligned}(\nabla_\lambda \underline{\omega}^\mu)(\vec{e_\nu}) &= -\underline{\omega}^\mu(\nabla_\lambda \vec{e_\nu}) \\ &= -\underline{\omega}^\mu(\Gamma^\alpha{}_{\nu\lambda}\vec{e_\alpha}) \\ &= -\Gamma^\alpha{}_{\nu\lambda}\underline{\omega}^\mu(\vec{e_\alpha}) \\ &= -\Gamma^\alpha{}_{\nu\lambda}\delta^\mu{}_\alpha \\ &= -\Gamma^\mu{}_{\nu\lambda}\end{aligned} \tag{4.105}$$

The contraction between a one-form and a basis vector gives the components of the one-form, $\underline{\alpha}(\vec{e_\nu}) = \alpha_\nu$. Equation (4.105) tells us that the ν-component of $\nabla_\lambda \underline{\omega}^\mu$ is equal to $-\Gamma^\mu{}_{\nu\lambda}$, and that we therefore have

$$\boxed{\nabla_\lambda \underline{\omega}^\mu = -\Gamma^\mu{}_{\nu\lambda}\underline{\omega}^\nu} \tag{4.106}$$

Equation (4.106) gives the directional derivatives of the basis forms. Using the product of differentiation gives

$$\begin{aligned}\nabla_\lambda \underline{\alpha} &= \nabla_\lambda(\alpha_\mu \underline{\omega}^\mu) \\ &= \nabla_\lambda(\alpha_\mu)\underline{\omega}^\mu + \alpha_\mu \nabla_\lambda \underline{\omega}^\mu \\ &= \vec{e_\lambda}(\alpha_\mu)\underline{\omega}^\mu - \alpha_\mu \Gamma^\mu{}_{\nu\lambda}\underline{\omega}^\nu\end{aligned} \tag{4.107}$$

Definition 4.8.3 (Covariant derivative of a one-form)
The covariant derivative of a one-form $\underline{\alpha} = \alpha_\mu \underline{\omega}^\mu$ is therefore given by Equation (4.108) below, when we let $\mu \to \nu$ in the first term on the right hand side in (4.107):

$$\boxed{\nabla_\lambda \underline{\alpha} = [\vec{e_\lambda}(\alpha_\nu) - \alpha_\mu \Gamma^\mu{}_{\nu\lambda}]\underline{\omega}^\nu} \tag{4.108}$$

The covariant derivative of the one-form components α_μ are denoted by $\alpha_{\nu;\lambda}$ and are defined by

$$\nabla_\lambda \underline{\alpha} \equiv \alpha_{\nu;\lambda}\underline{\omega}^\nu \tag{4.109}$$

It then follows that

$$\boxed{\alpha_{\nu;\lambda} = \vec{e_\lambda}(\alpha_\nu) - \alpha_\mu \Gamma^\mu{}_{\nu\lambda}} \tag{4.110}$$

It is worth to note that $\Gamma^\mu{}_{\nu\lambda}$ in Equation (4.110) are not Christoffel symbols. In coordinate basis we get:

$$\alpha_{\nu;\lambda} = \alpha_{\nu,\lambda} - \alpha_\nu \Gamma^\mu{}_{\lambda\nu} \tag{4.111}$$

where $\Gamma^\mu{}_{\lambda\nu} = \Gamma^\mu{}_{\nu\lambda}$ are Christoffel symbols.

4.8.3 Generalization for tensors of higher rank

Definition 4.8.4 (Covariant derivative of a tensor)
Let A and B be two tensors of arbitrary rank. The covariant directional derivative along a basis vector $\vec{e_\alpha}$ of a tensor $A \otimes B$ of arbitrary rank is defined by:

$$\nabla_\lambda (A \otimes B) \equiv (\nabla_\lambda A) \otimes B + A \otimes (\nabla_\lambda B) \tag{4.112}$$

We will use (4.112) to find the formula for the covariant derivative of the components of a tensor of rank 2:

$$\begin{aligned}\nabla_\alpha S &= \nabla_\alpha (S_{\mu\nu} \underline{\omega}^\mu \otimes \underline{\omega}^\nu) \\ &= (\nabla_\alpha S_{\mu\nu}) \underline{\omega}^\mu \otimes \underline{\omega}^\nu + S_{\mu\nu} (\nabla_\alpha \underline{\omega}^\mu) \otimes \underline{\omega}^\nu + S_{\mu\nu} \underline{\omega}^\mu \otimes (\nabla_\alpha \underline{\omega}^\nu) \\ &= (S_{\mu\nu,\alpha} - S_{\beta\nu} \Gamma^\beta_{\ \mu\alpha} - S_{\mu\beta} \Gamma^\beta_{\ \nu\alpha}) \underline{\omega}^\mu \otimes \underline{\omega}^\nu\end{aligned} \tag{4.113}$$

where $S_{\mu\nu,\alpha} = \vec{e}_\alpha (S_{\mu\nu})$. Defining the covariant derivative $S_{\mu\nu;\alpha}$ by

$$\nabla_\alpha S = S_{\mu\nu;\alpha} \underline{\omega}^\mu \otimes \underline{\omega}^\nu \tag{4.114}$$

we get

$$S_{\mu\nu;\alpha} = S_{\mu\nu,\alpha} - S_{\beta\nu} \Gamma^\beta_{\ \mu\alpha} - S_{\mu\beta} \Gamma^\beta_{\ \nu\alpha} \tag{4.115}$$

For the metric tensor we get

$$g_{\mu\nu;\alpha} = g_{\mu\nu,\alpha} - g_{\beta\nu} \Gamma^\beta_{\ \mu\alpha} - g_{\mu\beta} \Gamma^\beta_{\ \nu\alpha} \tag{4.116}$$

From

$$g_{\mu\nu} = \vec{e_\mu} \cdot \vec{e_\nu} \tag{4.117}$$

we get:

$$\begin{aligned}g_{\mu\nu,\alpha} &= (\nabla_\alpha \vec{e_\mu}) \cdot \vec{e_\nu} + \vec{e_\mu} (\nabla_\alpha \vec{e_\nu}) \\ &= \Gamma^\beta_{\ \mu\alpha} \vec{e_\beta} \cdot \vec{e_\nu} + \vec{e_\mu} \cdot \Gamma^\beta_{\ \nu\alpha} \vec{e_\beta} \\ &= g_{\beta\nu} \Gamma^\beta_{\ \mu\alpha} + g_{\mu\beta} \Gamma^\beta_{\ \nu\alpha}\end{aligned} \tag{4.118}$$

This means that

$$g_{\mu\nu;\alpha} = 0 \tag{4.119}$$

So the metric tensor is a (covariant) constant tensor.

4.9 The Cartan connection

Definition 4.9.1 (Exterior derivative of a basis vector)

$$\underline{d} \vec{e}_\mu \equiv \Gamma^\nu_{\ \mu\alpha} \vec{e}_\nu \otimes \underline{\omega}^\alpha \tag{4.120}$$

Exterior derivative of a vector field:

$$\underline{d}\vec{A} = \underline{d}(\vec{e}_\mu A^\mu) = \vec{e}_\nu \otimes \underline{d}A^\nu + A^\mu \underline{d}\vec{e}_\mu \tag{4.121}$$

In arbitrary basis:

$$\underline{d}A^\nu = \vec{e}_\lambda(A^\nu)\underline{\omega}^\lambda \tag{4.122}$$

(In coordinate basis, $\vec{e}_\lambda(A^\nu) = \frac{\partial}{\partial x^\lambda}(A^\nu) = A^\nu_{,\lambda}$)
giving:

$$\begin{aligned}\underline{d}\vec{A} &= \vec{e}_\nu \otimes [\vec{e}_\lambda(A^\nu)\underline{\omega}^\lambda] + A^\mu \Gamma^\nu{}_{\mu\lambda}\vec{e}_\nu \otimes \underline{\omega}^\lambda \\ &= (\vec{e}_\lambda(A^\nu) + A^\mu\Gamma^\nu{}_{\mu\lambda})\vec{e}_\nu \otimes \underline{\omega}^\lambda\end{aligned} \tag{4.123}$$

$$\boxed{\underline{d}\vec{A} = A^\nu_{;\lambda}\vec{e}_\nu \otimes \underline{\omega}^\lambda} \tag{4.124}$$

Definition 4.9.2 (Connection forms $\underline{\Omega}^\nu{}_\mu$)
The connection forms $\underline{\Omega}^\nu{}_\mu$ are 1-forms, defined by:

$$\begin{aligned}&\underline{d}\vec{e}_\mu \equiv \vec{e}_\nu \otimes \underline{\Omega}^\nu{}_\mu \\ &\Gamma^\nu{}_{\mu\alpha}\vec{e}_\nu \otimes \underline{\omega}^\alpha = \vec{e}_\nu \otimes \Gamma^\nu{}_{\mu\alpha}\underline{\omega}^\alpha = \vec{e}_\nu \otimes \underline{\Omega}^\nu{}_\mu\end{aligned} \tag{4.125}$$

$$\boxed{\underline{\Omega}^\nu{}_\mu = \Gamma^\nu{}_{\mu\alpha}\underline{\omega}^\alpha} \tag{4.126}$$

The exterior derivatives of the components of the metric tensor:

$$\underline{d}g_{\mu\nu} = \underline{d}(\vec{e}_\mu \cdot \vec{e}_\nu) = \vec{e}_\mu \cdot \underline{d}\vec{e}_\nu + \vec{e}_\nu \cdot \underline{d}\vec{e}_\mu \tag{4.127}$$

where the meaning of the dot is defined as follows:

Definition 4.9.3 (Scalar product between vector and 1-form)
The scalar product between a vector $\vec{u}$ and a (vectorial) one form $\underline{A} = A^\mu{}_\nu \vec{e}_\mu \otimes \underline{\omega}^\nu$ is defined by:

$$\vec{u} \cdot \underline{A} \equiv u^\alpha A^\mu{}_\nu(\vec{e}_\alpha \cdot \vec{e}_\mu)\underline{\omega}^\nu \tag{4.128}$$

Using this definition, we get:

$$\begin{aligned}\underline{d}g_{\mu\nu} &= (\vec{e}_\mu \cdot \vec{e}_\lambda)\underline{\Omega}^\lambda{}_\nu + (\vec{e}_\nu \cdot \vec{e}_\gamma)\underline{\Omega}^\gamma{}_\mu \\ &= g_{\mu\lambda}\underline{\Omega}^\lambda{}_\nu + g_{\nu\gamma}\underline{\Omega}^\gamma{}_\mu\end{aligned} \tag{4.129}$$

Lowering an index gives

$$\underline{d}g_{\mu\nu} = \underline{\Omega}_{\mu\nu} + \underline{\Omega}_{\nu\mu} \tag{4.130}$$

In an orthonormal basis field there is Minkowski-metric:

$$g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} \tag{4.131}$$

which is constant. Then we have :

$$\underline{d}g_{\hat{\mu}\hat{\nu}} = 0 \Rightarrow \boxed{\underline{\Omega}_{\hat{\nu}\hat{\mu}} = -\underline{\Omega}_{\hat{\mu}\hat{\nu}}} \tag{4.132}$$

where we write $\underline{\Omega}_{\hat{\nu}\hat{\mu}} = \Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}}\underline{\omega}^{\hat{\alpha}}$. It follows that $\Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}} = -\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}}$.
It also follows that

$$\begin{aligned} \Gamma^{\hat{t}}{}_{\hat{i}\hat{j}} &= -\Gamma_{\hat{t}\hat{i}\hat{j}} = \Gamma_{\hat{i}\hat{t}\hat{j}} = \Gamma^{\hat{i}}{}_{\hat{t}\hat{j}} \\ \Gamma^{\hat{i}}{}_{\hat{j}\hat{k}} &= -\Gamma^{\hat{j}}{}_{\hat{i}\hat{k}} \end{aligned} \tag{4.133}$$

Cartans 1st structure equation (without proof):

$$\begin{aligned} \underline{d\omega}^{\rho} &= -\frac{1}{2}c^{\rho}{}_{\mu\nu}\underline{\omega}^{\mu} \wedge \underline{\omega}^{\nu} \\ &= -\frac{1}{2}(\Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu})\underline{\omega}^{\mu} \wedge \underline{\omega}^{\nu} \\ &= -\Gamma^{\rho}{}_{\nu\mu}\underline{\omega}^{\mu} \wedge \underline{\omega}^{\nu} \\ &= -\underline{\Omega}^{\rho}{}_{\nu} \wedge \underline{\omega}^{\nu} \end{aligned} \tag{4.134}$$

$$\boxed{\underline{d\omega}^{\rho} = -\underline{\Omega}^{\rho}{}_{\nu} \wedge \underline{\omega}^{\nu}} \quad \text{and} \quad \boxed{\underline{d\omega}^{\rho} = \Gamma^{\rho}{}_{\mu\nu}\underline{\omega}^{\mu} \wedge \underline{\omega}^{\nu}} \tag{4.135}$$

In coordinate basis, we have $\underline{\omega}^{\rho} = \underline{d}x^{\rho}$.
Thus, $\underline{d\omega}^{\rho} = \underline{d}^2 x^{\rho} = 0$.
We also have $c^{\rho}{}_{\mu\nu} = 0$, and C1 is reduced to an identity.This formalism can not be used in coordinate basis!

`Example` **4.9.1 (Cartan-connection in an orthonormal basis field in plane polar coord.)**

$$ds^2 = dr^2 + r^2 d\theta^2$$

Introducing basis forms in an orthonormal basis field (where the metric is $g_{\hat{r}\hat{r}} = g_{\hat{\theta}\hat{\theta}} = 1$):

$$ds^2 = g_{\hat{r}\hat{r}}\underline{\omega}^{\hat{r}} \otimes \underline{\omega}^{\hat{r}} + g_{\hat{\theta}\hat{\theta}}\underline{\omega}^{\hat{\theta}} \otimes \underline{\omega}^{\hat{\theta}} = \underline{\omega}^{\hat{r}} \otimes \underline{\omega}^{\hat{r}} + \underline{\omega}^{\hat{\theta}} \otimes \underline{\omega}^{\hat{\theta}}$$

$$\Rightarrow \underline{\omega}^{\hat{r}} = \underline{d}r, \underline{\omega}^{\hat{\theta}} = r\underline{d}\theta$$

Exterior differentiation gives:

$$\underline{d\omega}^{\hat{r}} = \underline{d}^2 r = 0, \underline{d\omega}^{\hat{\theta}} = \underline{d}r \wedge \underline{d}\theta = \frac{1}{r}\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\theta}}$$

C1:

$$\underline{d\omega}^{\hat{\mu}} = -\underline{\Omega}^{\hat{\mu}}_{\ \hat{\nu}} \wedge \underline{\omega}^{\hat{\nu}}$$
$$= -\underline{\Omega}^{\hat{\mu}}_{\ \hat{r}} \wedge \underline{\omega}^{\hat{r}} - \underline{\Omega}^{\hat{\mu}}_{\ \hat{\theta}} \wedge \underline{\omega}^{\hat{\theta}}$$

We have that $\underline{d\omega}^{\hat{r}} = 0$, which gives:

$$\underline{\Omega}^{\hat{r}}_{\ \hat{\theta}} = \Gamma^{\hat{r}}_{\ \hat{\theta}\hat{\theta}}\underline{\omega}^{\hat{\theta}} \tag{4.136}$$

since $\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\theta}} = 0$. ($\underline{\Omega}^{\hat{r}}_{\ \hat{r}}$=0 because of the antisymmetry $\underline{\Omega}_{\hat{\nu}\hat{\mu}} = -\underline{\Omega}_{\hat{\mu}\hat{\nu}}$.)
We also have: $\underline{d\omega}^{\hat{\theta}} = -\frac{1}{r}\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{r}}$. C1:

$$\underline{d\omega}^{\hat{\theta}} = -\underline{\Omega}^{\hat{\theta}}_{\ \hat{r}} \wedge \underline{\omega}^{\hat{r}} - \underbrace{\underline{\Omega}^{\hat{\theta}}_{\ \hat{\theta}}}_{=0} \wedge \underline{\omega}^{\hat{\theta}}$$

$$\underline{\Omega}^{\hat{\theta}}_{\ \hat{r}} = \Gamma^{\hat{\theta}}_{\ \hat{r}\hat{\theta}}\underline{\omega}^{\hat{\theta}} + \Gamma^{\hat{\theta}}_{\ \hat{r}\hat{r}}\underline{\omega}^{\hat{r}} \tag{4.137}$$

giving $\Gamma^{\hat{\theta}}_{\ \hat{r}\hat{\theta}} = \frac{1}{r}$.
We have: $\underline{\Omega}^{\hat{r}}_{\ \hat{\theta}} = -\underline{\Omega}^{\hat{\theta}}_{\ \hat{r}}$. Using equations 4.136 and 4.137 we get:

$$\Gamma^{\hat{\theta}}_{\ \hat{r}\hat{r}} = 0$$

$$\Rightarrow \Gamma^{\hat{r}}_{\ \hat{\theta}\hat{\theta}} = -\frac{1}{r}$$

giving $\underline{\Omega}^{\hat{r}}_{\ \hat{\theta}} = -\underline{\Omega}^{\hat{\theta}}_{\ \hat{r}} = -\frac{1}{r}\underline{\omega}^{\hat{\theta}}$.

Chapter 5

Curvature

5.1 The Riemann curvature tensor

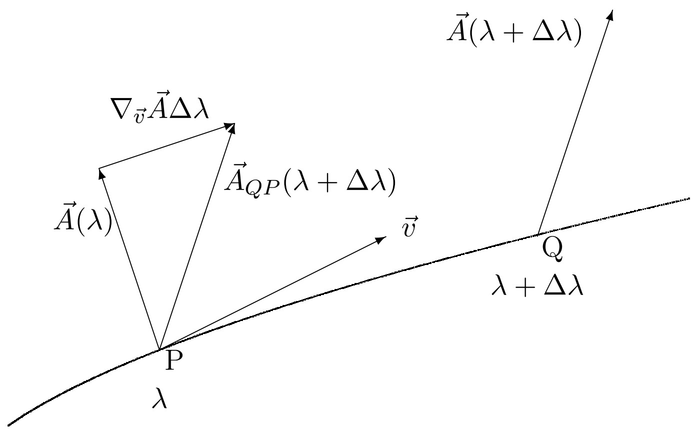


Figure 5.1: Parallel transport of $\vec{A}$

The covariant directional derivative of a vector field $\vec{A}$ along a vector $\vec{u}$ was defined and interpreted geometrically in section 4.2, as follows

$$\begin{aligned}\nabla_{\vec{v}}\vec{A} &= \frac{d\vec{A}}{d\lambda} = A^{\mu}{}_{;\nu}v^{\nu}\vec{e}_{\mu} \\ &= \lim_{\Delta\lambda\to 0}\frac{\vec{A}_{QP}(\lambda+\Delta\lambda)-\vec{A}(\lambda)}{\Delta\lambda}\end{aligned} \tag{5.1}$$

Let $\vec{A}_{QP}$ be the parallel transported of $\vec{A}$ from Q to P. Then to first order in $\Delta\lambda$ we have: $\vec{A}_{QP} = \vec{A}_P + (\nabla_{\vec{v}}\vec{A})_P\Delta\lambda$ and

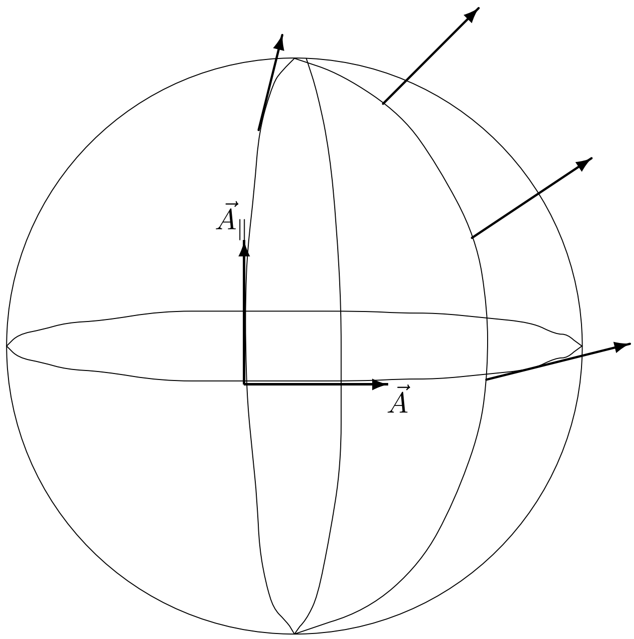


Figure 5.2: Parallel transport of a vector along a triangle of angles 90° is rotated 90°

$$\vec{A}_{PQ} = \vec{A}_Q - (\nabla_{\vec{v}}\vec{A})_Q \Delta\lambda \tag{5.2}$$

To second order in $\Delta\lambda$ we have:

$$\vec{A}_{PQ} = (1 - \nabla_{\vec{v}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2)\vec{A}_Q \tag{5.3}$$

If $\vec{A}_{PQ}$ is parallel transported further on to R we get

$$\begin{aligned}\vec{A}_{PQR} = (1 - \nabla_{\vec{u}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{u}}\nabla_{\vec{u}}(\Delta\lambda)^2) \\ \cdot (1 - \nabla_{\vec{v}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2)\vec{A}_R\end{aligned} \tag{5.4}$$

where $\vec{A}_Q$ is replaced by $\vec{A}_R$ because the differential operator always shall be applied to the vector in the first position. If we parallel transport $\vec{A}$ around the whole polygon in figure 5.3 we get:

$$\begin{aligned}\vec{A}_{PQRSTP} = (1 + \nabla_{\vec{u}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{u}}\nabla_{\vec{u}}(\Delta\lambda)^2) \\ \cdot (1 + \nabla_{\vec{v}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2) \\ \cdot (1 - \nabla_{[\vec{u},\vec{v}]}(\Delta\lambda)^2) \cdot (1 - \nabla_{\vec{u}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{u}}\nabla_{\vec{u}}(\Delta\lambda)^2) \\ \cdot (1 - \nabla_{\vec{v}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2)\vec{A}_P\end{aligned} \tag{5.5}$$

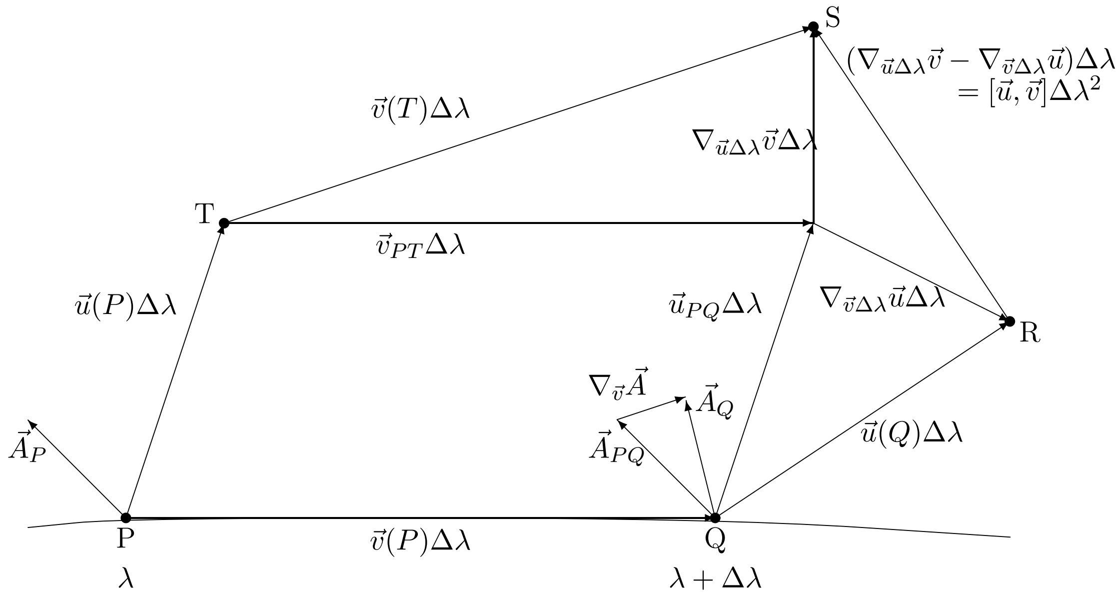


Figure 5.3: Geometrically implied curvature from non-zero differences between vectors along a curve (parameterized by λ) and their parallel transported equivalents

Calculating to 2. order in $\Delta\lambda$ gives:

$$\vec{A}_{PQRSTP} = \vec{A}_P + ([\nabla_{\vec{u}}, \nabla_{\vec{v}}] - \nabla_{[\vec{u},\vec{v}]})(\Delta\lambda)^2 \vec{A}_P \tag{5.6}$$

There is a variation of the vector under parallel transport around the closed polygon:

$$\delta\vec{A} = \vec{A}_{PQRSTP} - \vec{A}_P = ([\nabla_{\vec{u}}, \nabla_{\vec{v}}] - \nabla_{[\vec{u},\vec{v}]})\vec{A}_P(\Delta\lambda)^2 \tag{5.7}$$

We now introduce the Riemann's curvature tensor as:

$$R(\quad, \vec{A}, \vec{u}, \vec{v}) \equiv ([\nabla_{\vec{u}}, \nabla_{\vec{v}}] - \nabla_{[\vec{u},\vec{v}]})(\vec{A}) \tag{5.8}$$

The components of the Riemann curvature tensor is defined by applying the tensor on basis vectors,

$$R^{\mu}_{\ \nu\alpha\beta}\vec{e}_\mu \equiv ([\nabla_\alpha, \nabla_\beta] - \nabla_{[\vec{e}_\alpha,\vec{e}_\beta]})(\vec{e}_\nu) \tag{5.9}$$

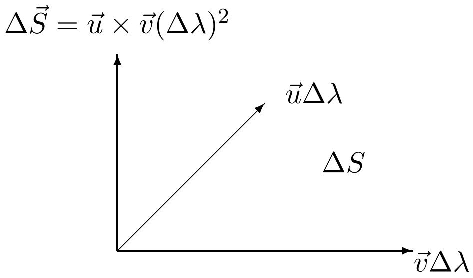
Anti-symmetry follows from the definition:

$$R^{\mu}_{\ \nu\beta\alpha} = -R^{\mu}_{\ \nu\alpha\beta} \tag{5.10}$$

The expression for the variation of $\vec{A}$ under parallel transport around the polygon, Eq. (5.7), can now be written as:

$$
\begin{aligned}
\delta\vec{A} &= R(\quad, \vec{A}, \vec{u}, \vec{v})(\Delta\lambda)^2 \\
&= R(\quad, A^\nu \vec{e}_\nu, u^\alpha \vec{e}_\alpha, v^\beta \vec{e}_\beta)(\Delta\lambda)^2 \\
&= \vec{e}_\mu R^\mu{}_{\nu\alpha\beta} A^\nu u^\alpha v^\beta \cdot (\Delta\lambda)^2 \\
&= \frac{1}{2}\vec{e}_\mu R^\mu{}_{\nu\alpha\beta} A^\nu (u^\alpha v^\beta - u^\beta v^\alpha)(\Delta\lambda)^2
\end{aligned}
\tag{5.11}
$$

The area of the parallellogram defined by the vectors $\vec{u}\Delta\lambda$ and $\vec{v}\Delta\lambda$ is



$$\Delta\vec{S} = \vec{u} \times \vec{v}(\Delta\lambda)^2 .$$

Using that

$$(\vec{u} \times \vec{v})^{\alpha\beta} = u^\alpha v^\beta - u^\beta v^\alpha \; .$$

we can write Eq. (5.11) as:

$$\boxed{\delta\vec{A} = \frac{1}{2} A^\nu R^\mu{}_{\nu\alpha\beta} \Delta S^{\alpha\beta} \vec{e}_\mu} \; . \tag{5.12}$$

The components of the Riemann tensor expressed by the connection- and structure-coefficients are given below:

$$
\begin{aligned}
\vec{e}_\mu R^\mu{}_{\nu\alpha\beta} &= [\nabla_\alpha, \nabla_\beta]\vec{e}_\nu - \nabla_{[\vec{e}_\alpha, \vec{e}_\beta]}\vec{e}_\nu \\
&= (\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha - c^\rho{}_{\alpha\beta}\nabla_\rho)\vec{e}_\nu \\
&= \nabla_\alpha\nabla_\beta\vec{e}_\nu - \nabla_\beta\nabla_\alpha\vec{e}_\nu - c^\rho{}_{\alpha\beta}\nabla_\rho\vec{e}_\nu \\
\text{(Kozul-connection)} \quad &= \nabla_\alpha\Gamma^\mu{}_{\nu\beta}\vec{e}_\mu - \nabla_\beta\Gamma^\mu{}_{\nu\alpha}\vec{e}_\mu - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho}\vec{e}_\mu \\
&= (\nabla_\alpha\Gamma^\mu{}_{\nu\beta})\vec{e}_\mu + \Gamma^\mu{}_{\nu\beta}\nabla_\alpha\vec{e}_\mu \\
&\quad - (\nabla_\beta\Gamma^\mu{}_{\nu\alpha})\vec{e}_\mu - \Gamma^\mu{}_{\nu\alpha}\nabla_\beta\vec{e}_\mu - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho}\vec{e}_\mu \\
&= \vec{e}_\alpha(\Gamma^\mu{}_{\nu\beta})\vec{e}_\mu + \Gamma^\rho{}_{\nu\beta}\Gamma^\mu{}_{\rho\alpha}\vec{e}_\mu \\
&\quad - \vec{e}_\beta(\Gamma^\mu{}_{\nu\alpha})\vec{e}_\mu - \Gamma^\rho{}_{\nu\alpha}\Gamma^\mu{}_{\rho\beta}\vec{e}_\mu - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho}\vec{e}_\mu \; .
\end{aligned}
\tag{5.13}
$$

This gives (in arbitrary basis):

$$
\begin{aligned}
R^\mu{}_{\nu\alpha\beta} &= \vec{e}_\alpha(\Gamma^\mu{}_{\nu\beta}) - \vec{e}_\beta(\Gamma^\mu{}_{\nu\alpha}) \\
&\quad + \Gamma^\rho{}_{\nu\beta}\Gamma^\mu{}_{\rho\alpha} - \Gamma^\rho{}_{\nu\alpha}\Gamma^\mu{}_{\rho\beta} - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho} \; .
\end{aligned}
\tag{5.14}
$$

In coordinate basis eq. (5.14) is reduced to:

$$R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}{}_{\nu\beta,\alpha} - \Gamma^{\mu}{}_{\nu\alpha,\beta} + \Gamma^{\rho}{}_{\nu\beta}\Gamma^{\mu}{}_{\rho\alpha} - \Gamma^{\rho}{}_{\nu\alpha}\Gamma^{\mu}{}_{\rho\beta} \ , \tag{5.15}$$

where $\Gamma^{\mu}{}_{\nu\beta} = \Gamma^{\mu}{}_{\beta\nu}$ are the Christoffel symbols.

Due to the antisymmetry (5.10) we can define a matrix of *curvature-forms*

$$\underline{R}^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\alpha\beta}\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} \tag{5.16}$$

Inserting the components of the Riemann tensor from eq. (5.14) gives

$$\underline{R}^{\mu}{}_{\nu} = (\vec{e}_{\alpha}(\Gamma^{\mu}{}_{\nu\beta}) + \Gamma^{\rho}{}_{\nu\beta}\Gamma^{\mu}{}_{\rho\alpha} - \frac{1}{2}c^{\rho}{}_{\alpha\beta}\Gamma^{\mu}{}_{\nu\rho})\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} \tag{5.17}$$

The connection forms:

$$\underline{\Omega}^{\mu}{}_{\nu} = \Gamma^{\mu}{}_{\nu\alpha}\underline{\omega}^{\alpha} \tag{5.18}$$

Exterior derivatives of basis forms:

$$\underline{d\omega}^{\rho} = -\frac{1}{2}c^{\rho}{}_{\alpha\beta}\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} \tag{5.19}$$

Exterior derivatives of connection forms (C1: $\underline{d\omega}^{\rho} = -\underline{\Omega}^{\rho}{}_{\alpha} \wedge \underline{\omega}^{\alpha}$) :

$$\begin{aligned} \underline{d\Omega}^{\mu}{}_{\nu} &= \underline{d}\Gamma^{\mu}{}_{\nu\beta} \wedge \underline{\omega}^{\beta} + \Gamma^{\mu}{}_{\nu\rho}\underline{d\omega}^{\rho} \\ &= \vec{e}_{\alpha}(\Gamma^{\mu}{}_{\nu\beta})\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} - \frac{1}{2}c^{\rho}{}_{\alpha\beta}\Gamma^{\mu}{}_{\nu\rho}\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} \end{aligned} \tag{5.20}$$

The curvature forms can now be written as:

$$\boxed{\underline{R}^{\mu}{}_{\nu} = \underline{d\Omega}^{\mu}{}_{\nu} + \underline{\Omega}^{\mu}{}_{\lambda} \wedge \underline{\Omega}^{\lambda}{}_{\nu}} \tag{5.21}$$

This is Cartans 2nd structure equation.

5.2 Differential geometry of surfaces

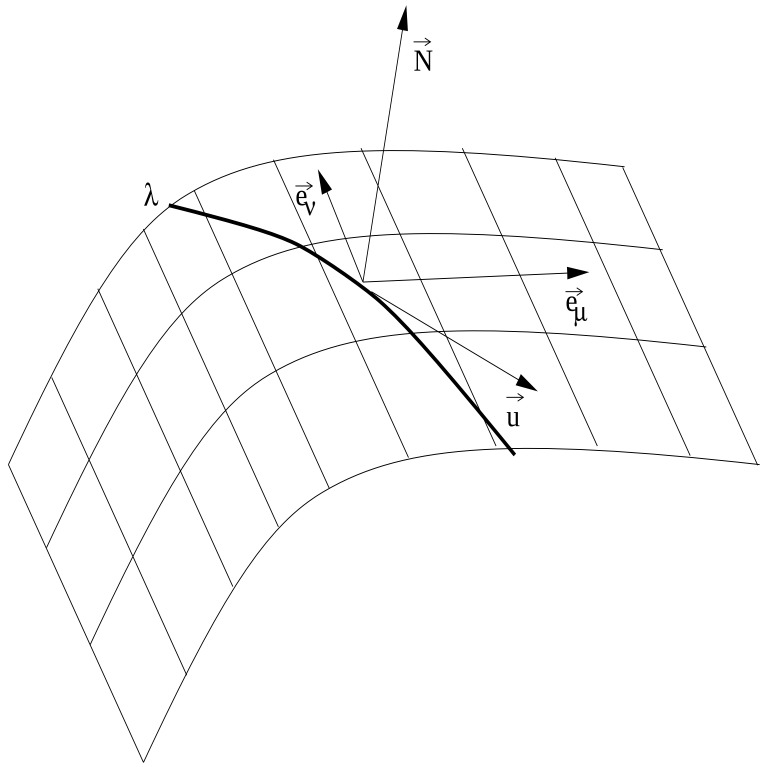


Figure 5.4: The geometry of a surface. We see the normal vector and the unit vectors of the tangent plane of a point on the surface.

Imagine an arbitrary surface embedded in an Euclidian 3 dimensional space. (See figure 5.4). Coordinate vectors on the surface :

$$\vec{e}_u = \frac{\partial}{\partial u}, \vec{e}_v = \frac{\partial}{\partial v} \tag{5.22}$$

where u and v are coordinates on the surface.
Line element on the surface:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{5.23}$$

with $x^1 = u$ and $x^2 = v$.
(1st fundamental form)
The directional derivatives of the basis vectors are written

$$\vec{e}_{\mu,\nu} = \Gamma^\alpha_{\mu\nu} \vec{e}_\alpha + K_{\mu\nu} \vec{N}, \alpha = 1, 2 \tag{5.24}$$

Greek indices run through the surface coordinates, $\vec{N}$ is a unit vector orthogonal to the surface.

The equation above is called Gauss' equation. We have: $K_{\mu\nu} = \vec{e}_{\mu,\nu} \cdot \vec{N}$. In coordinate basis, we have $\vec{e}_{\mu,\nu} = \frac{\partial^2}{\partial x^\mu \partial x^\nu} = \frac{\partial^2}{\partial x^\nu \partial x^\mu} = \vec{e}_{\nu,\mu}$. It follows that

$$K_{\mu\nu} = K_{\nu\mu} \tag{5.25}$$

Let $\vec{u}$ be the unit tangent vector to a curve on the surface, parametrised by λ. Differentiating $\vec{u}$ along the curve:

$$\frac{d\vec{u}}{d\lambda} = u^\mu{}_{;\nu} u^\nu \vec{e}_\mu + \underbrace{K_{\mu\nu} u^\mu u^\nu}_{\text{2nd fundamental form}} \vec{N} \tag{5.26}$$

We define κ_g and κ_N by:

$$\frac{d\vec{u}}{d\lambda} = \kappa_g \vec{e} + \kappa_N \vec{N} \tag{5.27}$$

κ_g is called <u>geodesic curvature</u>. κ_N is called <u>normal curvature</u> (external curvature). $\kappa_g = 0$ for geodesic curves on the surface.

$$\begin{aligned} \kappa_g \vec{e} &= u^\mu{}_{;\nu} u^\nu \vec{e}_\mu = \nabla_{\vec{u}} \vec{u} \\ \kappa_N &= K_{\mu\nu} u^\mu u^\nu \\ \text{And :}\kappa_N &= \frac{d\vec{u}}{d\lambda} \cdot \vec{N} \end{aligned} \tag{5.28}$$

We also have that $\vec{u} \cdot \vec{N} = 0$ along the whole curve. Differentiation gives:

$$\frac{d\vec{u}}{d\lambda} \cdot \vec{N} + \vec{u} \cdot \frac{d\vec{N}}{d\lambda} = 0 \tag{5.29}$$

gives:

$$\kappa_N = -\vec{u} \cdot \frac{d\vec{N}}{d\lambda} \tag{5.30}$$

which is called Weingarten's equation.

κ_g and κ_N together give a complete description of the geometry of a surface in a flat 3 dimensional space. We are now going to consider geodesic curves through a point on the surface. Tangent vector $\vec{u} = u^\mu \vec{e}_\mu$ with $\vec{u} \cdot \vec{u} = g_{\mu\nu} u^\mu u^\nu = 1$. Directions with maximum and minimum values for the normal curvatures are found, by extremalizing κ_N under the condition $g_{\mu\nu} u^\mu u^\nu = 1$. We then solve the variation problem $\delta F = 0$ for arbitrary u^μ, where $F = K_{\mu\nu} u^\mu u^\nu - k(g_{\mu\nu} u^\mu u^\nu - 1)$. Here k is the Lagrange multiplicator. Variation with respect to u^μ gives:

$$\begin{aligned} &\delta F = 2(K_{\mu\nu} - k g_{\mu\nu}) u^\nu \delta u^\mu \\ &\delta F = 0 \text{ for arbitrary } \delta u^\mu \text{ demands:} \end{aligned}$$

$$(K_{\mu\nu} - k g_{\mu\nu}) u^\nu = 0 \tag{5.31}$$

For this system of equations to have nonzero solutions, we must have:

$$\det(K_{\mu\nu} - k g_{\mu\nu}) = 0 \tag{5.32}$$

$$\begin{vmatrix} K_{11} - kg_{11} & K_{12} - kg_{12} \\ K_{21} - kg_{21} & K_{22} - kg_{22} \end{vmatrix} = 0 \tag{5.33}$$

This gives the following quadratic equation for k:

$$k^2 \det(g_{\mu\nu}) - (g_{11}K_{22} - 2g_{12}K_{12} + g_{22}K_{11})k + \det(K_{\mu\nu}) = 0 \tag{5.34}$$
$$(K \text{ symmetric } K_{12} = K_{21})$$

The equation has two solutions, k_1 and k_2. These are the extremal values of k. To find the meaning of k, we multiply eq.5.31 by u^μ:

$$\begin{aligned} 0 &= (K_{\mu\nu} - kg_{\mu\nu})u^\mu u^\nu \\ &= K_{\mu\nu}u^\mu u^\nu - kg_{\mu\nu}u^\mu u^\nu \\ &= \kappa_N - k \Rightarrow k = \kappa_N \end{aligned} \tag{5.35}$$

The extremal values of κ_N are called the principal curvatures of the surface. Let the directions of the geodesics with extreme normal curvature be given by the tangent vectors $\vec{u}$ and $\vec{v}$.Eq.5.31 gives:

$$K_{\mu\nu}u^\nu = kg_{\mu\nu}u^\nu \tag{5.36}$$

We then get:

$$\begin{aligned} K_{\mu\nu}u^\nu v^\mu &= k_1 g_{\mu\nu}u^\nu v^\mu \\ &= k_1 u_\mu v^\mu = k_1(\vec{u} \cdot \vec{v}) \\ K_{\mu\nu}v^\nu u^\mu &= k_2 g_{\mu\nu}v^\nu u^\mu = k_2(\vec{u} \cdot \vec{v}) \end{aligned}$$

$$\begin{aligned} \text{gives } (k_1 - k_2)(\vec{u} \cdot \vec{v}) &= K_{\mu\nu}(u^\nu v^\mu - v^\nu u^\mu) \\ &= 2K_{\mu\nu}u^{[\nu}v^{\mu]} \end{aligned} \tag{5.37}$$

$K_{\mu\nu}$ is symmetric in μ and ν. So we get:$(k_1 - k_2)(\vec{u} \cdot \vec{v}) = 0$. For $k_1 \neq k_2$ we have to demand $\vec{u} \cdot \vec{v} = 0$. So the geodesics with extremal normal curvature, are orthogonal to each other.

The Gaussian curvature (at a point) is defined as:

$$\boxed{K = \kappa_{N1} \cdot \kappa_{N2}} \tag{5.38}$$

Since κ_{N1} and κ_{N2} are solutions of the quadratic equation above, we get:

$$K = \frac{\det(K_{\mu\nu})}{\det(g_{\mu\nu})} \tag{5.39}$$

5.2.1 Surface curvature, using the Cartan formalism

In each point on the surface we have an orthonormal set of basis vectors. Greek indices run through the surface coordinates (two dimensional) and Latin indices through the space coordinates (three dimensional):

$$\vec{e_a} = (\vec{e_1}, \vec{e_2}, \vec{N}) \,, \quad \vec{e_\mu} = \{\vec{e_1}, \vec{e_2}\} \tag{5.40}$$

Using the exterior derivative and form formalism, we find how the unit vectors on the surface change:

$$\begin{aligned} \underline{d}\vec{e_\nu} &= \vec{e_a} \otimes \underline{\Omega}^a{}_\nu \\ &= \vec{e_\alpha} \otimes \underline{\Omega}^\alpha{}_\nu + \vec{N} \otimes \underline{\Omega}^3{}_\nu \,, \end{aligned} \tag{5.41}$$

where $\underline{\Omega}^\mu{}_\nu = \Gamma^\mu{}_{\nu\alpha}\underline{\omega}^\alpha$ are the connection forms on the surface, i.e. the intrinsic connection forms. The extrinsic connection forms are

$$\underline{\Omega}^3{}_\nu = K_{\nu\alpha}\underline{\omega}^\alpha \,, \underline{\Omega}^\mu{}_3 = K^\mu{}_\alpha\underline{\omega}^\alpha \tag{5.42}$$

We let the surface be embedded in an Euclidean (flat) 3-dimensional space. This means that the curvature forms of the 3-dimensional space are zero:

$$\underline{R}^a{}_{3b} = 0 = \underline{d}\,\underline{\Omega}^a{}_b + \underline{\Omega}^a{}_k \wedge \underline{\Omega}^k{}_b \tag{5.43}$$

which gives:

$$\begin{aligned} R^\mu{}_{3\nu} &= 0 = \underline{d}\,\underline{\Omega}^\mu{}_\nu + \underline{\Omega}^\mu{}_\alpha \wedge \underline{\Omega}^\alpha{}_\nu + \underline{\Omega}^\mu{}_3 \wedge \underline{\Omega}^3{}_\nu \\ &= \underline{R}^\mu{}_\nu + \underline{\Omega}^\mu{}_3 \wedge \underline{\Omega}^3{}_\nu \,, \end{aligned} \tag{5.44}$$

where $\underline{R}^\mu{}_\nu$ are the **curvature forms of the surface**. We then have:

$$\frac{1}{2}R^\mu{}_{\nu\alpha\beta}\underline{\omega}^\alpha \wedge \underline{\omega}^\beta = -\underline{\Omega}^\mu{}_3 \wedge \underline{\Omega}^3{}_\nu \tag{5.45}$$

Inserting the components of the extrinsic connection forms, we get: (when we remember the anti symmetry of α and β in $\underline{R}^\mu{}_{\nu\alpha\beta}$)

$$R^\mu{}_{\nu\alpha\beta} = K^\mu{}_\alpha K_{\nu\beta} - K^\mu{}_\beta K_{\nu\alpha} \tag{5.46}$$

We now lower the first index:

$$R_{\mu\nu\alpha\beta} = K_{\mu\alpha}K_{\nu\beta} - K_{\mu\beta}K_{\nu\alpha} \tag{5.47}$$

$R_{\mu\nu\alpha\beta}$ are the components of a curvature tensor which **only** refer to the dimensions of the surface. In particular we have:

$$R_{1212} = K_{11}K_{22} - K_{12}K_{21} = \det K \tag{5.48}$$

We then have the following connection between this component of the Riemann curvature tensor of the surface and the Gaussian curvature of the surface:

$$K = \kappa_{N1} \cdot \kappa_{N2} = \frac{\det K_{\mu\nu}}{\det g_{\mu\nu}} = \frac{R_{1212}}{\det g_{\mu\nu}} \tag{5.49}$$

Since the right hand side refers to the intrinsic curvature and the metric on the surface, we have proved that the Gaussian curvature of a surface is an intrinsic quantity. It can be measured by observers on the surface without embedding the surface in a three-dimensional space. This is the contents of Gauss' **theorema egregium**.

5.3 The Ricci identity

$$\vec{e_\mu} R^\mu{}_{\nu\alpha\beta} A^\nu = (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha - \nabla_{[\vec{e_\alpha}, \vec{e_\beta}]})(\vec{A}) \tag{5.50}$$

In coordinate basis this is reduced to

$$\vec{e_\mu} R^\mu{}_{\nu\alpha\beta} A^\nu = (A^\mu{}_{;\beta\alpha} - A^\mu{}_{;\alpha\beta})\vec{e_\mu}\ , \tag{5.51}$$

where

$$A^\mu{}_{;\alpha\beta} \equiv (A^\mu{}_{;\beta})_{;\alpha} \tag{5.52}$$

The **Ricci identity** on component form is:

$$A^\nu R^\mu{}_{\nu\alpha\beta} = A^\mu{}_{;\beta\alpha} - A^\mu{}_{;\alpha\beta} \tag{5.53}$$

We can write this as:

$$\underline{d}^2 \vec{A} = \frac{1}{2} R^\mu{}_{\nu\alpha\beta} A^\nu \vec{e_\mu} \otimes \underline{\omega}^\alpha \wedge \underline{\omega}^\beta \tag{5.54}$$

This shows us that the 2nd exterior derivative of a vector is equal to zero only in a *flat* space. Equations (5.53) and (5.54) *both* represents the Ricci identity.

5.4 Bianchi's 1st identity

Cartan's 1st structure equation:

$$\underline{d}\,\underline{\omega}^\mu = -\underline{\Omega}^\mu{}_\nu \wedge \underline{\omega}^\nu \tag{5.55}$$

Cartan's 2nd structure equation:

$$\underline{R}^\mu{}_\nu = \underline{d}\,\underline{\Omega}^\mu{}_\nu + \underline{\Omega}^\mu{}_\lambda \wedge \underline{\Omega}^\lambda{}_\nu \tag{5.56}$$

Exterior differentiation of (5.55) and use of $Poincaré's$ lemma (4.16) gives: ($\underline{d}^2\,\underline{\omega}^\mu = 0$)

$$0 = \underline{d}\,\underline{\Omega}^\mu{}_\nu \wedge \underline{\omega}^\nu - \underline{\Omega}^\mu{}_\lambda \wedge \underline{d}\,\underline{\omega}^\lambda \tag{5.57}$$

Use of (5.55) gives:

$$\underline{d}\,\underline{\Omega}^\mu{}_\nu \wedge \underline{\omega}^\nu + \underline{\Omega}^\mu{}_\lambda \wedge \underline{\Omega}^\lambda{}_\nu \wedge \underline{\omega}^\nu = 0 \tag{5.58}$$

From this we see that

$$(\underline{d}\,\underline{\Omega}^\mu{}_\nu + \underline{\Omega}^\mu{}_\lambda \wedge \underline{\Omega}^\lambda{}_\nu) \wedge \underline{\omega}^\nu = 0 \tag{5.59}$$

We now get **Bianchi's 1st identity**:

$$\boxed{\underline{R}^\mu{}_\nu \wedge \underline{\omega}^\nu = 0} \tag{5.60}$$

On component form Bianchi's 1st identity is

$$\underbrace{\frac{1}{2}R^{\mu}{}_{\nu\alpha\beta}\underline{\omega}^{\alpha}\wedge\underline{\omega}^{\beta}}_{R^{\mu}{}_{\nu}}\wedge\underline{\omega}^{\nu}=0 \tag{5.61}$$

The component equation is: (remember the anti symmetry in α and β)

$$R^{\mu}{}_{[\nu\alpha\beta]}=0 \tag{5.62}$$

or

$$R^{\mu}{}_{\nu\alpha\beta}+R^{\mu}{}_{\alpha\beta\nu}+R^{\mu}{}_{\beta\nu\alpha}+=0 \tag{5.63}$$

where the anti symmetry $R^{\mu}{}_{\nu\alpha\beta}=-R^{\mu}{}_{\nu\beta\alpha}$ has been used. Without this anti symmetry we would have gotten six, and not three, terms in this equation.

5.5 Bianchi's 2nd identity

Exterior differentiation of (5.56) $\Rightarrow$

$$\begin{aligned}\underline{d}\,\underline{R}^{\mu}{}_{\nu}&=\underline{R}^{\mu}{}_{\lambda}\wedge\underline{\Omega}^{\lambda}{}_{\nu}-\underline{\Omega}^{\mu}{}_{\rho}\wedge\underline{\Omega}^{\rho}{}_{\lambda}\wedge\underline{\Omega}^{\lambda}{}_{\nu}-\underline{\Omega}^{\mu}{}_{\lambda}\wedge\underline{R}^{\lambda}{}_{\nu}+\underline{\Omega}^{\mu}{}_{\lambda}\wedge\underline{\Omega}^{\lambda}{}_{\rho}\wedge\underline{\Omega}^{\rho}{}_{\nu}\\&=\underline{R}^{\mu}{}_{\lambda}\wedge\underline{\Omega}^{\lambda}{}_{\nu}-\underline{\Omega}^{\mu}{}_{\lambda}\wedge\underline{R}^{\lambda}{}_{\nu}\end{aligned} \tag{5.64}$$

We now have **Bianchi's 2nd identity** as a form equation:

$$\boxed{\underline{d}\,\underline{R}^{\mu}{}_{\nu}+\underline{\Omega}^{\mu}{}_{\lambda}\wedge\underline{R}^{\lambda}{}_{\nu}-\underline{R}^{\mu}{}_{\lambda}\wedge\underline{\Omega}^{\lambda}{}_{\nu}=0} \tag{5.65}$$

As a component equation Bianchi's 2nd identity is given by

$$R^{\mu}{}_{\nu[\alpha\beta;\gamma]}=0 \tag{5.66}$$

Definition 5.5.1 (Contraction)
'Contraction' is a tensor operation defined by

$$R_{\nu\beta}\equiv R^{\mu}{}_{\nu\mu\beta} \tag{5.67}$$

We must here have summation over μ. What we do, then, is constructing a new tensor from another given tensor, with a rank 2 lower than the given one.

The tensor with components $R_{\nu\beta}$ is called **the Ricci curvature tensor**. Another contraction gives **the Ricci curvature scalar**, $R=R^{\mu}{}_{\mu}$.

Riemann curvature tensor has four symmetries. The definition of the Riemann tensor implies that $R^{\mu}_{\ \nu\alpha\beta} = -R^{\mu}_{\ \nu\beta\alpha}$
Bianchi's 1st identity: $R^{\mu}_{\ [\nu\alpha\beta]} = 0$

From Cartan's 2nd structure equation follows

$$\begin{aligned} \underline{R}_{\ \mu\nu} &= \underline{d\Omega}_{\ \mu\nu} + \underline{\Omega}_{\mu\lambda} \wedge \underline{\Omega}^{\lambda}_{\ \nu} \\ \Rightarrow R_{\ \mu\nu\alpha\beta} &= -R_{\ \nu\mu\alpha\beta} \end{aligned} \tag{5.68}$$

By choosing a locally Cartesian coordinate system in an inertial frame we get the following expression for the components of the Riemann curvature tensor:

$$R_{\ \mu\nu\alpha\beta} = \frac{1}{2}(g_{\mu\beta,\nu\alpha} - g_{\mu\alpha,\nu\beta} + g_{\nu\alpha,\mu\beta} - g_{\nu\beta,\mu\alpha}) \tag{5.69}$$

from which it follows that $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$. Contraction of μ and α leads to:

$$\begin{aligned} R^{\alpha}_{\ \nu\alpha\beta} &= R^{\alpha}_{\ \beta\alpha\nu} \\ \Rightarrow R_{\nu\beta} &= R_{\beta\nu} \end{aligned} \tag{5.70}$$

i.e. the Ricci tensor is symmetric. In 4-D the Ricci tensor has 10 independent components.

Chapter 6

Einstein's Field Equations

6.1 Energy-momentum conservation

6.1.1 Newtonian fluid

Energy-momentum conservation for a Newtonian fluid in terms of the divergence of the energy momentum tensor can be shown as follows. The total derivative of a velocity field is

$$\frac{D\vec{v}}{Dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \tag{6.1}$$

$\frac{\partial \vec{v}}{\partial t}$ is the local derivative which gives the change in $\vec{v}$ as a function of time at a given point in space. $(\vec{v} \cdot \vec{\nabla})\vec{v}$ is called the **convective** derivative of $\vec{v}$. It represents the change of $\vec{v}$ for a moving fluid particle due to the inhomogeneity of the fluid velocity field. In component notation the above become

$$\frac{Dv^i}{Dt} \equiv \frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} \tag{6.2}$$

The continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v^i)}{\partial x^i} = 0 \tag{6.3}$$

Euler's equation of motion (ignoring gravity)

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p \quad \text{or} \quad \rho \left(\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} \right) = -\frac{\partial p}{\partial x^i} \tag{6.4}$$

The **energy momentum tensor** is a symmetric tensor of rank 2 that describes material characteristics.

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} \tag{6.5}$$

$c \equiv 1$

T^{00} represents energy density.

T^{i0} represents momentum density.

T^{ii} represents pressure ($T^{ii} > 0$).

T^{ii} represents stress ($T^{ii} < 0$).

T^{ij} represents shear forces ($i \neq j$).

`Example` **6.1.1 (Energy momentum tensor for a Newtonian fluid)**

$$\begin{aligned} T^{00} &= \rho \qquad T^{i0} = \rho v^i \\ T^{ij} &= \rho v^i v^j + p\delta^{ij} \end{aligned} \tag{6.6}$$

where p is pressure, assumed isotropic here. We choose a locally Cartesian coordinate system in an inertial frame such that the covariant derivatives are reduced to partial derivatives. The divergence of the momentum energy tensor, $T^{\mu\nu}_{;\nu}$ has 4 components, one for each value of μ.

The zeroth component is

$$\begin{aligned} T^{0\nu}_{;\nu} &= T^{0\nu}_{,\nu} = T^{00}_{,0} + T^{0i}_{,i} \\ &= \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i} \end{aligned} \tag{6.7}$$

which by comparison to Newtonian hydrodynamics implies that $T^{0\nu}_{;\nu} = 0$ is the continuity equation. This equation represents the conservation of energy.

The ith component of the divergence is

$$\begin{aligned} T^{i\nu}_{,\nu} &= T^{i0}_{,0} + T^{ij}_{,j} \\ &= \frac{\partial(\rho v^i)}{\partial t} + \frac{\partial(\rho v^i v^j + p\delta^{ij})}{\partial x^j} \\ &= \rho\frac{\partial v^i}{\partial t} + v^i\frac{\partial \rho}{\partial t} + v^i\frac{\partial \rho v^j}{\partial x^j} + \rho v^j\frac{\partial v^i}{\partial x^j} + \frac{\partial p}{\partial x^i} \end{aligned} \tag{6.8}$$

now, according to the continuity equation

$$\begin{aligned} \frac{\partial(\rho v^i)}{\partial x^i} &= -\frac{\partial \rho}{\partial t} \\ \Rightarrow \quad T^{i\nu}_{,\nu} &= \rho\frac{\partial v^i}{\partial t} + v^i\frac{\partial \rho}{\partial t} - v^i\frac{\partial \rho}{\partial t} + \rho v^j\frac{\partial v^i}{\partial x^j} + \frac{\partial p}{\partial x^i} \\ &= \rho\frac{Dv^i}{Dt} + \frac{\partial p}{\partial x^i} \\ \therefore \quad T^{i\nu}_{;\nu} &= 0 \Rightarrow \rho\frac{Dv^i}{Dt} = -\frac{\partial p}{\partial x^i} \end{aligned} \tag{6.9}$$

which is Euler's equation of motion. It expresses the conservation of momentum.

The equations $T^{\mu\nu}_{;\nu} = 0$ are general expressions for energy and momentum conservation.

6.1.2 Perfect fluids

A perfect fluid is a fluid with no viscosity and is given by the energy-momentum tensor

$$T_{\mu\nu} = (\rho + \frac{p}{c^2})u_\mu u_\nu + pg_{\mu\nu} \tag{6.10}$$

where ρ and p are the mass density and the stress, respectively, measured in the fluids rest frame, u_μ are the components of the 4-velocity of the fluid.

In a comoving orthonormal basis the components of the 4-velocity are $u^{\hat{\mu}} = (c, 0, 0, 0)$. Then the energy-momentum tensor is given by

$$T_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{6.11}$$

where $p > 0$ is pressure and $p < 0$ is tension.

There are three different types of perfect fluids that are useful.

1. **dust** or non-relativistic gas is given by $p = 0$ and the energy-momemtum tensor $T_{\mu\nu} = \rho u_\mu u_\nu$.

2. **radiation** or ultra-relativistic gas is given by a traceless energy-momemtum tensor, i.e. $T^\mu{}_\mu = 0$. It follows that $p = \frac{1}{3}\rho c^2$.

3. **vacuum energy**: If we assume that no velocity can be measured relatively to vacuum, then all the components of the energy-momentum tensor must be Lorentz-invariant. It follows that $T_{\mu\nu} \propto g_{\mu\nu}$. If vacuum is defined as a perfect fluid we get $p = -\rho c^2$ so that $T_{\mu\nu} = pg_{\mu\nu} = -\rho c^2 g_{\mu\nu}$.

6.2 Einstein's curvature tensor

The field equations are assumed to have the form:

space-time curvature $\propto$ momentum-energy tensor

Also, it is demanded that energy and momentum conservation should follow as a consequence of the field equation. This puts the following constraints on the curvature tensor: It must be a symmetric, divergence free tensor of rank 2.

Bianchi's 2nd identity:

$$R^\mu{}_{\nu\alpha\beta;\sigma} + R^\mu{}_{\nu\sigma\alpha;\beta} + R^\mu{}_{\nu\beta\sigma;\alpha} = 0 \tag{6.12}$$

contraction of μ and $\alpha \Rightarrow$

$$\begin{aligned} R^\mu{}_{\nu\mu\beta;\sigma} - R^\mu{}_{\nu\mu\sigma;\beta} + R^\mu{}_{\nu\beta\sigma;\mu} &= 0 \\ R_{\ \nu\beta;\sigma} - R_{\ \nu\sigma;\beta} + R^\mu{}_{\nu\beta\sigma;\mu} &= 0 \end{aligned} \tag{6.13}$$

further contraction of ν and σ gives

$$\begin{aligned} R^{\sigma}{}_{\beta;\sigma} - R^{\sigma}{}_{\sigma;\beta} + R^{\sigma\mu}{}_{\sigma\beta;\mu} &= 0 \\ R^{\sigma}{}_{\beta;\sigma} - R_{\ ;\beta} + R^{\sigma}{}_{\beta;\sigma} &= 0 \\ \therefore \quad 2R^{\sigma}{}_{\beta;\sigma} &= R_{\ ;\beta} \end{aligned} \tag{6.14}$$

Thus, we have calculated the divergence of the Ricci tensor,

$$R^{\sigma}{}_{\beta;\sigma} = \frac{1}{2} R_{\ ;\beta} \tag{6.15}$$

Now we use this expression together with the fact that the metric tensor is covariant and divergence free to construct a new divergence free curvature tensor.

$$R^{\sigma}{}_{\beta;\sigma} - \frac{1}{2} R_{\ ;\beta} = 0 \tag{6.16}$$

Keeping in mind that $(g^{\sigma}{}_{\beta} R)_{;\sigma} = g^{\sigma}{}_{\beta} R_{;\sigma}$ we multiply (6.16) by $g^{\beta}_{\ \alpha}$ to get

$$\begin{aligned} g^{\beta}{}_{\alpha} R^{\sigma}{}_{\beta;\sigma} - g^{\beta}{}_{\alpha} \frac{1}{2} R_{\ ;\beta} &= 0 \\ \left(g^{\beta}{}_{\alpha} R^{\sigma}{}_{\beta} \right)_{;\sigma} - \frac{1}{2} \left(g^{\beta}{}_{\alpha} R \right)_{;\beta} &= 0 \end{aligned} \tag{6.17}$$

interchanging σ and β in the first term of the last equation:

$$\begin{aligned} \left(g^{\sigma}{}_{\alpha} R^{\beta}{}_{\sigma} \right)_{;\beta} - \frac{1}{2} \left(g^{\beta}{}_{\alpha} R \right)_{;\beta} &= 0 \\ \Rightarrow \quad \left(R^{\beta}_{\ \alpha} - \frac{1}{2} \delta^{\beta}{}_{\alpha} R \right)_{;\beta} &= 0 \end{aligned} \tag{6.18}$$

since $g^{\sigma}{}_{\alpha} R^{\beta}{}_{\sigma} = \delta^{\sigma}{}_{\alpha} R^{\beta}{}_{\sigma} = R^{\beta}_{\ \alpha}$. So that $R^{\beta}_{\ \alpha} - \frac{1}{2} \delta^{\beta}{}_{\alpha} R$ is the divergence free curvature tensor desired.

This tensor is called the Einstein tensor and its covariant components are denoted by $E_{\alpha\beta}$. That is

$$\boxed{E_{\alpha\beta} = R_{\ \alpha\beta} - \frac{1}{2} g_{\ \alpha\beta} R} \tag{6.19}$$

NOTE THAT: $E^{\mu\nu}_{;\nu} = 0 \rightarrow 4$ equations, giving only 6 equations from $E_{\ \mu\nu}$, which secures a free choice of coordinate system.

6.3 Einstein's field equations

Einstein's field equations:

$$E_{\mu\nu} = \kappa T_{\mu\nu} \tag{6.20}$$

or

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{6.21}$$

Contraction gives:

$$\begin{aligned} R - \frac{1}{2}4R &= \kappa T \;, \qquad \text{where } T \equiv T^{\mu}_{\mu} \\ R &= -\kappa T \end{aligned} \tag{6.22}$$

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}(-\kappa T) + \kappa T_{\mu\nu} \;, \tag{6.23}$$

Thus the field equations may be written in the form

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{6.24}$$

In the Newtonian limit the metric may be written

$$ds^2 = -\left(1 + \frac{2\phi}{c^2}\right)dt^2 + (1 + h_{ii})(dx^2 + dy^2 + dz^2) \tag{6.25}$$

where the Newtonian potential $|\phi| \ll c^2$. We also have $T_{00} \gg T_{kk}$ and $T \approx -T_{00}$. Then the 00-component of the field equations becomes

$$R_{00} \approx \frac{\kappa}{2}T_{00} \tag{6.26}$$

Furthermore we have

$$\begin{aligned} R_{00} = R^{\mu}{}_{0\mu0} &= R^{i}{}_{0i0} \\ &= \Gamma^{i}{}_{00,i} - \Gamma^{i}{}_{0i,0} \\ &= \frac{\partial \Gamma^{k}{}_{00}}{\partial x^k} = \frac{1}{c^2}\nabla^2\phi \end{aligned} \tag{6.27}$$

Since $T_{00} \approx \rho c^2$ eq.(6.26) can be written $\nabla^2\phi = \frac{1}{2}\kappa c^4\rho$. Comparing this equation with the Newtonian law of gravitation on local form: $\nabla^2\phi = 4\pi G\rho$, we see that $\kappa = \frac{8\pi G}{c^4}$.

In classical vacuum we have : $T_{\mu\nu} = 0$, which gives

$$\boxed{E_{\mu\nu} = 0 \quad \text{or} \quad R_{\mu\nu} = 0 \;.} \tag{6.28}$$

These are the "vacuum field equations". Note that $R_{\mu\nu} = 0$ does *not* imply $R_{\mu\nu\alpha\beta} = 0$.

`Digression` **6.3.1 (Lagrange (variation principle))**
It was shown by Hilbert that the field equations may be deduced from a variation principle with action

$$\int R\sqrt{-g}d^4x \ , \tag{6.29}$$

where $R\sqrt{-g}$ is the Lagrange density. One may also include a so-called cosmological constant Λ:

$$\int (R+2\Lambda)\sqrt{-g}d^4x \tag{6.30}$$

The field equations with cosmological constant are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{6.31}$$

6.4 The "geodesic postulate" as a consequence of the field equations

The principle that free particles follow geodesic curves has been called the "geodesic postulate". We shall now show that the "geodesic postulate" follows as a consequence of the field equations.

Consider a system of free particles in curved space-time. This system can be regarded as a pressure-free gas. Such a gas is called *dust*. It is described by an energy-momentum tensor

$$T^{\mu\nu} = \rho u^\mu u^\nu \tag{6.32}$$

where ρ is the rest density of the dust as measured by an observer at rest in the dust and u^μ are the components of the four-velocity of the dust particles.

Einstein's field equations as applied to space-time filled with dust, take the form

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa\rho u^\mu u^\nu \tag{6.33}$$

Because the divergence of the left hand side is zero, the divergence of the right hand side must be zero, too

$$(\rho u^\mu u^\nu)_{;\nu} = 0 \tag{6.34}$$

or

$$(\rho u^\nu u^\mu)_{;\nu} = 0 \tag{6.35}$$

we now regard the quantity in the parenthesis as a product of ρu^ν and u^μ. By the rule for differentiating a product we get

$$(\rho u^\nu)_{;\nu}u^\mu + \rho u^\nu u^\mu{}_{;\nu} = 0 \tag{6.36}$$

Since the four-velocity of any object has a magnitude equal to the velocity of light we have

$$u_\mu u^\mu = -c^2 \tag{6.37}$$

Differentiation gives

$$(u_\mu u^\mu)_{;\nu} = 0 \tag{6.38}$$

Using, again, the rule for differentiating a product, we get

$$u_{\mu;\nu} u^\mu + u_\mu u^\mu{}_{;\nu} = 0 \tag{6.39}$$

From the rule for raising an index and the freedom of changing a summation index from α to μ, say, we get

$$u_{\ \mu;\nu} u^\mu = u^\mu u_{\mu;\nu} = g^{\mu\alpha} u_\alpha u_{\mu;\nu} = u_\alpha g^{\mu\alpha} u_{\mu;\nu} = u_\alpha u^\alpha{}_{;\nu} = u_\mu u^\mu{}_{;\nu} \tag{6.40}$$

Thus the two terms of eq.(6.39) are equal. It follows that each of them are equal to zero. So we have

$$u_\mu u^\mu{}_{;\nu} = 0 \tag{6.41}$$

Multiplying eq.(6.36) by u_μ, we get

$$(\rho u^\nu)_{;\nu} u_\mu u^\mu + \rho u^\nu u_\mu u^\mu{}_{;\nu} = 0 \tag{6.42}$$

Using eq.(6.37) in the first term, and eq.(6.41) in the last term, which then vanishes, we get

$$(\rho u^\nu)_{;\nu} = 0 \tag{6.43}$$

Thus the first term in eq.(6.36) vanishes and we get

$$\rho u^\nu u^\mu{}_{;\nu} = 0 \tag{6.44}$$

Since $\rho \neq 0$ we must have

$$u^\nu u^\mu{}_{;\nu} = 0 \tag{6.45}$$

This is just the geodesic equation. Conclusion: *It follows from Einstein's field equations that free particles move along paths corresponding to geodesic curves of space-time.*

Chapter 7

The Schwarzschild spacetime

7.1 Schwarzschild's exterior solution

This is a solution of the vacuum field equations $E_{\mu\nu} = 0$ for a static spherically symmetric spacetime. One can then *choose* the following form of the line element (employing units so that c=1),

$$\begin{aligned} ds^2 &= -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \\ d\Omega^2 &= d\theta^2 + \sin^2\theta d\phi^2 \end{aligned} \tag{7.1}$$

These coordinates are chosen so that the area of a sphere with radius r is $4\pi r^2$.

Physical distance in radial direction, corresponding to a coordinate distance dr, is $dl_r = \sqrt{g_{rr}}dr = e^{\beta(r)}dr$.

Here follows a stepwise algorithm to determine the components of the Einstein tensor by using the Cartan formalism:

1. Using orthonormal basis we find

$$\underline{\omega}^{\hat{t}} = e^{\alpha(r)}\underline{d}t \ , \quad \underline{\omega}^{\hat{r}} = e^{\beta(r)}\underline{d}r \ , \quad \underline{\omega}^{\hat{\theta}} = r\underline{d}\theta \ , \quad \underline{\omega}^{\hat{\phi}} = r\sin\theta\underline{d}\phi \tag{7.2}$$

2. Computing the connection forms by applying Cartan's 1. structure equations

$$\underline{d\omega}^{\hat{\mu}} = -\underline{\Omega}^{\hat{\mu}}{}_{\hat{\nu}} \wedge \underline{\omega}^{\hat{\nu}} \tag{7.3}$$

$$\begin{aligned} \underline{d\omega}^{\hat{t}} &= e^{\alpha}\alpha'\underline{d}r \wedge \underline{d}t \\ &= e^{\alpha}\alpha' e^{-\beta}\underline{\omega}^{\hat{r}} \wedge e^{-\alpha}\underline{\omega}^{\hat{t}} \\ &= -e^{-\beta}\alpha'\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}} \\ &= -\underline{\Omega}^{\hat{t}}{}_{\hat{r}} \wedge \underline{\omega}^{\hat{r}} \end{aligned} \tag{7.4}$$

$$\therefore \underline{\Omega}^{\hat{t}}{}_{\hat{r}} = e^{-\beta}\alpha'\underline{\omega}^{\hat{t}} + f_1\underline{\omega}^{\hat{r}} \tag{7.5}$$

3. To determine the f-functions we apply the anti-symmetry

$$\underline{\Omega}_{\hat{\mu}\hat{\nu}} = -\underline{\Omega}_{\hat{\nu}\hat{\mu}} \tag{7.6}$$

This gives:

$$\begin{aligned}
\underline{\Omega}^{\hat{r}}{}_{\hat{\phi}} &= -\underline{\Omega}^{\hat{\phi}}{}_{\hat{r}} = -\frac{1}{r}e^{-\beta}\underline{\omega}^{\hat{\phi}} \\
\underline{\Omega}^{\hat{\theta}}{}_{\hat{\phi}} &= -\underline{\Omega}^{\hat{\phi}}{}_{\hat{\theta}} = -\frac{1}{r}\cot\theta\underline{\omega}^{\hat{\phi}} \\
\underline{\Omega}^{\hat{t}}{}_{\hat{r}} &= +\underline{\Omega}^{\hat{r}}{}_{\hat{t}} = e^{-\beta}\alpha'\underline{\omega}^{\hat{t}} \\
\underline{\Omega}^{\hat{r}}{}_{\hat{\theta}} &= -\underline{\Omega}^{\hat{\theta}}{}_{\hat{r}} = -\frac{1}{r}e^{-\beta}\underline{\omega}^{\hat{\theta}}
\end{aligned} \tag{7.7}$$

4. We then proceed to determine the curvature forms by applying Cartan's 2nd structure equations

$$\underline{R}^{\hat{\mu}}{}_{\hat{\nu}} = \underline{d\Omega}^{\hat{\mu}}{}_{\hat{\nu}} + \underline{\Omega}^{\hat{\mu}}{}_{\hat{\alpha}} \wedge \underline{\Omega}^{\hat{\alpha}}{}_{\hat{\nu}} \tag{7.8}$$

which gives:

$$\begin{aligned}
\underline{R}^{\hat{t}}{}_{\hat{r}} &= -e^{-2\beta}(\alpha'' + \alpha'^2 - \alpha'\beta')\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}} \\
\underline{R}^{\hat{t}}{}_{\hat{\theta}} &= -\frac{1}{r}e^{-2\beta}\alpha'\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\theta}} \\
\underline{R}^{\hat{t}}{}_{\hat{\phi}} &= -\frac{1}{r}e^{-2\beta}\alpha'\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\phi}} \\
\underline{R}^{\hat{r}}{}_{\hat{\theta}} &= \frac{1}{r}e^{-2\beta}\beta'\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\theta}} \\
\underline{R}^{\hat{r}}{}_{\hat{\phi}} &= \frac{1}{r}e^{-2\beta}\beta'\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\phi}} \\
\underline{R}^{\hat{\theta}}{}_{\hat{\phi}} &= \frac{1}{r^2}(1 - e^{-2\beta})\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\phi}}
\end{aligned} \tag{7.9}$$

5. By applying the following relation

$$\underline{R}^{\hat{\mu}}{}_{\hat{\nu}} = \frac{1}{2}R^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}\hat{\beta}}\underline{\omega}^{\hat{\alpha}} \wedge \underline{\omega}^{\hat{\beta}} \tag{7.10}$$

we find the components of Riemann's curvature tensor.

6. Contraction gives the components of Ricci's curvature tensor

$$R_{\hat{\mu}\hat{\nu}} \equiv R^{\hat{\alpha}}{}_{\hat{\mu}\hat{\alpha}\hat{\nu}} \tag{7.11}$$

7. A new contraction gives Ricci's curvature scalar

$$R \equiv R^{\hat{\mu}}{}_{\hat{\mu}} \tag{7.12}$$

8. The components of the Einstein tensor can then be found

$$E_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2}\eta_{\hat{\mu}\hat{\nu}}R \ , \tag{7.13}$$

where $\eta_{\hat{\mu}\hat{\nu}} = diag(-1,1,1,1)$. We then have:

$$\begin{aligned} E_{\hat{t}\hat{t}} &= \frac{2}{r}e^{-2\beta}\beta' + \frac{1}{r^2}(1-e^{-2\beta}) \\ E_{\hat{r}\hat{r}} &= \frac{2}{r}e^{-2\beta}\alpha' - \frac{1}{r^2}(1-e^{-2\beta}) \\ E_{\hat{\theta}\hat{\theta}} = E_{\hat{\phi}\hat{\phi}} &= e^{-2\beta}(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{\alpha'}{r} - \frac{\beta'}{r}) \end{aligned} \tag{7.14}$$

We want to solve the equations $E_{\hat{\mu}\hat{\nu}} = 0$. We get only 2 independent equations, and choose to solve those:

$$E_{\hat{t}\hat{t}} = 0 \qquad \text{and} \qquad E_{\hat{r}\hat{r}} = 0 \tag{7.15}$$

By adding the 2 equations we get:

$$E_{\hat{t}\hat{t}} + E_{\hat{r}\hat{r}} = 0$$

$$\Rightarrow \frac{2}{r}e^{-2\beta}(\beta' + \alpha') = 0$$

$$\Rightarrow (\alpha+\beta)' = 0 \Rightarrow \alpha + \beta = K_1 \quad \text{(const)} \tag{7.16}$$

We now have:

$$ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2 d\Omega^2 \tag{7.17}$$

By choosing a suitable coordinate time, we can achieve

$$K_1 = 0 \Rightarrow \alpha = -\beta$$

Since we have $ds^2 = -e^{2\alpha}dt^2 + e^{-2\alpha}dr^2 + r^2 d\Omega^2$, this means that $g_{rr} = -\frac{1}{g_{tt}}$. We still have to solve one more equation to get the complete solution, and choose the equation $E_{\hat{t}\hat{t}} = 0$, which gives

$$\frac{2}{r}e^{-2\beta}\beta' + \frac{1}{r^2}(1-e^{-2\beta}) = 0$$

This equation can be written:

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}[r(1-e^{-2\beta})] &= 0 \\ \therefore r(1-e^{-2\beta}) &= K_2 \qquad \text{(const)} \end{aligned} \tag{7.18}$$

If we choose $K_2 = 0$ we get $\beta = 0$ giving $\alpha = 0$ and

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \ , \tag{7.19}$$

which is the Minkowski space-time described in spherical coordinates. In general, $K_2 \neq 0$ and $1 - e^{-2\beta} = \frac{K_2}{r} \equiv \frac{K}{r}$, giving

$$e^{2\alpha} = e^{-2\beta} = 1 - \frac{K}{r}$$

and

$$ds^2 = -(1 - \frac{K}{r})dt^2 + \frac{dr^2}{1 - \frac{K}{r}} + r^2 d\Omega^2 \tag{7.20}$$

We can find K by going to the Newtonian limit. We calculate the gravitational acceleration (that is, the acceleration of a free particle instantanously at rest) in the limit of a weak field of a particle at a distance r from a spherical mass M. Newtonian:

$$g = \frac{d^2 r}{dt^2} = -\frac{GM}{r^2} \tag{7.21}$$

We anticipate that $r >> K$. Then the proper time τ of a particle will be approximately equal to the coordinate time, since $d\tau = \sqrt{1 - \frac{K}{r}} dt$
The acceleration of a particle in 3-space, is given by the geodesic equation:

$$\begin{aligned} \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta &= 0 \\ u^\alpha &= \frac{dx^\alpha}{d\tau} \end{aligned} \tag{7.22}$$

For a particle instantanously at rest in a weak field, we have $d\tau \approx dt$. Using $u^\mu = (1, 0, 0, 0)$, we get:

$$g = \frac{d^2 r}{dt^2} = -\Gamma^r{}_{tt} \tag{7.23}$$

This equation gives a physical interpretation of $\Gamma^r{}_{tt}$ as the gravitational acceleration. This is a mathematical way to express the principle of equivalence: The gravitational acceleration can be transformed to 0, since the Christoffel symbols always can be transformed to 0 locally, in a freely falling non-rotating frame, i.e. a local inertial frame.

$$\begin{aligned} \Gamma^r{}_{tt} &= \frac{1}{2} \underbrace{g^{r\alpha}}_{\frac{1}{g_{r\alpha}}} (\underbrace{\frac{\partial g_{\alpha t}}{\partial t}}_{=0} + \underbrace{\frac{\partial g_{\alpha t}}{\partial t}}_{=0} - \frac{\partial g_{tt}}{\partial x^\alpha}) \\ &= -\frac{1}{2 g_{rr}} \frac{\partial g_{tt}}{\partial r} \\ g_{tt} &= -(1 - \frac{K}{r}) \ , \ \frac{\partial g_{tt}}{\partial r} = -\frac{K}{r^2} \\ g &= -\Gamma^r{}_{tt} = -\frac{K}{2r^2} = -\frac{GM}{r^2} \\ &\text{gives } K = 2GM \\ &\text{or with c: } K = \frac{2GM}{c^2} \end{aligned} \tag{7.24}$$

Then we have the line element of the exterior Schwarzschild metric:

$$ds^2 = -(1 - \frac{2GM}{c^2 r})c^2 dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2 d\Omega^2 \tag{7.25}$$

$R_S \equiv \frac{2GM}{c^2}$ is the Schwarzschild radius of a mass M.
Weak field: $r >> R_S$.
For the earth: $R_S \sim 0.9cm$
For the sun: $R_S \sim 3km$
A standard clock at rest in the Schwarzschild spacetime shows a proper time τ:

$$d\tau = \sqrt{1 - \frac{R_S}{r}} dt \tag{7.26}$$

So the coordinate clocks showing t, are ticking with the same rate as the standard clocks far from M. Coordinate clocks are running equally fast no matter where they are. If they hadn't, the spatial distance between simultanous events with given spatial coordinates, would depend on the time of the measuring of the distance. Then the metric would be time dependent. Time is not running at the Schwarzschild radius.

Definition 7.1.1 (Physical singularity)
A physical singularity is a point where the curvature is infinitely large.

Definition 7.1.2 (Coordinate singularity)
A coordinate singularity is a point (or a surface) where at least one of the components of the metric tensor is infinitely large, but where the curvature of spacetime is finite.

Kretschmann's curvature scalar is $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. From the Schwarzschild metric, we get:

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{48G^2M^2}{r^8} \tag{7.27}$$

which diverges only at the origin. Since there is no physical singularity at $r = R_S$, the singularity here is just a coordinate singularity, and can be removed by a transformation to a coordinate system falling inward. (Eddington - Finkelstein coordinates, Kruskal - Szekers analytical extension of the description of Schwarzschild spacetime to include the area inside R_S).

7.2 Radial free fall in Schwarzschild spacetime

The Lagrangian function of a particle moving radially in Schwarzschild spacetime

$$L = -\frac{1}{2}(1 - \frac{R_S}{r})c^2\dot{t}^2 + \frac{1}{2}\frac{\dot{r}^2}{(1 - \frac{R_S}{r})} \quad , \quad \dot{} \equiv \frac{d}{d\tau} \tag{7.28}$$

where τ is the time measured on a standard clock which the particle is carrying. The momentum conjugate p_t of the cyclic coordinate t, is a constant of motion.

$$p_t = \frac{\partial L}{\partial \dot{t}} = -(1 - \frac{R_S}{r})c^2\dot{t} \tag{7.29}$$

4-velocity identity: $u_\mu u^\mu = -c^2$:

$$-(1 - \frac{R_S}{r})c^2\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{R_S}{r}} = -c^2 \tag{7.30}$$

Inserting the expression for $\dot{t}$ gives:

$$\dot{r}^2 - \frac{p_t^2}{c^2} = -(1 - \frac{R_S}{r})c^2 \tag{7.31}$$

Boundary conditions: the particle is falling from rest at $r = r_0$.

$$p_t = -(1 - \frac{R_S}{r_0})\underbrace{\frac{c^2}{\sqrt{1 - \frac{R_S}{r_0}}}}_{\dot{t}(r=r_0)} = -\sqrt{1 - \frac{R_S}{r_0}}c^2 \tag{7.32}$$

giving

$$\dot{r} = \frac{dr}{d\tau} = -c\sqrt{\frac{R_S}{r_0}}\sqrt{\frac{r_0 - r}{r}} \tag{7.33}$$

$$\int \frac{dr}{\sqrt{\frac{r_0 - r}{r}}} = -c\sqrt{\frac{R_S}{r_0}}\tau \tag{7.34}$$

Integration with $\tau = 0$ for $r = 0$ gives:

$$\tau = -\frac{r_0}{c}\sqrt{\frac{r_0}{R_S}}(\arcsin\sqrt{\frac{r}{r_0}} - \sqrt{\frac{r}{r_0}}\sqrt{1 - \frac{r}{r_0}}) \tag{7.35}$$

τ is the proper time that the particle spends on the part of the fall which is from r to r=0. To the lowest order in $\frac{r}{r_0}$ we get:

$$\tau = -\frac{2}{3}\sqrt{\frac{r}{R_S}}\frac{r}{c} \tag{7.36}$$

Travelling time from $r = R_S$ to $r = 0$ for $R_S = 2km$ is then:

$$|\tau(R_S)| = \frac{2}{3}\frac{R_S}{c} = 4 \times 10^{-6}s \tag{7.37}$$

7.3 Light cones in Schwarzschild spacetime

The Schwarzschild line-element (with $c = 1$) is

$$ds^2 = -(1 - \frac{R_S}{r})dt^2 + \frac{dr^2}{(1 - \frac{R_S}{r})} + r^2 d\Omega^2 \tag{7.38}$$

We will look at **radially moving photons** ($ds^2 = d\Omega^2 = 0$). We then get

$$\begin{aligned} \frac{dr}{\sqrt{1 - \frac{R_S}{r}}} &= \pm\sqrt{1 - \frac{R_S}{r}}dt \Leftrightarrow \frac{r^{\frac{1}{2}}dr}{\sqrt{r - R_S}} = \pm\frac{\sqrt{r - R_S}}{r^{\frac{1}{2}}}dt \\ \frac{rdr}{r - R_S} &= \pm dt \end{aligned} \tag{7.39}$$

with $+$ for outward motion and $-$ for inward motion. For inwardly moving photons, integration yields

$$r + t + R_S \ln|\frac{r}{R_S} - 1| = k = constant \tag{7.40}$$

We now introduce a new time coordinate t' such that the equation of motion for photons moving **inwards** takes the following form

$$\begin{aligned} r + t' = k \Rightarrow \frac{dr}{dt'} &= -1 \\ \therefore \quad t' = t + R_S \ln|\frac{r}{R_S} - 1| \end{aligned} \tag{7.41}$$

The coordinate t' is called an ingoing Eddington-Finkelstein coordinate. The photons here always move with the *local* velocity of light, c. For photons moving **outwards** we have

$$r + R_S \ln|\frac{r}{R_S} - 1| = t + k \tag{7.42}$$

Making use of $t = t' - R_S \ln|\frac{r}{R_S} - 1|$ we get

$$\begin{aligned} & r + 2R_S \ln|\frac{r}{R_S} - 1| = t' + k \\ \Rightarrow \quad & \frac{dr}{dt'} + \frac{2R_S}{r - R_S}\frac{dr}{dt'} = 1 \Leftrightarrow \frac{r + R_S}{r - R_S}\frac{dr}{dt'} = 1 \\ \Leftrightarrow \quad & \frac{dr}{dt'} = \frac{r - R_S}{r + R_S} \end{aligned} \tag{7.43}$$

Making use of ordinary Schwarzschild coordinates we would have gotten the following coordinate velocities for inn- and outwardly moving photons:

$$\frac{dr}{dt} = \pm(1 - \frac{R_S}{r}) \tag{7.44}$$

which shows us how light is decelerated in a gravitational field. Figure 7.1 shows how this is viewed by a non-moving observer located far away from the mass. In Figure 7.2 we have instead used the alternative time coordinate t'. The special theory of relativity is valid locally, and all material particles thus have to remain inside the light cone.

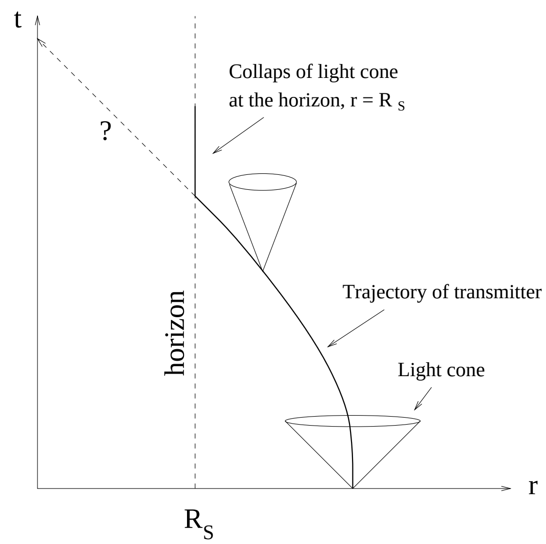


Figure 7.1: At a radius $r = R_S$ the light cones collapse, and nothing can any longer escape, when we use the Schwarzschild coordinate time.

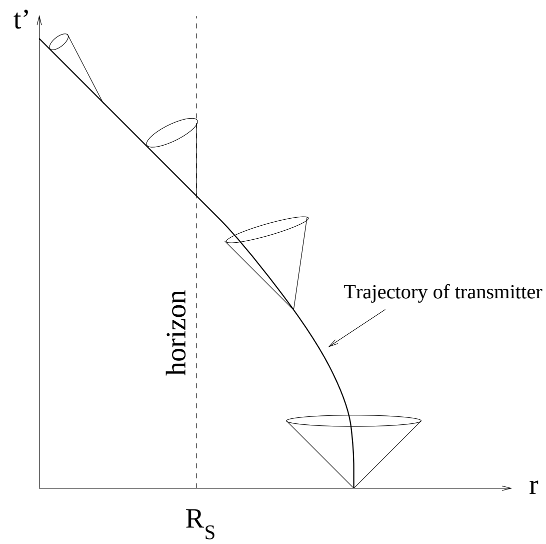


Figure 7.2: Using the ingoing Eddingto n-Finkelstein time coordinate there is no collapse of the light cone at $r = R_S$. Instead we get a collapse at the singularity at $r = 0$. The angle between the left part of the light cone and the t'-axis is always 45 degrees. We also see that once the transmitter gets inside the horizon at $r = R_S$, no particles can escape.

7.4 Analytical extension of the Schwarzschild spacetime

The Schwarzschild coordinates are comoving with a static reference frame outside a spherical mass distribution. If the mass has collapsed to a black hole there exist a horizon at the Schwarzschild radius. As we have seen in section 7.3 there do not exist static observers at finite radii inside the horizon. Hence, the Schwarzschild coordinates are well defined only outside the horizon.

Also the rr-component of the metric tensor has a coordinate singularity at the Schwarzschild radius. The curvature of spacetime is finite here.

Kruskal and Szekeres have introduced new coordinates that are well defined inside as well as outside the Schwarzschild radius, and with the property that the metric tensor is non-singular for all $r > 0$.

In order to arrive at these coordinates we start by considering a photon moving radially inwards. From eq. (7.40) we then have

$$t = -r - R_S \ln\left|\frac{r}{R_S} - 1\right| + v \tag{7.45}$$

where v is a constant along the world line of the photon. We introduce a new radial coordinate

$$r^* \equiv r + R_S \ln\left|\frac{r}{R_S} - 1\right| \tag{7.46}$$

Then the equation of the worldline of the photon takes the form

$$t + r^* = v \tag{7.47}$$

The value of the constant v does only depend upon the point of time when the photon was emitted. We may therefore use v as a new time coordinate.

For an outgoing photon we get in the same way

$$t - r^* = u \tag{7.48}$$

where u is a constant of integration, which may be used as a new time coordinate for outgoing photons. The coordinates u and v are the generalization of the **light cone coordinates** of Minkowski spacetime to the Schwarzschild spacetime.

From eqs. (7.47) and (7.48) we get

$$dt = \frac{1}{2}(dv + du) \tag{7.49}$$

$$dr^* = \frac{1}{2}(dv - du) \tag{7.50}$$

and from eq. (7.46)

$$dr = \left(1 - \frac{R_s}{r}\right) dr^* \tag{7.51}$$

Inserting these differentials into eq. (7.38) we arrive at a new form of the Schwarzschild line-element,

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) du\, dv + r^2 d\Omega^2 \tag{7.52}$$

The metric is still not well behaved at the horizon. Introducing the coordinates

$$U = -e^{-\frac{u}{2R_s}} \tag{7.53}$$
$$V = e^{\frac{v}{2R_s}} \tag{7.54}$$

gives

$$UV = -e^{\frac{v-u}{2R_s}} = -e^{\frac{r^*}{R_s}} = -\left|\frac{R_s}{r} - 1\right| e^{\frac{r}{R_s}} \tag{7.55}$$

and

$$du\,dv = -4R_s^2 \frac{dUdV}{UV} \tag{7.56}$$

The line-element (7.52) then takes the form

$$ds^2 = -\frac{4R_s^3}{r} e^{-\frac{r}{R_s}} dUdV + r^2 d\Omega^2 \tag{7.57}$$

This is the first form of the Kruskal-Szekeres line-element. Here is no coordinate singularity, only a physical singularity at $r = 0$.

We may furthermore introduce two new coordinates

$$T = \frac{1}{2}(V + U) = \left(\frac{r}{R_s} - 1\right) e^{\frac{r}{2R_s}} \sinh \frac{t}{2R_s} \tag{7.58}$$
$$Z = \frac{1}{2}(V - U) = \left(\frac{r}{R_s} - 1\right) e^{\frac{r}{2R_s}} \cosh \frac{t}{2R_s} \tag{7.59}$$

Hence

$$V = T + Z \tag{7.60}$$
$$U = T - Z \tag{7.61}$$

giving

$$dUdV = dT^2 - dZ^2 \tag{7.62}$$

Inserting this into eq. (7.57) we arrive at the second form of the Kruskal-Szekeres line-element

$$ds^2 = -\frac{4R_s^3}{r} e^{-\frac{r}{R_s}} \left(dT^2 - dZ^2\right) + r^2 d\Omega^2 \tag{7.63}$$

The inverse transformations of eqs. (7.58) and (7.59) is

$$\left(1 - \frac{r}{R_s}\right) e^{\frac{r}{R_s}} = T^2 - Z^2 \tag{7.64}$$
$$\tanh \frac{t}{2R_s} = \frac{T}{Z} \tag{7.65}$$

Note from eq. (7.63) that with the Kruskal-Szekeres coordinates T and Z the equation of the radial null geodesics has the same form as in flat spacetime

$$Z = \pm T + \text{constant} \tag{7.66}$$

7.5 Embedding of the Schwarzschild metric

We will now look at a static, spherically symmetric space. A curved simultaneity plane ($dt = 0$) through the equatorial plane ($d\theta = 0$) has the line element

$$ds^2 = g_{rr}dr^2 + r^2 d\phi^2 \tag{7.67}$$

with a radial coordinate such that a circle with radius r has a circumference of length $2\pi r$.

We now embed this surface in a flat 3-dimensional space with cylinder coordinates (z, r, ϕ) and line element

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \tag{7.68}$$

The surface described by the line element in (7.67) has the equation $z = z(r)$. The line element in (7.68) is therefore written as

$$ds^2 = [1 + (\frac{dz}{dr})^2]dr^2 + r^2 d\phi^2 \tag{7.69}$$

Demanding that (7.69) is in agreement with (7.67) we get

$$g_{rr} = 1 + (\frac{dz}{dr})^2 \Leftrightarrow \frac{dz}{dr} = \pm\sqrt{g_{rr} - 1} \tag{7.70}$$

Choosing the positive solution gives

$$\boxed{dz = \sqrt{g_{rr} - 1}dr} \tag{7.71}$$

In the Schwarzschild spacetime we have

$$g_{rr} = \frac{1}{1 - \frac{R_S}{r}} \tag{7.72}$$

Making use of this we find z:

$$z = \int_{R_S}^{r} \frac{dr}{\sqrt{1 - \frac{R_S}{r}}} = \sqrt{4R_S(r - R_S)} \tag{7.73}$$

This is shown in Figure 7.3.

7.6 Deceleration of light

The radial speed of light in Schwarzschild coordinates found by putting $ds^2 = d\Omega^2 = 0$ in eq. (7.38) is

$$\bar{c} = 1 - \frac{R_S}{r} \tag{7.74}$$

To measure this effect one can look at how long it takes for light to get from Mercury to the Earth. This is illustrated in Figure 7.4. The travel time from

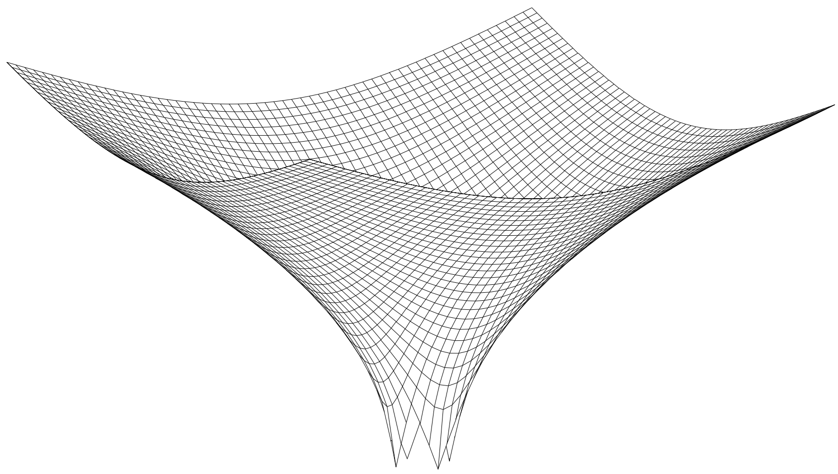


Figure 7.3: Embedding of the Schwarzschild metric.

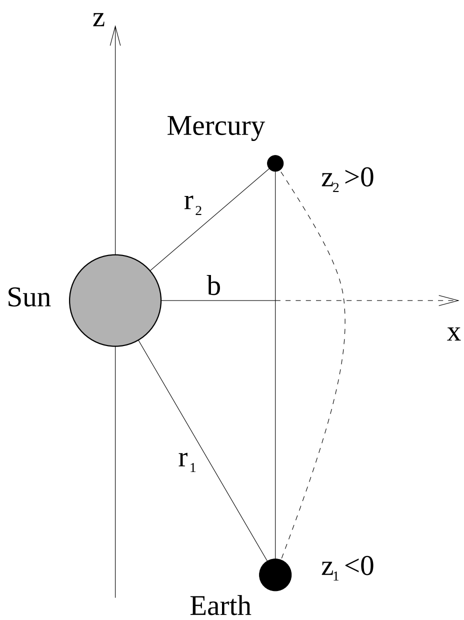


Figure 7.4: General relativity predicts that light traveling from Mercury to the Earth will be delayed due to the effect of the Suns gravity field on the speed of light. This effect has been measured.

z_1 to z_2 is

$$\begin{aligned}\Delta t &= \int_{z_1}^{z_2} \frac{dz}{1-\frac{R_S}{r}} \approx \int_{z_1}^{z_2} (1+\frac{R_S}{r})dz = \int_{z_1}^{z_2} (1+\frac{R_S}{\sqrt{b^2+z^2}})dz \\ &= z_2 + |z_1| + R_S \ln \frac{\sqrt{{z_2}^2+b^2}+z_2}{\sqrt{{z_1}^2+b^2}-|z_1|}\end{aligned} \tag{7.75}$$

where R_S is the Schwarzschild radius of the Sun.

The deceleration is greatest when Earth and Mercury (where the light is reflected) are on nearly opposite sides of the Sun. The impact parameter b is then small. A series expansion to the lowest order of b/z gives

$$\Delta t = z_2 + |z_1| + R_S \ln \frac{4|z_1|z_2}{b^2} \tag{7.76}$$

The last term represents the extra traveling time due to the effect of the Suns gravity field on the speed of light. The journey takes longer time:

$$\begin{array}{lll} R_S = & \text{the Schwarzschild radius of the Sun} & \sim 2km \\ |z_1| = & \text{the radius of Earth's orbit} & = 15 \times 10^{10} m \\ z_2 = & \text{the radius of Mercury's orbit} & = 5.8 \times 10^{10} m \\ b = & R_\odot = 7 \times 10^8 m & \end{array}$$

give a delay of $1.1 \times 10^{-4} s$. In addition to this one must also, of course, take into account among other things the effects of the curvature of spacetime near the Sun and atmospheric effects on Earth.

7.7 Particle trajectories in Schwarzschild 3-space

$$\begin{aligned} L &= \frac{1}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \\ &= -\frac{1}{2}\left(1-\frac{R_s}{r}\right)\dot{t}^2 + \frac{\frac{1}{2}\dot{r}^2}{1-\frac{R_s}{r}} + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}r^2\sin^2\theta\dot{\phi}^2 \end{aligned} \tag{7.77}$$

Since t is a cyclic coordinate

$$-p_t = -\frac{\partial L}{\partial \dot{t}} = \left(1-\frac{R_s}{r}\right)\dot{t} = constant = E \tag{7.78}$$

where E is the particle's energy as measured by an observer "far away" ($r \gg R_s$). Also ϕ is a cyclic coordinate so that

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = r^2 \sin^2\theta \dot{\phi} = constant \tag{7.79}$$

where p_ϕ is the particle's orbital angular momentum.

Making use of the 4-velocity identity $\vec{U}^2 = g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = -1$ we transform the above to get

$$-\left(1-\frac{R_s}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-\frac{R_s}{r}} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 = -1 \tag{7.80}$$

which on substitution for $\dot{t} = \frac{E}{1-\frac{R_s}{r}}$ and $\dot{\phi} = \frac{p_\phi}{r^2\sin^2\theta}$ becomes

$$-\frac{E^2}{1-\frac{R_s}{r}} + \frac{\dot{r}^2}{1-\frac{R_s}{r}} + r^2\dot{\theta}^2 + \frac{p_\phi^2}{r^2\sin^2\theta} = -1 \tag{7.81}$$

Now, refering back to the Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{X}^\mu}\right) - \frac{\partial L}{\partial X^\mu} = 0 \tag{7.82}$$

we get, for θ

$$\begin{aligned}(r^2\dot{\theta})^\bullet &= r^2\sin\theta\cos\theta\dot{\phi}^2\\ &= \frac{p_\phi^2\cos\theta}{r^2\sin^3\theta}\end{aligned} \tag{7.83}$$

Multiplying this by $r^2\dot{\theta}$ we get

$$(r^2\dot{\theta})(r^2\dot{\theta})^\bullet = \frac{\cos\theta\dot{\theta}}{\sin^3\theta}p_\phi^2 \tag{7.84}$$

which, on integration, gives

$$(r^2\dot{\theta})^2 = k - \left(\frac{p_\phi}{\sin\theta}\right)^2 \tag{7.85}$$

where k is the constant of integration.

Because of the spherical geometry we are free to choose a coordinate system such that the particle moves in the equatorial plane and along the equator at a given time $t = 0$. That is $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$ at time $t = 0$. This determines the constant of integration and $k = p_\phi^2$ such that

$$(r^2\dot{\theta})^2 = p_\phi^2\left(1 - \frac{1}{\sin^2\theta}\right) \tag{7.86}$$

The RHS is negative for all $\theta \neq \frac{\pi}{2}$. It follows that the particle cannot deviate from its original (equatorial) trajectory. Also, since this particular choice of trajectory was arbitrary we can conclude, quite generally, that any motion of free particles in a spherically symmetric gravitational field is planar motion.

7.7.1 Motion in the equatorial plane

$$-\frac{E^2}{1-\frac{R_s}{r}}+\frac{\dot{r}^2}{1-\frac{R_s}{r}}+\frac{p_\phi^2}{r^2}=-1 \tag{7.87}$$

that is

$$\dot{r}^2=E^2-\left(1-\frac{R_s}{r}\right)\left(1+\frac{p_\phi^2}{r^2}\right) \tag{7.88}$$

This corresponds to an energy equation with an effective potential $V(r)$ given by

$$\begin{aligned}
V^2(r) &= \left(1-\frac{R_s}{r}\right)\left(1+\frac{p_\phi^2}{r^2}\right)\\
\dot{r}^2+V^2(r) &= E^2\\
\Rightarrow V &= \sqrt{1-\frac{r_s}{r}+\frac{p_\phi^2}{r^2}-\frac{R_s p_\phi^2}{r^3}}\\
&\approxeq 1-\frac{1}{2}\frac{R_s}{r}+\frac{1}{2}\frac{p_\phi^2}{r^2}
\end{aligned} \tag{7.89}$$

Newtonian potential V_N is defined by using the last expression in

$$V_N=V-1\Rightarrow V_N=-\frac{GM}{r}+\frac{p_\phi^2}{2r^2} \tag{7.90}$$

The possible trajectories of particles in the Schwarzschild 3-space are shown schematically in Figure 7.5 as functions of position and energy of the particle in the Newtonian limit.

To take into account the relativistic effects the above picture must be modified. We introduce dimensionless variables

$$X=\frac{r}{GM} \quad \text{and} \quad k=\frac{p_\phi}{GMm} \tag{7.91}$$

The potential $V^2(r)$ now take the form

$$V=\left(1-\frac{2}{X}+\frac{k^2}{X^2}-\frac{2k^2}{X^3}\right)^{1/2} \tag{7.92}$$

For r equal to the Schwarzschild radius ($X=2$) we have

$$V(2)=\sqrt{1-1+\frac{k^2}{4}-\frac{2k^2}{8}}=0 \tag{7.93}$$

For $k^2<12$ particles will fall in towards $r=0$.

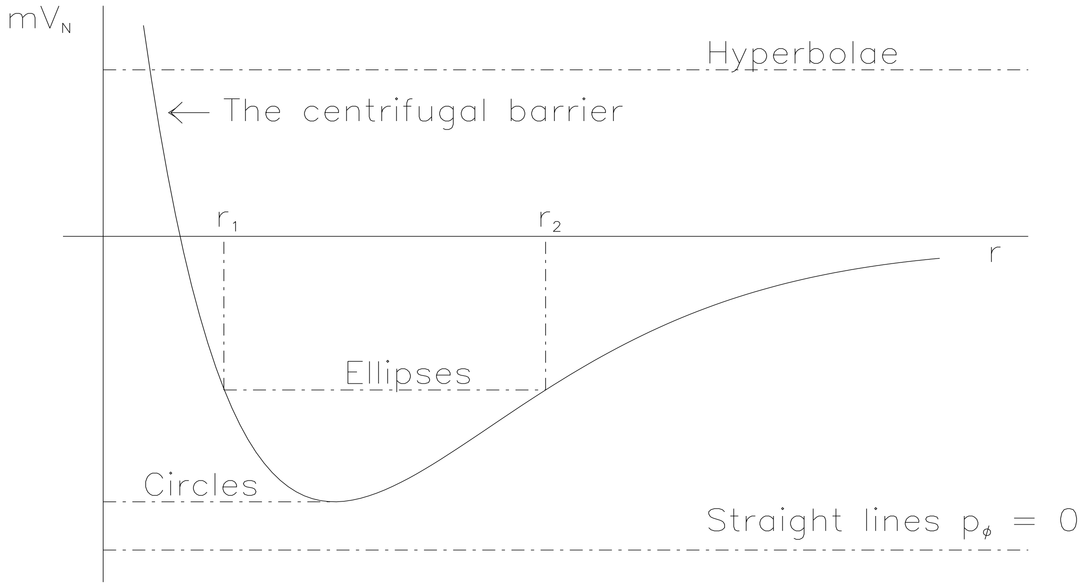


Figure 7.5: Newtonian particle trajectories are functions of the position and energy of the particle. Note the **centrifugal barrier**. Due to this particles with $p_\phi \neq 0$ cannot arrive at $r = 0$.

An **orbit equation** is one which connects r and ϕ. So for motion in the equatorial plane for weaks fields we have

$$\frac{d\phi}{dt} = \frac{p_\phi}{mr^2} \qquad \bullet \equiv \frac{d}{d\tau} = \frac{p_\phi}{mr^2}\frac{d}{d\phi} \tag{7.94}$$

Introducing the new radial coordinate $u \equiv \frac{1}{r}$ our equations transform to

$$\begin{aligned} \frac{du}{d\phi} &= -\frac{1}{r^2}\frac{dr}{d\phi} = -\frac{1}{r^2}\frac{mr^2}{p_\phi}\frac{dr}{dt} = -\frac{m}{p_\phi}\dot{r} \\ \Rightarrow \dot{r} &= -\frac{p_\phi}{m}\frac{du}{d\phi} \end{aligned} \tag{7.95}$$

Substitution from above for $\dot{r}$ in the energy equation yields the orbit equation,

$$\left(\frac{du}{d\phi}\right)^2 + (1 - 2GMu)\left(u^2 + \frac{m^2}{p_\phi^2}\right) = \frac{E^2}{p_\phi^2}. \tag{7.96}$$

Differentiating this, we find

$$\frac{d^2u}{d\phi^2} + u = \frac{GMm^2}{p_\phi^2} + 3GMu^2 \tag{7.97}$$

The last term on the RHS is a relativistic correction term.

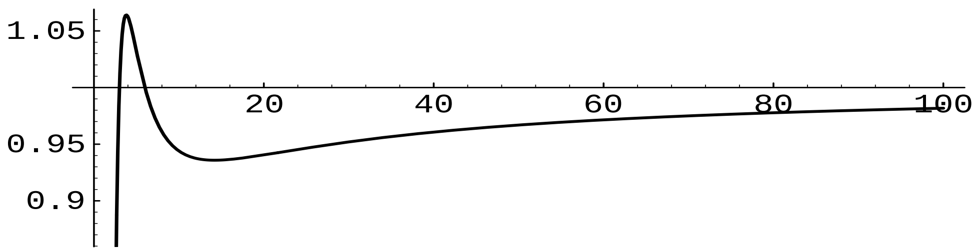


Figure 7.6: When relativistic effects are included there is no longer a limit to the values that r can take and collapse to a singularity is "possible". Note that V^2 is plotted here.

7.8 Classical tests of Einstein's general theory of relativity

7.8.1 The Hafele-Keating experiment

Hafele and Keating measured the difference in time shown on moving and stationary atomic clocks. This was done by flying around the Earth in the East-West direction comparing the time on the clock in the plane with the time on a clock on the ground.

The proper time interval measured on a clock moving with a velocity $v^i = \frac{dx^i}{dt}$ in an arbitrary coordinate system with metric tensor $g_{\mu\nu}$ is given by

$$\begin{aligned} d\tau &= (-\frac{g_{\mu\nu}}{c^2}dx^\mu dx^\nu)^{\frac{1}{2}}\,, \quad dx^0 = cdt \\ &= (-g_{00} - 2g_{i0}\frac{v^i}{c} - \frac{v^2}{c^2})^{\frac{1}{2}}dt \\ v^2 &\equiv g_{ij}v^i v^j \end{aligned} \tag{7.98}$$

For a diagonal metric tensor ($g_{i0} = 0$) we get

$$d\tau = (-g_{00} - \frac{v^2}{c^2})^{\frac{1}{2}} dt \ , \quad v^2 = g_{ii}(v^i)^2 \tag{7.99}$$

We now look at an idealized situation where a plane flies at constant altitude and with constant speed along the equator.

$$d\tau = (1 - \frac{R_S}{r} - \frac{v^2}{c^2})^{\frac{1}{2}} dt \ , \quad r = R + h \tag{7.100}$$

To the lowest order in $\frac{R_S}{r}$ and $\frac{v^2}{c^2}$ we get

$$d\tau = (1 - \frac{R_S}{2r} - \frac{1}{2}\frac{v^2}{c^2})dt \tag{7.101}$$

The speed of the moving clock is

$$v = (R + h)\Omega + u \tag{7.102}$$

where Ω is the angular velocity of the Earth and u is the speed of the plane. A series expansion and use of this value for v gives

$$\Delta\tau = (1 - \frac{GM}{Rc^2} - \frac{1}{2}\frac{R^2\Omega^2}{c^2} + \frac{gh}{c^2} - \frac{2R\Omega u + u^2}{2c^2})\Delta t \ , \quad g = \frac{GM}{R^2} - R\Omega^2 \tag{7.103}$$

$u > 0$ when flying in the direction of the Earth's rotation, i.e. eastwards. For a clock that is left on the airport (stationary, $h = u = 0$) we get

$$\Delta\tau_0 = (1 - \frac{GM}{Rc^2} - \frac{1}{2}\frac{R^2\Omega^2}{c^2})\Delta t \tag{7.104}$$

To the lowest order the relative difference in travel time is

$$\boxed{k = \frac{\Delta\tau - \Delta\tau_0}{\Delta\tau_0} \cong \frac{gh}{c^2} - \frac{2R\Omega u + u^2}{2c^2}} \tag{7.105}$$

Measurements:

Travel time:	$\Delta\tau_0$	$= 1.2 \times 10^5 s$ (a little over 24h)
Traveling eastwards:	k_e	$= -1.0 \times 10^{-12}$
Traveling westwards:	k_w	$= \ \ 2.1 \times 10^{-12}$
$(\Delta\tau - \Delta\tau_0)_e$		$= -1.2 \times 10^{-7}s \approx -120ns$
$(\Delta\tau - \Delta\tau_0)_w$		$= \ \ 2.5 \times 10^{-7}s \approx 250ns$

7.8.2 Mercury's perihelion precession

The orbit equation for a planet orbiting a star of mass M is given by equation (7.97),

$$\frac{d^2u}{d\phi^2} + u = \frac{GMm^2}{{p_\phi}^2} + ku^2 \tag{7.106}$$

where $k = 3GM$. We will be slightly more general, and allow k to be a theory- or situation dependent term. This equation has a circular solution, such that

$$u_0 = \frac{GMm^2}{{p_\phi}^2} + k{u_0}^2 \tag{7.107}$$

With a small perturbation from the circular motion u is changed by u_1, where $u_1 \ll u_0$. To lowest order in u_1 we have

$$\frac{d^2u_1}{d\phi^2} + u_0 + u_1 = \frac{GMm^2}{{p_\phi}^2} + k{u_0}^2 + 2ku_0u_1 \tag{7.108}$$

or

$$\frac{d^2u_1}{d\phi^2} + u_1 = 2ku_0u_1 \quad \Leftrightarrow \quad \frac{d^2u_1}{d\phi^2} + (1-2ku_0)u_1 = 0 \tag{7.109}$$

For $ku_0 \ll 1$ the equilibrium orbit is stable and we get a periodic solution:

$$u_1 = \epsilon u_0 \cos[\sqrt{1-2ku_0}(\phi - \phi_0)] \tag{7.110}$$

where ϵ and ϕ_0 are integration constants. ϵ is the *eccentricity* of the orbit. We can choose $\phi_0 = 0$ and then have

$$\frac{1}{r} = u = u_0 + u_1 = u_0[1 + \epsilon\cos(\sqrt{1-2ku_0}\phi) \tag{7.111}$$

Let $f \equiv \sqrt{1-2ku_0} \ \Rightarrow$

$$\frac{1}{r} = \frac{1}{r_0}(1 + \epsilon\cos f\phi) \tag{7.112}$$

For $f = 1$ ($k = 0$, no relativistic term) this expression describes a non-precessing elliptic orbit (a Kepler-orbit).

For $f < 1$ ($k > 0$) the ellipse is not closed. To give the same value for r as on a given starting point, ϕ has to increase by $\frac{2\pi}{f} > 2\pi$. The extra angle per rotation is $2\pi(\frac{1}{f} - 1) = \Delta\phi_1$.

$$\Delta\phi_1 = 2\pi(\frac{1}{\sqrt{1-2ku_0}} - 1) \approx 2\pi ku_0 \tag{7.113}$$

Using general relativity we get for Mercury

$$k = 3GM \quad \Rightarrow \quad \Delta\phi = 6\pi GMu_0 \approx 6\pi GM\frac{GMm^2}{{p_\phi}^2} \tag{7.114}$$

$$\boxed{\Delta\phi = 6\pi(\frac{GMm}{p_\phi})^2 \text{per orbit.}} \tag{7.115}$$

which in Mercury's case amounts to $(\Delta\phi)_{\text{century}} = 43''$

7.8.3 Deflection of light

The orbit equation for a free particle with mass $m = 0$ is

$$\frac{d^2u}{d\phi^2} + u = ku^2 \tag{7.116}$$

If light is not deflected it will follow the straight line

$$\cos\phi = \frac{b}{r} = b\,u_0 \tag{7.117}$$

where b is the impact parameter of the path. This is the horizontal dashed line in Figure 7.7. The 0'th order solution (7.117) fullfills

$$\frac{d^2u_0}{d\phi^2} + u_0 = 0 \tag{7.118}$$

Hence it is a solution of (7.116) with $k = 0$.

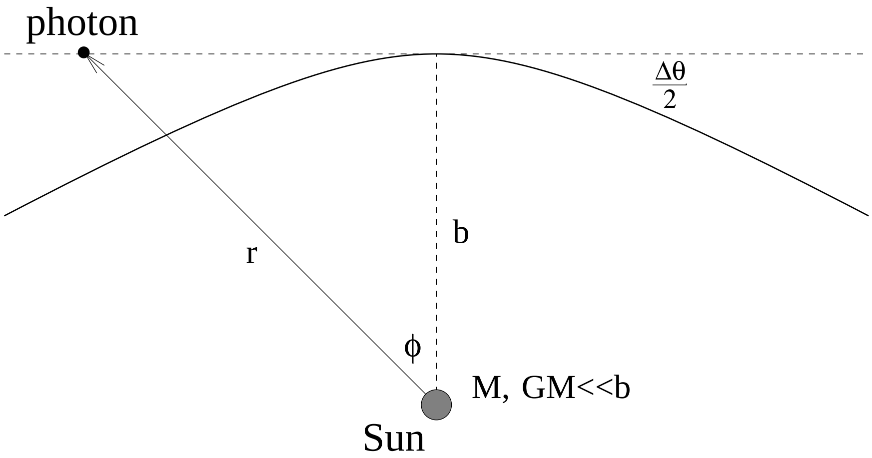


Figure 7.7: Light traveling close to a massive object is deflected.

The perturbed solution is

$$u = u_0 + u_1\,, \quad |u_1| \ll u_0 \tag{7.119}$$

Inserting this into the orbit equation gives

$$\frac{d^2u_0}{d\phi^2} + \frac{d^2u_1}{d\phi^2} + u_0 + u_1 = ku_0^2 + 2ku_0u_1 + ku_1^2 \tag{7.120}$$

The first and third term at the left hand side cancel each other due to eq. (7.118] and the last term at the right hand side is small to second order in u_1 and will be neglected. Hence we get

$$\frac{d^2u_1}{d\phi^2} + u_1 = ku_0^2 + 2ku_0u_1 \tag{7.121}$$

The last term at the right hand side is much smaller then the first, and will also be neglected. Inserting for u_0 from (7.117) we then get

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{k}{b^2}\cos^2\phi \tag{7.122}$$

This equation has a particular solution of the form

$$u_{1p} = A + B\cos^2\phi \tag{7.123}$$

Inserting this into (7.122) we find

$$A = \frac{2k}{3b^2}\,, \quad B = -\frac{k}{3b^2} \tag{7.124}$$

Hence

$$u_{1p} = \frac{k}{3b^2}\left(2 - \cos^2\phi\right) \tag{7.125}$$

giving

$$\frac{1}{r} = u = u_0 + u_1 = \frac{\cos\phi}{b} + \frac{k}{3b^2}\left(2 - \cos^2\phi\right) \tag{7.126}$$

The deflection of the light $\Delta\theta$ is assumed to be small. We therefore put $\phi = \frac{n}{2} + \frac{\Delta\theta}{2}$ where $\Delta\theta \ll \pi$, (see Figure 7.7). Hence

$$\cos\phi = \cos\left(\frac{n}{2} + \frac{\Delta\theta}{2}\right) = -\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2} \tag{7.127}$$

Thus, the term $\cos^2\phi$ in (7.126) can be neglected. Furthermore, the deflection of the light is found by letting $r \to \infty$, i. e. $u \to 0$. Then we get

$$\Delta\theta = \frac{4k}{3b} \tag{7.128}$$

For motion in the Schwarzschild spacetime outside the Sun, $k = \frac{3}{2}R_S$ where R_S is the Schwarzschild radius of the Sun. And for light passing the surface of the Sun $b = R_\odot$ where $R_\odot$ is the actual radius of the Sun. The deflection is then

$$\Delta\theta = 2\frac{R_S}{R_\odot} = 1.75'' \tag{7.129}$$

Chapter 8

Black Holes

8.1 'Surface gravity':gravitational acceleration on the horizon of a black hole

Surface gravity is denoted by κ_1 and is defined by

$$\kappa = \lim_{r \to r_+} \frac{a}{u^t} \qquad a = \sqrt{a_\mu a^\mu} \tag{8.1}$$

where r_+ is the horizon radius, $r_+ = R_S$ for the Schwarzschild spacetime, u^t is the time component of the 4-velocity.

The 4-velocity of a free particle instantanously at rest in the Schwarzschild spacetime:

$$\vec{u} = u^t \vec{e}_t = \frac{dt}{d\tau}\vec{e}_t = \frac{1}{\sqrt{-g_{tt}}}\vec{e}_t = \frac{\vec{e}_t}{\sqrt{1 - \frac{R_S}{r}}} \tag{8.2}$$

The only component of the 4-acceleration different from zero, is a_r. The 4-acceleration:$\vec{a} = \nabla_{\vec{u}}\vec{u} = u^\mu{}_{;\nu} u^\nu \vec{e}_\mu = (u^\mu{}_{,\nu} + \Gamma^\mu{}_{\alpha\nu} u^\alpha) u^\nu \vec{e}_\mu$.

$$\begin{aligned}
a_r &= (u_{r,\nu} + \Gamma_{r\alpha\nu} u^\alpha) u^\nu \\
&= \underbrace{u_{r,\nu} u^\nu}_{=0} + \Gamma_{rtt} (u^t)^2 \\
&= \frac{\Gamma_{rtt}}{1 - \frac{R_S}{r}} \\
\Gamma_{rtt} &= -\frac{1}{2}\frac{\partial g_{tt}}{\partial r} = -\frac{R_S}{2r^2} \\
a_r &= \frac{\frac{R_S}{2r^2}}{1 - \frac{R_S}{r}} \\
a^r &= g^{rr} a_r = \frac{a_r}{g_{rr}} = (1 - \frac{R_S}{r}) a_r = \frac{R_S}{2r^2}
\end{aligned} \tag{8.3}$$

The acceleration scalar: $a = \sqrt{a_r a^r} = \frac{\frac{R_S}{2r^2}}{\sqrt{1 - \frac{R_S}{r}}}$ (measured with standard instruments: at the horizon, time is not running).

$$\frac{a}{u^t} = \frac{R_S}{2r^2} \tag{8.4}$$

With c:

$$\frac{a}{u^t} = \frac{c^2 R_S}{2r^2} = \frac{GM}{r^2} \tag{8.5}$$

$$\kappa = \lim_{r \to R_S} \frac{a}{u^t} = \frac{1}{2R_S} = \frac{1}{4GM} \tag{8.6}$$

Including c the expression is $\kappa = \frac{c^2}{4GM}$. On the horizon of a black hole with one solar mass, we get $\kappa_\odot = 2 \times 10^{13} \frac{m}{s^2}$.

8.2 Hawking radiation:radiation from a black hole (1973)

The radiation from a black hole has a thermal spectrum. We are going to 'find' the temperature of a Schwarzschild black hole of mass M. The Planck spectrum has an intensity maximum at a wavelength given by Wien's displacement law.

$$\Lambda = \frac{N\hbar c}{kT} \quad \text{where k is the Boltzmann constant, and N=0.2014}$$

For radiation emitted from a black hole, Hawking derived the following expression for the wavelength at a maximum intensity

$$\Lambda = 4\pi N R_S = \frac{8\pi NGM}{c^2} \tag{8.7}$$

Inserting Λ from Wien's displacement law, gives:

$$T = \frac{\hbar c^3}{8\pi GkM} = \frac{\hbar c}{2\pi k}\kappa \tag{8.8}$$

Inserting values for $\hbar$, c and k gives:

$$T \approx \frac{2 \times 10^{-4} m}{R_S} K \tag{8.9}$$

For a black hole with one solar mass,we have $T_\odot \approx 10^{-7}$. When the mass is decreasing because of the radiation, the temperature is *increasing*.So a black hole has a negative heat capacity. The energy loss of a black hole because of radiation, is given by the Stefan-Boltzmann law:

$$-\frac{dM}{dt} = \sigma T^4 \frac{A}{c^2} \tag{8.10}$$

where A is the surface of the horizon.

$$A = 4\pi R_S^2 = \frac{16\pi G^2 M^2}{c^4} \tag{8.11}$$

gives:

$$-\frac{dM}{dt} = \frac{1}{15360\pi}\frac{\hbar c^6}{G^2M^2} \equiv \frac{Q}{M^2}$$
$$M(t) = (M_0^3 - 3Qt)^{1/3}, \quad M_0 = M(0) \tag{8.12}$$

A black hole with mass M_0 early in the history of the universe which is about to explode now, had to have a starting mass

$$M_0 = (3Qt_0)^{1/3} \approx 10^{12} kg \tag{8.13}$$

about the mass of a mountain. They are called 'mini black holes'.

8.3 Rotating Black Holes: The Kerr metric

This solution was found by Roy Kerr in 1963.

A time-independent, time-orthogonal metric is known as a **static** metric. A time-independent metric is known as a **stationary** metric. A stationary metric allows rotation.

Consider a stationary metric which describes a axial-symmetric space

$$ds^2 = -e^{2\nu}dt^2 + e^{2\mu}dr^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\lambda}d\theta^2 \ , \tag{8.14}$$

where ν, μ, ψ, λ and ω are functions of r and θ.

By solving the vacuum field equations for this line-element, Kerr found the solution:

$$\begin{aligned} &e^{2\nu} = \frac{\rho^2\Delta}{\Sigma^2} \ , \quad e^{2\mu} = \frac{\rho^2}{\Delta} \ , \quad e^{2\psi} = \frac{\Sigma^2}{\rho^2}\sin^2\theta \ , \quad e^{2\lambda} = \rho^2 \ , \\ &\omega = \frac{2Mar}{\Sigma^2} \ , \quad \text{where} \quad \rho^2 = r^2 + a^2\cos^2\theta \\ &\Delta = r^2 + a^2 - 2Mr \\ &\Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\theta \end{aligned} \tag{8.15}$$

This is the Kerr solution expressed in Boyer-Lindquist coordinates. The function ω is the angular-velocity. The Kerr-solution is the metric for space-time outside a rotating mass-distribution. The constant a is spin per mass-unit for the mass-distribution and M is its mass.

Line-element:

$$\begin{aligned} ds^2 = &-(1 - \frac{2Mr}{\rho^2})dt^2 + \frac{\rho^2}{\Delta}dr^2 - \frac{4Mar}{\rho^2}\sin^2\theta dt d\phi + \rho^2 d\theta^2 \\ &+ (r^2 + a^2 + \frac{2Ma^2r}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2 \end{aligned} \tag{8.16}$$

(Here M is a measure of the mass so that $M = G\cdot\text{mass}$, ie. $G = 1$)

Light emitted from the surface, $r = r_0$, where $g_{tt} = 0$ is infinitely redshifted further out. Observed from the outside time stands still.

$$\begin{aligned} \rho^2 = 2Mr_0 \quad &\Rightarrow \quad r_0^2 + a^2\cos^2\theta = 2Mr_0 \\ &r_0 = M \pm \sqrt{M^2 - a^2\cos^2\theta} \end{aligned} \tag{8.17}$$

This is the equation for the surface which represents infinite redshift.

8.3.1 Zero-angular-momentum-observers (ZAMO's)

The Lagrange function of a free particle in the equator plane, $\theta = \frac{\pi}{2}$

$$L = -\frac{1}{2}(e^{2\nu} - \omega^2 e^{2\psi})\dot{t}^2 + \frac{1}{2}e^{2\mu}\dot{r}^2 + \frac{1}{2}e^{2\psi}\dot{\phi}^2 + \frac{1}{2}e^{2\lambda}\dot{\theta}^2 - \omega e^{2\psi}\dot{t}\dot{\phi} \tag{8.18}$$

Here $\dot{\theta} = 0$. The momentum p_ϕ of the cyclic coordinates ϕ:

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = e^{2\psi}(\dot{\phi} - \omega\dot{t}) \ , \quad \dot{t} = \frac{dt}{d\tau} \ , \quad \dot{\phi} = \frac{d\phi}{d\tau} \tag{8.19}$$

The angular speed of the particle relative to the coordinate system:

$$\begin{aligned} \Omega = \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} \ , \quad \dot{\phi} = \Omega\dot{t} \\ \Rightarrow p_\phi = e^{2\psi}\dot{t}(\Omega - \omega) \end{aligned} \tag{8.20}$$

p_ϕ is conserved during the movement.

$$\begin{aligned} \omega = \frac{2Mar}{(r^2+a^2)^2 - a^2(r^2+a^2-2Mr)} \ , \\ \omega \to 0 \quad \text{when} \quad r \to \infty \end{aligned} \tag{8.21}$$

When studying the Kerr metric one finds that Kerr $\to$ Minkowski for large r. The coordinate clocks in the Kerr space-time show the same time as the standard-clocks at rest in the asymptotic Minkowski space-time.

A ZAMO is per definition a particle or observer with $p_\phi = 0$. Consider a far away observer who let a stone fall with vanishing initial velocity. p_ϕ is a constant of motion, so the stone remains a ZAMO during the movement. A local reference frame which coincides with the stone is a local inertial frame.

$$p_\phi = 0 \Rightarrow \Omega = \frac{d\phi}{dt} = \omega \tag{8.22}$$

That is, the local inertial frame obtains an angular speed relative to the BL-system (Boyer-Lindquist system).

Since the Kerr metric is time independent, the BL-system is stiff. The distant observer has no motion relative to the BL-system. To this observer the BL-system will appear stiff and non-rotating. The observer will observe that the local inertial system of the stone obtains an angular speed

`a is spin per mass unity and Ma is spin`

$$\frac{d\phi}{dt} = \omega = \frac{2Mar}{(r^2+a^2)^2 - a^2(r^2+a^2-2Mr)} \tag{8.23}$$

In other words, inertial systems at finite distances from the rotating mass M are dragged with it in the same direction. This is known as **inertial dragging** or the Lense-Thirring effect (about 1920).

8.3.2 Does the Kerr space have a horizon?

Definition 8.3.1 (Horizon)
a surface one can enter, but not exit.

Consider a particle in an orbit with constant r and θ. It's 4-velocity is:

$$\begin{aligned}\vec{u} &= \frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{x}}{dt} \\ &= (-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2)^{-\frac{1}{2}}(1, \Omega) \ , \qquad \text{where } \Omega = \frac{d\phi}{dt}\end{aligned} \tag{8.24}$$

To have stationary orbits the following must be true

$$g_{\phi\phi}\Omega^2 + 2g_{t\phi}\Omega + g_{tt} < 0 \tag{8.25}$$

This implies that Ω must be in the interval

$$\Omega_{min} < \Omega < \Omega_{max} \ , \tag{8.26}$$

where $\Omega_{min} = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}}$, $\Omega_{max} = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}}$ since $g_{t\phi} = -\omega g_{\phi\phi}$.

Outside the surface with infinite redshift $g_{tt} < 0$. That is Ω can be negative, zero and positive. Inside the surface $r = r_0$ with infinite redshift $g_{tt} > 0$. Here $\Omega_{min} > 0$ and static particles, $\Omega = 0$, cannot exist. This is due to the inertial dragging effect. The surface $r = r_0$ is therefore known as "the static border".

The interval of Ω, where stationary orbits are allowed, is reduced to zero when $\Omega_{min} = \Omega_{max}$, that is $\omega^2 = \frac{g_{tt}}{g_{\phi\phi}} \Rightarrow g_{tt} = \omega^2 g_{\phi\phi}$ (equation for the horizon).

For the Kerr metric we have:

$$g_{tt} = \omega^2 g_{\phi\phi} - e^{2\nu} \tag{8.27}$$

Therefore the horizon equation becomes

$$e^{2\nu} = 0 \quad \Rightarrow \Delta = 0 \qquad \therefore r^2 - 2mr + a^2 = 0 \tag{8.28}$$

The largest solution is $r_+ = M + \sqrt{M^2 - a^2}$ and this is the equation for a spherical surface. The static border is $r_0 = M + \sqrt{M^2 - a^2 \cos\theta}$.

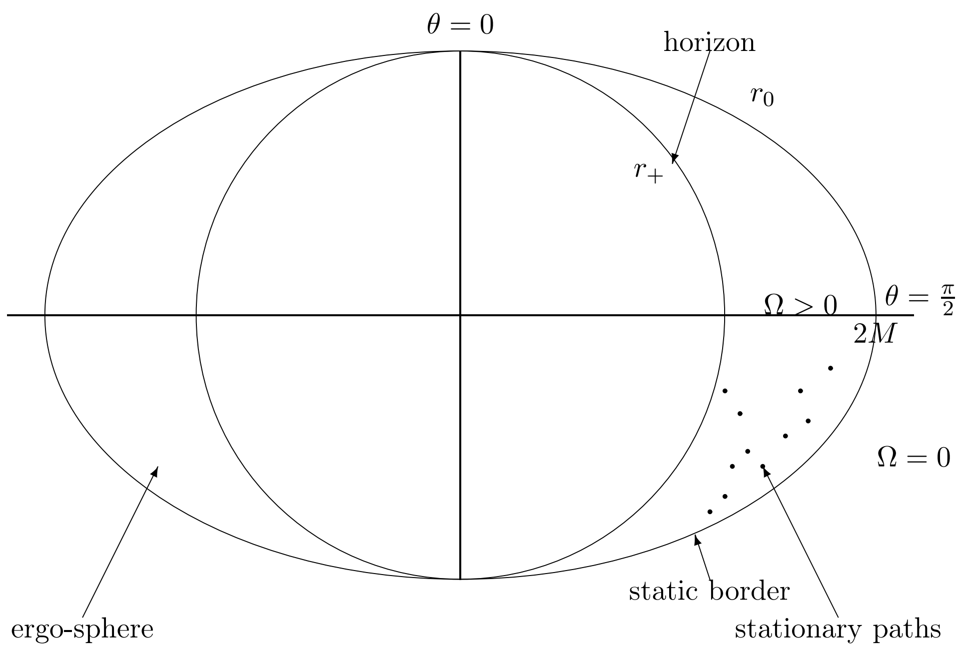


Figure 8.1: Static border and horizon of a Kerr black hole

Chapter 9

Schwarzschild's Interior Solution

9.1 Newtonian incompressible star

$$\begin{aligned} \nabla^2\phi &= 4\pi G\rho, \quad \phi = \phi(r) \\ \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\phi}{dr}) &= 4\pi G\rho \end{aligned} \tag{9.1}$$

Assuming $\rho = constant$.

$$\begin{aligned} d(r^2\frac{d\phi}{dr}) &= 4\pi G\rho r^2 dr \\ r^2\frac{d\phi}{dr} &= \frac{4\pi}{3}G\rho r^3 + K \\ &= M(r) + K \end{aligned} \tag{9.2}$$

Gravitational acceleration: $\vec{g} = -\nabla\phi = -\frac{d\phi}{dr}\vec{e}_r$

$$g = \frac{M(r)}{r^2} + \frac{K_1}{r^2} = \frac{4\pi}{3}G\rho r + \frac{K_1}{r^2} \tag{9.3}$$

Finite g in $r = 0$ demands $K_1 = 0$.

$$g = \frac{4\pi}{3}G\rho r, \qquad \frac{d\phi}{dr} = \frac{4\pi}{3}G\rho r \tag{9.4}$$

Assume that the massdistribution has a radius R.

$$\phi = \frac{2\pi}{3}G\rho r^2 + K_2 \tag{9.5}$$

Demands continuous potensial at $r = R$.

$$\begin{aligned} \frac{2\pi}{3}G\rho R^2 + K_2 = \frac{M(R)}{R} &= -\frac{4\pi}{3}G\rho R^2 \\ \Rightarrow K_2 &= -2\pi G\rho R^2 \end{aligned} \tag{9.6}$$

(with zero level at infinite distance). Gives the potensial inside the mass distribution:

$$\phi = \frac{2\pi}{3}G\rho(r^2 - 3R^2) \tag{9.7}$$

The star is in hydrostatic equilibrium, that is, the pressure forces are in equilibrium with the gravitational forces.

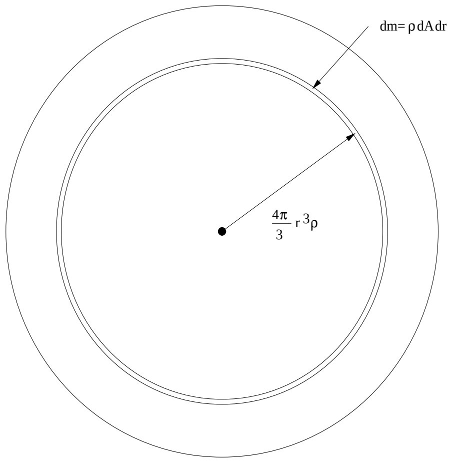


Figure 9.1: The shell with thickness dr, is affected by both gravitational and pressure forces.

Consider a mass element, $dm = \rho dV = \rho dAdr$, in the shell depicted in figure 9.1. The pressure force on the mass element is $dF = dAdp$, and the gravitational force is

$$dG = gdm = \frac{Gm(r)}{r^2}dm \tag{9.8}$$

where $m(r)$ is the mass inside the shell. With constant density $m(r) = (4\pi/3)\rho r^3$. Hence

$$dG = gdm = \frac{4\pi}{3}G\rho^2 rdAdr \tag{9.9}$$

Equilibrium, $dF = -dG$, demands that

$$dp = -\frac{4\pi}{3}G\rho^2 rdr \tag{9.10}$$

Integrating this gives

$$p = K_3 - \frac{2\pi G}{3}\rho^2 r^2 \tag{9.11}$$

$p(R) = 0$ gives the value of the constant of integration K_3

$$K_3 = \frac{2\pi G}{3}\rho^2 R^2 \tag{9.12}$$

and we find

$$p(r) = \frac{2\pi G}{3}\rho^2(R^2 - r^2) \tag{9.13}$$

No matter how massive the star is, it is possible for the pressure forces to keep the equilibrium with gravity. In Newtonian theory, gravitational collapse is not a necessity.

9.2 The pressure contribution to the gravitational mass of a static, spherical symmetric system

We now give a new definition of the gravitational acceleration (not equivalent to (7.23))

$$g = +\frac{a}{u^t} \ , \qquad a = \sqrt{a_\mu a^\mu} \tag{9.14}$$

We have the line element:

$$\begin{gathered} ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \\ g_{tt} = -e^{2\alpha} \ , \quad g_{rr} = e^{2\beta} \end{gathered} \tag{9.15}$$

gives (because of the gravitational acceleration)

$$g = +e^{\alpha-\beta}\alpha' \tag{9.16}$$

From the expressions for $E_{\hat{t}\hat{t}}$, $E_{\hat{r}\hat{r}}$, $E_{\hat{\theta}\hat{\theta}}$, $E_{\hat{\phi}\hat{\phi}}$ follow (see Section 7.1)

$$E^{\hat{t}}_{\ \hat{t}} - E^{\hat{r}}_{\ \hat{r}} - E^{\hat{\theta}}_{\ \hat{\theta}} - E^{\hat{\phi}}_{\ \hat{\phi}} = -2e^{2\beta}(\frac{2\alpha'}{r} + \alpha'' + \alpha'^2 - \alpha'\beta') \ . \tag{9.17}$$

We also have

$$(r^2 e^{\alpha-\beta}\alpha')' = r^2 e^{\alpha-\beta}(\frac{2\alpha'}{r} + \alpha'' + \alpha'^2 - \alpha'\beta') \ , \tag{9.18}$$

which gives

$$g = +\frac{1}{2r^2}\int (E^{\hat{t}}_{\ \hat{t}} - E^{\hat{r}}_{\ \hat{r}} - E^{\hat{\theta}}_{\ \hat{\theta}} - E^{\hat{\phi}}_{\ \hat{\phi}}) r^2 e^{\alpha+\beta} dr \ . \tag{9.19}$$

By applying Einstein's field equations

$$E^{\hat{\mu}}_{\ \hat{\nu}} = 8\pi G T^{\hat{\mu}}_{\ \hat{\nu}} \tag{9.20}$$

we get

$$g = +\frac{4\pi G}{r^2}\int (T^{\hat{t}}_{\ \hat{t}} - T^{\hat{r}}_{\ \hat{r}} - T^{\hat{\theta}}_{\ \hat{\theta}} - T^{\hat{\phi}}_{\ \hat{\phi}}) r^2 e^{\alpha+\beta} dr \ . \tag{9.21}$$

This is the Tolman-Whittaker expression for gravitational acceleration.

The corresponding Newtonian expression is :

$$g_N = -\frac{4\pi G}{r^2}\int \rho r^2 dr \tag{9.22}$$

The relativistic gravitational mass density is therefore defined as

$$\rho_G = -T^{\hat{t}}_{\ \hat{t}} + T^{\hat{r}}_{\ \hat{r}} + T^{\hat{\theta}}_{\ \hat{\theta}} + T^{\hat{\phi}}_{\ \hat{\phi}} \tag{9.23}$$

For an isotropic fluid with

$$T^{\hat{t}}_{\hat{t}} = -\rho \ , \quad T^{\hat{r}}_{\hat{r}} = T^{\hat{\theta}}_{\hat{\theta}} = T^{\hat{\phi}}_{\hat{\phi}} = p \tag{9.24}$$

we get $\rho_G = \rho + 3p$ (with $c = 1$), which becomes

$$\boxed{\rho_G = \rho + \frac{3p}{c^2}} \tag{9.25}$$

It follows that in relativity, pressure has a gravitational effect. Greater pressure gives increasing gravitational attraction. Strain ($p < 0$) decreases the gravitational attraction.

In the Newtonian limit, $c \to \infty$, pressure has no gravitational effect.

9.3 The Tolman-Oppenheimer-Volkov equation

With spherical symmetry the spacetime line-element may be written

$$\begin{aligned} ds^2 &= -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \\ E_{\hat{t}\hat{t}} &= 8\pi G T_{\hat{t}\hat{t}} \ , \quad T^{\hat{\mu}}_{\hat{\nu}} = diag(-\rho, p, p, p) \end{aligned} \tag{9.26}$$

From $E_{\hat{t}\hat{t}}$ we get

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}[r(1 - e^{-2\beta})] &= 8\pi G\rho \\ r(1 - e^{-2\beta}) &= 2G\int_0^r 4\pi\rho r^2 dr \ , \end{aligned} \tag{9.27}$$

where $m(r) = \int_0^r 4\pi\rho r^2 dr$ giving

$$e^{-2\beta} = 1 - \frac{2Gm(r)}{r} = \frac{1}{g_{rr}} \tag{9.28}$$

From $E_{\hat{r}\hat{r}}$ we have

$$\begin{aligned} E_{\hat{r}\hat{r}} &= 8\pi G T_{\hat{r}\hat{r}} \\ \frac{2}{r}\frac{d\alpha}{dr}e^{-2\beta} - \frac{1}{r^2}(1 - e^{-2\beta}) &= 8\pi G p \end{aligned} \tag{9.29}$$

We get

$$\frac{2}{r}\frac{d\alpha}{dr}(1 - \frac{2Gm(r)}{r}) - \frac{2Gm(r)}{r^3} = 8\pi G p$$

$$\frac{d\alpha}{dr} = G\frac{m(r) + 4\pi r^3 p(r)}{r(r - 2Gm(r))} \tag{9.30}$$

The relativistic generalized equation for hydrostatic equilibrium is $T^{\hat{r}\hat{\nu}}_{;\hat{\nu}} = 0$, giving

$$\begin{aligned} T^{\hat{r}\hat{\nu}}_{,\hat{\nu}} + \Gamma^{\hat{\nu}}_{\hat{\alpha}\hat{\nu}}T^{\hat{r}\hat{\alpha}} + \Gamma^{\hat{r}}_{\hat{\alpha}\hat{\nu}}T^{\hat{\alpha}\hat{\nu}} &= 0 \\ T^{\hat{r}\hat{\nu}}_{,\hat{\nu}} = T^{\hat{r}\hat{r}}_{,\hat{r}} = p_{,\hat{r}} &= \frac{1}{\sqrt{g_{rr}}}\frac{\partial p}{\partial r} \\ T^{\hat{r}\hat{\nu}}_{,\hat{\nu}} &= e^{-\beta}\frac{dp}{dr} \\ \Gamma^{\hat{\nu}}_{\hat{\alpha}\hat{\nu}}T^{\hat{r}\hat{\alpha}} = \Gamma^{\hat{\nu}}_{\hat{r}\hat{\nu}}p &= \Gamma^{\hat{t}}_{\hat{r}\hat{t}}p + \Gamma^{\hat{\alpha}}_{\hat{r}\hat{\alpha}}p \\ \Gamma^{\hat{r}}_{\hat{\alpha}\hat{\nu}}T^{\hat{\alpha}\hat{\nu}} = \Gamma^{\hat{r}}_{\hat{\nu}\hat{\nu}}T^{\hat{\nu}\hat{\nu}} &= \Gamma^{\hat{r}}_{\hat{t}\hat{t}}\rho + \Gamma^{\hat{r}}_{\hat{\alpha}\hat{\alpha}}p \end{aligned} \tag{9.31}$$

In orthonormal basis we have

$$\begin{aligned} \underline{\Omega}_{\hat{\nu}\hat{\mu}} = -\underline{\Omega}_{\hat{\mu}\hat{\nu}} &\Rightarrow \Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} = -\Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}} \\ \Gamma^{\hat{\alpha}}_{\hat{r}\hat{\alpha}} = \Gamma_{\hat{\alpha}\hat{r}\hat{\alpha}} &= -\Gamma_{\hat{r}\hat{\alpha}\hat{\alpha}} = -\Gamma^{\hat{r}}_{\hat{\alpha}\hat{\alpha}} \end{aligned} \tag{9.32}$$

$T^{\hat{r}\hat{\nu}}_{;\hat{\nu}} = 0$ now takes the form:

$$e^{-\beta}\frac{dp}{dr} + \Gamma^{\hat{t}}_{\hat{r}\hat{t}}p + \Gamma^{\hat{r}}_{\hat{t}\hat{t}}\rho \tag{9.33}$$

We have

$$\Gamma^{\hat{t}}_{\hat{r}\hat{t}} = -\Gamma_{\hat{t}\hat{r}\hat{t}} = \Gamma_{\hat{r}\hat{t}\hat{t}} = \Gamma^{\hat{r}}_{\hat{t}\hat{t}} \tag{9.34}$$

and we also have $\Gamma^{\hat{r}}_{\hat{t}\hat{t}} = e^{-\beta}\frac{d\alpha}{dr}$, giving:

$$\frac{dp}{dr} + (p+\rho)\frac{d\alpha}{dr} = 0 \tag{9.35}$$

Inserting Equation 9.30 into Equation 9.35 gives

$$\boxed{\frac{dp}{dr} = -G(\rho+p)\frac{m(r) + 4\pi r^3 p(r)}{r(r-2Gm(r))}} \tag{9.36}$$

This is the Tolman-Oppenheimer-Volkov (TOV) equation. The component $g_{tt} = -e^{2\alpha(r)}$ may now be calculated as follows

$$\begin{aligned} \frac{dp}{\rho+p} &= -d\alpha\ , \qquad \rho = \text{constant} \\ ln(\rho+p) &= K - \alpha \\ \rho + p &= K_1 e^{-\alpha}\ , \quad p = K_1 e^{-\alpha} - \rho \end{aligned} \tag{9.37}$$

Hence

$$e^{\alpha} = e^{\alpha(R)}(1 + \frac{p}{\rho})^{-1} \tag{9.38}$$

where R is the radius of the mass distribution.

9.4 An exact solution for incompressible stars - Schwarzschild's interior solution

The mass inside a radius r for an incompressiable star is

$$m(r) = \frac{4}{3}\pi\rho r^3 \tag{9.39}$$

$$e^{-2\beta} = 1 - \frac{2Gm(r)}{r} \equiv 1 - \frac{r^2}{a^2} \tag{9.40}$$

where

$$a^2 = \frac{3}{8\pi G\rho}, \quad m(r) = \frac{r^3}{2Ga^2}, \quad r_s = 2Gm = \frac{r^2}{a^2}r \tag{9.41}$$

TOV equation:

$$\begin{aligned}
\frac{dp}{dr} &= -G\frac{\frac{4}{3}\pi\rho r^3 + 4\pi r^3 p(r)}{r(r - 2G\frac{4}{3}\pi\rho r^3)}(\rho + p(r)) \\
&= -G\frac{4}{3}\pi\frac{\rho + 3p(r)}{1 - G\frac{8}{3}\pi\rho r^2}r(\rho + p(r)) \\
&= -\frac{1}{2a^2\rho}\frac{\rho + 3p(r)}{1 - \frac{r^2}{a^2}}r(\rho + p(r)) \\
\Rightarrow \int_0^p \frac{dp}{(\rho + 3p)(\rho + p} &= -\frac{1}{2a^2\rho}\int_R^r \frac{r}{1 - \frac{r^2}{a^2}}dr \\
\frac{p + \rho}{3p + \rho} &= \sqrt{\frac{a^2 - R^2}{a^2 - r^2}}
\end{aligned} \tag{9.42}$$

So the relativistic pressure distribution is

$$p(r) = \frac{\sqrt{a^2 - r^2} - \sqrt{a^2 - R^2}}{3\sqrt{a^2 - R^2} - \sqrt{a^2 - r^2}}\rho, \quad \forall r \leq R \tag{9.43}$$

also

$$a^2 = \frac{3}{8\pi G\rho}, \quad \frac{a^2}{r^2} = \frac{r}{r_s} > 1 \Rightarrow a > r \tag{9.44}$$

To satisfy the condition for hydrostatic equilibrium we must have $p > 0$ or $p(0) > 0$ which gives

$$p(0) \equiv p_c = \frac{a - \sqrt{a^2 - R^2}}{3\sqrt{a^2 - R^2} - a} > 0 \tag{9.45}$$

in which the numerator is positive so that

$$\begin{aligned}
3\sqrt{a^2 - R^2} &> a \\
9a^2 - 9R^2 &> a^2 \\
R &< \sqrt{\frac{8}{9}}a \\
R^2 &< \frac{8}{9}a = \frac{8}{9}\frac{3}{8\pi G\rho} = \frac{1}{3\pi G\rho}
\end{aligned} \tag{9.46}$$

Stellar mass:

$$
\begin{aligned}
M = \frac{4}{3}\pi\rho R^3 &< \frac{4}{3}\pi\rho R \frac{1}{3\pi G\rho} = \frac{4R}{9G} \\
M &< \frac{4}{9G}\frac{1}{\sqrt{3\pi G\rho}}
\end{aligned} \tag{9.47}
$$

For a neutron star we can use $\rho \approx 10^{17}\ \ g/cm^3$. An upper limit on the mass is then $M < 2.5\ M_\odot$ Substitution for p in the expression for e^α gives

$$
e^\alpha = \frac{3}{2}\sqrt{1-\frac{R_s}{R}} - \frac{1}{2}\sqrt{1-\frac{R_s}{R^3}r^2} \tag{9.48}
$$

The line element for the interior Schwarzschild solution is

$$
ds^2 = -\left(\frac{3}{2}\sqrt{1-\frac{R_s}{R}} - \frac{1}{2}\sqrt{1-\frac{R_s}{R^3}r^2}\right)^2 dt^2 + \frac{dr^2}{1-\frac{R_s}{R^3}r^2} + r^2 d\Omega, \quad r \leq R \tag{9.49}
$$

Chapter 10

Cosmology

10.1 Comoving coordinate system

We will consider expanding homogenous and isotropic models of the universe. We introduce an expanding frame of reference with the galactic clusters as reference particles. Then we introduce a 'comoving coordinate system' in this frame of reference with spatial coordinates χ, θ, ϕ. We use time measured on standard clocks carried by the galactic clusters as coordinate time (cosmic time). The line element can then be written in the form:

$$ds^2 = -dt^2 + a(t)^2[d\chi^2 + r(\chi)^2 d\Omega^2] \tag{10.1}$$

(For standard clocks at rest in the expanding system, $d\chi = d\Omega = 0$ and $ds^2 = -d\tau^2 = -dt^2$). The function $a(t)$ is called the expansion factor, and t is called cosmic time.

The physical distance to a galaxy with coordinate distance $d\chi$ from an observer at the origin, is:

$$dl_x = \sqrt{g_{\chi\chi}} d\chi = a(t) d\chi \tag{10.2}$$

Even if the galactic clusters have no coordinate velocity, they do have a radial velocity expressed by the expansion factor.

The value χ determines which cluster we are observing and $a(t)$ how it is moving. 4-velocity of a reference particle (galactic cluster):

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} = (1, 0, 0, 0) \tag{10.3}$$

This applies at an abritrary time, that is $\frac{du^\mu}{dt} = 0$. Geodesic equation: $\frac{du^\mu}{dt} + \Gamma^\mu_{\ \alpha\beta} u^\alpha u^\beta = 0$ which is reduces to: $\Gamma^\mu_{\ tt} = 0$

$$\Gamma^\mu_{\ tt} = \frac{1}{2} g^{\mu\nu} (\overbrace{g_{\nu t,t}}^{0} + \overbrace{g_{t\nu,t}}^{0} + \overbrace{g_{tt,\nu}}^{0}) = 0 \tag{10.4}$$

We have that $g_{tt} = -1$. This shows that the reference particles are freely falling.

10.2 Curvature isotropy - the Robertson-Walker metric

Introduce orthonormal form-basis:

$$\begin{aligned} &\underline{\omega}^{\hat{t}} = \underline{d}t \quad \underline{\omega}^{\hat{\chi}} = a(t)\underline{d}\chi \quad \underline{\omega}^{\hat{\theta}} = a(t)r(\chi)\underline{d}\theta \\ &\underline{\omega}^{\hat{\phi}} = a(t)r(\chi)\sin\theta \underline{d}\phi \end{aligned} \tag{10.5}$$

Using Cartans 1st equation:

$$\underline{d\omega}^{\hat{\mu}} = -\underline{\Omega}^{\hat{\mu}}{}_{\hat{\nu}} \wedge \underline{\omega}^{\hat{\nu}} \tag{10.6}$$

to find the connection forms. Then using Cartans 2nd structure equation to calculate the curvature forms:

$$\underline{R}^{\hat{\mu}}{}_{\hat{\nu}} = \underline{d\Omega}^{\hat{\mu}}{}_{\hat{\nu}} + \underline{\Omega}^{\hat{\mu}}{}_{\hat{\lambda}} \wedge \underline{\Omega}^{\hat{\lambda}}{}_{\hat{\nu}} \tag{10.7}$$

Calculations give: (notation: $\dot{} = \frac{d}{dt}$, $' = \frac{d}{d\chi}$)

$$\begin{aligned} \underline{R}^{\hat{t}}{}_{\hat{i}} &= \frac{\ddot{a}}{a}\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{i}}, \qquad \underline{\omega}^{\hat{i}} = \underline{\omega}^{\hat{\chi}}, \underline{\omega}^{\theta}, \underline{\omega}^{\phi} \\ \underline{R}^{\hat{\chi}}{}_{\hat{j}} &= (\frac{\dot{a}^2}{a^2} - \frac{r''}{ra^2})\underline{\omega}^{\hat{\chi}} \wedge \underline{\omega}^{\hat{j}}, \qquad \underline{\omega}^{\hat{j}} = \underline{\omega}^{\hat{\theta}}, \underline{\omega}^{\hat{\phi}} \\ \underline{R}^{\hat{\theta}}{}_{\hat{\phi}} &= (\frac{\dot{a}^2}{a^2} + \frac{1}{r^2a^2} - \frac{r'^2}{r^2a^2})\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\phi}} \end{aligned} \tag{10.8}$$

The curvature of 3-space ($dt = 0$) can be found by putting $a = 1$. That is:

$$\begin{aligned} {}_3\underline{R}^{\hat{\chi}}{}_{\hat{j}} &= -\frac{r''}{r}\underline{\omega}^{\hat{\chi}} \wedge \underline{\omega}^{\hat{j}} \\ {}_3\underline{R}^{\hat{\theta}}{}_{\hat{\phi}} &= (\frac{1}{r^2} - \frac{r'^2}{r^2})\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\phi}} \end{aligned} \tag{10.9}$$

The 3-space is assumed to be isotropic and homogenous. This demands

$$-\frac{r''}{r} = \frac{1 - r'^2}{r^2} = k \ , \tag{10.10}$$

where k represents the constant curvature of the 3-space.

$$\therefore \ r'' + kr = 0 \quad \text{and} \quad r' = \sqrt{1 - kr^2} \tag{10.11}$$

Solutions with $r(0) = 0$, $r'(0) = 1$:

$$\begin{aligned} \sqrt{-k}r &= \sinh(\sqrt{-k}\chi) \qquad (k < 0) \\ r &= \chi \qquad (k = 0) \\ \sqrt{k}r &= \sin(\sqrt{k}\chi) \qquad (k > 0) \end{aligned} \tag{10.12}$$

The solutions can be characterized by the following 3 cases:

$$\begin{aligned} r &= \sinh\chi, \qquad dr = \sqrt{1+r^2}d\chi, \qquad (k=-1) \\ r &= \chi, \qquad dr = d\chi, \qquad (k=0) \\ r &= \sin\chi, \qquad dr = \sqrt{1-r^2}d\chi, \qquad (k=1) \end{aligned} \tag{10.13}$$

In all three cases one may write $dr = \sqrt{1-kr^2}d\chi$, which is just the last equation above.

We now set $d\chi^2 = \frac{dr^2}{1-kr^2}$ into the line-element :

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\left(d\chi^2 + r^2(\chi)d\Omega^2\right) \\ &= -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \end{aligned} \tag{10.14}$$

The first expression is known as the standard form of the line-element, the second is called the Robertson-Walker line-element.

The 3-space has constant curvature. 3-space is spherical for $k=1$, Euclidean for $k=0$ and hyperbolic for $k=-1$.

Universe models with $k=1$ are known as 'closed' and models with $k=-1$ are known as 'open'. Models with $k=0$ are called 'flat' even though these models also have curved space-time.

10.3 Cosmic dynamics

10.3.1 Hubbles law

The observer is placed in origo of the coordinate-system; $\chi_0 = 0$. The proper distance to a galaxy with radial coordinate χ_e is $D = a(t)\chi_e$. The galaxy has a radial velocity:

$$v = \frac{dD}{dt} = \dot{a}\chi_e = \frac{\dot{a}}{a}D = HD \quad \text{where} H = \frac{\dot{a}}{a} \tag{10.15}$$

The expansion velocity v is proportional to the distance D. This is Hubbles law.

10.3.2 Cosmological redshift of light

Δt_e : the time interval in transmitter-position at transmission-time
Δt_0 : the time interval in receiver-position at receiving-time

Light follows curves with $ds^2 = 0$, with $d\theta = d\phi = 0$ we have :

$$dt = -a(t)d\chi \tag{10.16}$$

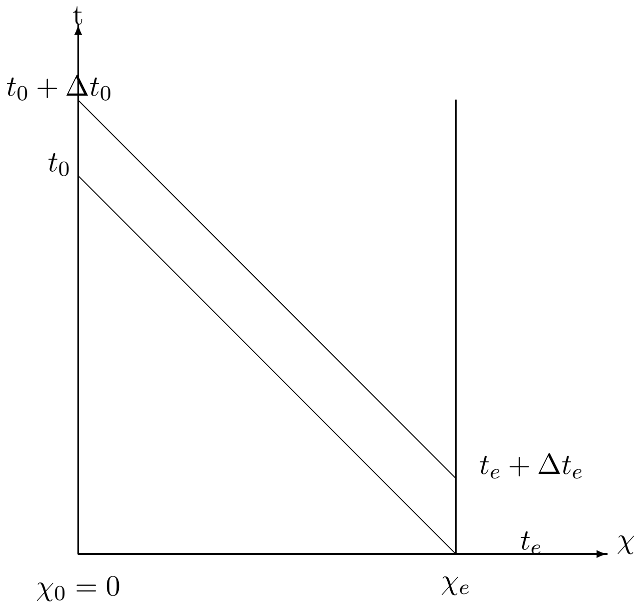


Figure 10.1: Schematic representation of cosmological redshift

Integration from transmitter-event to receiver-event :

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = -\int_{\chi_e}^{\chi_0} d\chi = \chi_e$$
$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} = -\int_{\chi_e}^{\chi_0} d\chi = \chi_e \ ,$$

which gives

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a} - \int_{t_e}^{t_0} \frac{dt}{a} = 0 \tag{10.17}$$

or

$$\int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a} - \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a} = 0 \tag{10.18}$$

Under the integration from t_e to $t_e + \Delta t_e$ the expansion factor $a(t)$ can be considered a constant with value $a(t_e)$ and under the integration from t_0 to $t_0 + \Delta t_0$ with value $a(t_0)$, giving:

$$\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_0}{a(t_0)} \tag{10.19}$$

Δt_0 and Δt_e are intervals of the light at the receiving and transmitting time. Since the wavelength of the light is $\lambda = c\Delta t$ we have:

$$\frac{\lambda_0}{a(t_0)} = \frac{\lambda_e}{a(t_e)} \tag{10.20}$$

This can be interpreted as a "stretching" of the electromagnetic waves due to the expansion of space. The cosmological redshift is denoted by z and is given by:

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{a(t_0)}{a(t_e)} - 1 \tag{10.21}$$

Using $a_0 \equiv a(t_0)$ we can write this as:

$$1 + z(t) = \frac{a_0}{a} \tag{10.22}$$

10.3.3 Cosmic fluids

The energy-momentum tensor for a perfect fluid (no viscosity and no thermal conductivity) is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{10.23}$$

In an orthonormal basis

$$T_{\hat{\mu}\hat{\nu}} = (\rho + p)u_{\hat{\mu}} u_{\hat{\nu}} + p\eta_{\hat{\mu}\hat{\nu}} \tag{10.24}$$

where $\eta_{\hat{\mu}\hat{\nu}}$ is the Minkowski metric. We consider 3 types of cosmic fluid:

1. dust: $p = 0$,
$$T_{\hat{\mu}\hat{\nu}} = \rho u_{\hat{\mu}} u_{\hat{\nu}} \tag{10.25}$$

2. radiation: $p = \frac{1}{3}\rho$,
$$\begin{aligned} T_{\hat{\mu}\hat{\nu}} &= \frac{4}{3}\rho u_{\hat{\mu}} u_{\hat{\nu}} + p\eta_{\hat{\mu}\hat{\nu}} \\ &= \frac{\rho}{3}(4u_{\hat{\mu}} u_{\hat{\nu}} + \eta_{\hat{\mu}\hat{\nu}}) \end{aligned} \tag{10.26}$$

   The trace
$$T = T^{\hat{\mu}}_{\ \hat{\mu}} = \frac{\rho}{3}(4u^{\hat{\mu}} u_{\hat{\mu}} + \delta^{\mu}_{\ \mu}) = 0 \tag{10.27}$$

3. vacuum: $p = -\rho$,
$$T_{\hat{\mu}\hat{\nu}} = -\rho\eta_{\hat{\mu}\hat{\nu}} \tag{10.28}$$

If vacuum can be described as a perfect fluid we have $p_v = -\rho_v$, where ρ is the energy density. It can be related to Einstein's cosmological constant, $\Lambda = 8\pi G\rho_v$.

One has also introduced a more general type of vacuum energy given by the equation of state $p_\phi = w\rho_\phi$, where ϕ denotes that the vacuum energy is connected to a scalar field ϕ. In a homogeneous universe the pressure and the density are given by

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \qquad \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{10.29}$$

where $V(\phi)$ is the potential for the scalar field. Then we have

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{10.30}$$

The special case $\dot{\phi} = 0$ gives the Lorentz invariant vacuum with $w = -1$. The more general vacuum is called "quintessence".

10.3.4 Isotropic and homogeneous universe models

We will discuss isotropic and homogenous universe models with perfect fluid and a non-vanishing cosmological constant Λ. Calculating the components of the Einstein tensor from the line-ement (10.14) we find in an orthonormal basis

$$E_{\hat{t}\hat{t}} = \frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} \tag{10.31}$$

$$E_{\hat{m}\hat{m}} = -\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2}. \tag{10.32}$$

The components of the energy-momentum tensor of a perfect fluid in a comoving orthonormal basis are

$$T_{\hat{t}\hat{t}} = \rho, \qquad T_{\hat{m}\hat{m}} = p. \tag{10.33}$$

Hence the $\hat{t}\hat{t}$ component of Einstein's field equations is

$$3\frac{\dot{a}^2 + k}{a^2} = 8\pi G\rho + \Lambda \tag{10.34}$$

$\hat{m}\hat{m}$ components:

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 8\pi G p - \Lambda \tag{10.35}$$

where ρ is the energy density and p is the pressure. The equations with vanishing cosmological constant are called the Friedmann equations. Inserting eq. (10.34) into eq. (10.35) gives:

$$\ddot{a} = -\frac{4\pi G}{3}a(\rho + 3p) \tag{10.36}$$

If we interpret ρ as the mass density and use the speed of light c, we get

$$\ddot{a} = -\frac{4\pi G}{3}a(\rho + 3p/c^2) \tag{10.37}$$

Inserting the gravitational mass density ρ_G from eq.(9.25) this equation takes the form

$$\ddot{a} = -\frac{4\pi G}{3}a\rho_G \tag{10.38}$$

Inserting $p = w\rho c^2$ into (9.25) gives

$$\rho_G = (1 + 3w)\rho \tag{10.39}$$

which is negative for $w < -1/3$, i.e. for $\dot{\phi}^2 < V(\phi)$. Special cases:

- dust: $w = 0$, $\rho_G = \rho$
- radiation: $w = \frac{1}{3}$, $\rho_G = 2\rho$
- Lorentz-invariant vacuum: $w = -1$, $\rho_G = -2\rho$

In a universe dominated by a Lorentz-invariant vacuum the acceleration of the cosmic expansion is

$$\ddot{a}_v = \frac{8\pi G}{3} a\rho_v > 0, \tag{10.40}$$

i.e. *accelerated expansion.* This means that vacuum acts upon itself with repulsive gravitation.

The field equations can be combined into

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{10.41}$$

where ρ_m is the density of matter, $\Lambda = 8\pi G\rho_\Lambda$ where ρ_Λ is the vacuum energy with constant density. $\rho = \rho_m + \rho_\Lambda$ is the total mass density. Then we may write

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{10.42}$$

The critical density ρ_{cr} is the density in a universe with euclidean spacelike geometry, $k = 0$, which gives

$$\rho_{cr} = \frac{3H^2}{8\pi G} \tag{10.43}$$

We introduce the relative densities

$$\Omega_m = \frac{\rho_m}{\rho_{cr}}, \qquad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}} \tag{10.44}$$

Furthermore we introduce a dimensionless parameter that describes the curvature of 3-space

$$\Omega_k = -\frac{k}{a^2 H^2} \tag{10.45}$$

Eq. (10.42) can now be written

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1 \tag{10.46}$$

From the Bianchi identity and Einstein's field equations follow that the energy-momentum density tensor is covariant divergence free. The time-component expresses the equation of continuity and takes the form

$$[(\rho + p)u^{\hat{t}}u^{\hat{\nu}}]_{;\hat{\nu}} + (p\eta^{\hat{t}\hat{\nu}})_{;\hat{\nu}} = 0 \tag{10.47}$$

Since $u^{\hat{t}} = 1$, $u^{\hat{m}} = 0$ and $\eta^{\hat{t}\hat{t}} = -1$, $\eta^{\hat{t}\hat{m}} = 0$, we get

$$(\rho + p)^{\cdot} + (\rho + p)u^{\hat{\nu}}{}_{;\hat{\nu}} - \dot{p} = 0 \tag{10.48}$$

or

$$\dot{\rho} + (\rho + p)(u^{\hat{\nu}}{}_{,\hat{\nu}} + \Gamma^{\hat{\nu}}{}_{\hat{t}\hat{\nu}}) = 0 \tag{10.49}$$

Here $u^{\nu}{}_{,\nu} = 0$ and $\Gamma^{\hat{t}}{}_{\hat{t}\hat{t}} = 0$. Calculating $\Gamma^{\hat{m}}{}_{\hat{t}\hat{m}}$ for $\underline{d\omega}^{\hat{\mu}} = \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\underline{\omega}^{\hat{\alpha}} \wedge \underline{\omega}^{\hat{\beta}}$ we get

$$\Gamma^{\hat{m}}{}_{\hat{t}\hat{m}} = \Gamma^{\hat{r}}{}_{\hat{t}\hat{r}} + \Gamma^{\hat{\theta}}{}_{\hat{t}\hat{\theta}} + \Gamma^{\hat{\phi}}{}_{\hat{t}\hat{\phi}} = 3\frac{\dot{a}}{a} \tag{10.50}$$

Hence

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0 \tag{10.51}$$

which may be written

$$(\rho a^3)^{\cdot} + p(a^3)^{\cdot} = 0 \tag{10.52}$$

Let $V = a^3$ be a comoving volume in the universe and $U = \rho V$ be the energy in the comoving volume. Then we may write

$$dU + pdV = 0 \tag{10.53}$$

This is the first law of thermodynamics for an adiabatic expansion. It follows that the universe expands adiabatically. The adiabatic equation can be written

$$\frac{\dot{\rho}}{\rho + p} = -3\frac{\dot{a}}{a} \tag{10.54}$$

Assuming $p = w\rho$ we get

$$\begin{aligned} \frac{d\rho}{\rho} &= -3(1+w)\frac{da}{a} \\ \ln\frac{\rho}{\rho_0} &= \ln\left(\frac{a}{a_0}\right)^{-3(1+w)} \end{aligned}$$

It follows that

$$\rho = \rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)} \tag{10.55}$$

This equation tells how the density of different types of matter depends on the expansion factor

$$\rho a^{3(1+w)} = constant \tag{10.56}$$

Special cases:

- dust: $w = 0$ gives $\rho_d a^3 = constant$
  Thus, the mass in a comoving volume is constant.

- radiation: $w = \frac{1}{3}$ gives $\rho_r a^4 = constant$
  Thus, the radiation energy density decreases faster than the case with dust when the universe is expanding. The energy in a comoving volume is decreasing because of the thermodynamic work on the surface. In a remote past, the density of radiation must have exceeded the density of dust:

$$\begin{aligned}\rho_{d0} a_0^3 =& \rho_d a^3 \\ \rho_{r0} a_0^4 =& \rho_r a^4 \\ \frac{\rho_r a^4}{\rho_d a^3} =& \frac{\rho_{r0} a_0^4}{\rho_{d0} a_0^3}\end{aligned}$$

  The expansion factor when $\rho_r = \rho_d$:

$$a(t_1) = \frac{\rho_{r0}}{\rho_{d0}} a_0$$

- Lorentz-invariant vacuum: $w = -1$ gives $\rho_\Lambda = constant$.
  The vacuum energy in a comoving volume is increasing $\propto a^3$.

10.4 Some cosmological models

10.4.1 Radiation dominated model

The energy-momentum tensor for radiation is trace free. According to the Einstein field equations the Einstein tensor must then be trace free:

$$\begin{aligned} a\ddot{a} + \dot{a}^2 + k = 0 \\ (a\dot{a} + kt)^{\cdot} = 0 \end{aligned} \tag{10.57}$$

Integration gives

$$a\dot{a} + kt = B \tag{10.58}$$

Another integration gives

$$\frac{1}{2}a^2 + \frac{1}{2}kt^2 = Bt + C \tag{10.59}$$

The initial condition $a(0) = 0$ gives $C = 0$. Hence

$$a = \sqrt{2Bt - kt^2} \tag{10.60}$$

For $k = 0$ we have

$$a = \sqrt{2Bt}\,, \quad \dot{a} = \sqrt{\frac{B}{2t}} \tag{10.61}$$

The expansion velocity reaches infinity at $t = 0$, ($\lim_{t\to 0} \dot{a} = \infty$)

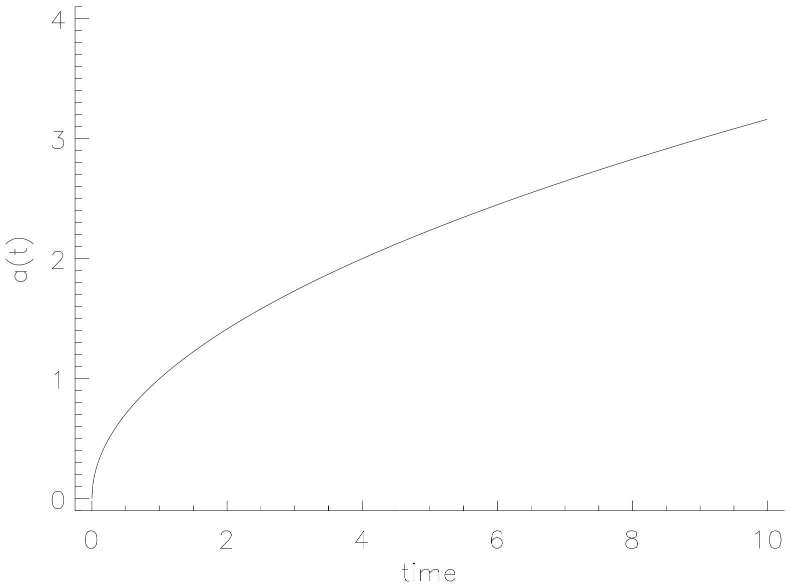


Figure 10.2: In a radiation dominated universe the expansion velocity reaches infinity at $t = 0$.

$$\begin{aligned} \rho_R a^4 = K \ , \quad a = \sqrt{2Bt} \\ 4\rho_R B^2 t^2 = K \end{aligned} \tag{10.62}$$

According to the Stefan-Boltzmann law we then have

$$\begin{aligned} &\rho_R = \sigma T^4 \rightarrow 4B^2\sigma T^4 t^2 = K \Rightarrow \\ &t = \frac{K_1}{T^2} \Leftrightarrow T = \sqrt{\frac{K_1}{t}} \end{aligned} \tag{10.63}$$

where T is the temperature of the background radiation.

10.4.2 Dust dominated model

From the first of the Friedmann equations we have

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 \tag{10.64}$$

We now introduce a time parameter η given by

$$\begin{aligned} \frac{dt}{d\eta} = a(\eta) \ \Rightarrow \ \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta} \\ \text{So:} \quad \dot{a} = \frac{da}{dt} = \frac{1}{a}\frac{da}{d\eta} \end{aligned} \tag{10.65}$$

We also introduce $A \equiv \frac{8\pi G}{3}\rho_0 {a_0}^3$. The first Friedmann equation then gives

$$a\dot{a}^2 + ka = \frac{8\pi G}{3}\rho a^3 = \frac{8\pi G}{3}\rho_0 {a_0}^3 = A \tag{10.66}$$

Using η we get

$$\begin{aligned} \frac{1}{a}(\frac{da}{d\eta})^2 &= A - ka \\ \frac{1}{a^2}(\frac{da}{d\eta})^2 &= \frac{A}{a} - k \\ \frac{1}{a}\frac{da}{d\eta} &= \sqrt{\frac{A}{a} - k} = \sqrt{\frac{A}{a}}\sqrt{1 - \frac{a}{A}k} \end{aligned} \tag{10.67}$$

where we chose the positive root. We now introduce u, given by $a = Au^2$, $u = \sqrt{\frac{a}{A}}$. We then get

$$\frac{da}{d\eta} = 2Au\frac{du}{d\eta} \tag{10.68}$$

which together with the equation above give

$$\begin{gathered} \frac{1}{Au^2}2Au\frac{du}{d\eta} = \frac{1}{u}\sqrt{1 - ku^2} \\ \Downarrow \\ \frac{du}{\sqrt{1 - ku^2}} = \frac{1}{2}d\eta \end{gathered} \tag{10.69}$$

This equation will first be integrated for $k < 0$. Then $k = -|k|$, so that

$$\int \frac{du}{\sqrt{1 + |k|u^2}} = \frac{\eta}{2} + K \tag{10.70}$$

or $\operatorname{arcsinh}(\sqrt{-k}u) = \frac{\eta}{2} + K$. The condition $u(0) = 0$ gives $K = 0$. Hence

$$-\frac{k}{A}a = \sinh^2\frac{\eta}{2} = \frac{1}{2}(\cosh\eta - 1) \tag{10.71}$$

or

$$a = -\frac{A}{2k}(\cosh\eta - 1) \tag{10.72}$$

From eqs. (10.43), (10.44) and (10.66) we have

$$A = \frac{8\pi G}{3}\rho_{m0} = H_0^2\frac{\rho_{m0}}{\rho_{cr0}} = H_0^2\Omega_{m0} \tag{10.73}$$

From egs. (10.45) and (10.46) we get

$$k = H_0^2(\Omega_{m0} - 1) \tag{10.74}$$

Hence, the scale factor of the negatively curved, dust dominated universe model is

$$a(\eta) = \frac{1}{2}\frac{\Omega_{m0}}{1 - \Omega_{m0}}(\cosh\eta - 1) \tag{10.75}$$

Inserting this into eq. (10.65) and integrating with $t(0) = \eta(0)$ leads to

$$t(\eta) = \frac{\Omega_{m0}}{2H_0(1-\Omega_{m0})^{3/2}}(\sinh\eta - \eta) \tag{10.76}$$

Integrating eg. (10.69) for $k = 0$ leads to an Einstein-deSitter universe

$$a(t) = (\frac{t}{t_0})^{\frac{2}{3}} \tag{10.77}$$

Finally integrating eg. (10.69) for $k > 0$ gives, in a similar way as for $k < 0$

$$a(\eta) = \frac{1}{2}\frac{\Omega_{m0}}{\Omega_{m0}-1}(1-\cos\eta) \tag{10.78}$$

$$t(\eta) = \frac{\Omega_{m0}}{2H_0(\Omega_{m0}-1)^{3/2}}(\eta - \sin\eta) \tag{10.79}$$

We see that this is a parametric representation of a cycloid.

In the Einstein-deSitter model the Hubble factor is

$$\boxed{H = \frac{\dot{a}}{a} = \frac{2}{3}\frac{1}{t}\ , \quad t = \frac{2}{3}\frac{1}{H} = \frac{2}{3}t_H} \tag{10.80}$$

The critical density in the Einstein-deSitter model is given by the first Friedmann equation:

$$\begin{aligned} H^2 = \frac{8\pi G}{3}\rho_{\mathrm{cr}}\ , &\quad k = 0 \\ &\Downarrow \\ \rho_{\mathrm{cr}} = \frac{3H^2}{8\pi G}\ , &\quad \Omega = \frac{\rho}{\rho_{\mathrm{cr}}} \end{aligned} \tag{10.81}$$

`Example` **10.4.1 (Age-redshift relation for dust dominated universe with $k = 0$)**

$$\begin{aligned} 1 + z = \frac{a_0}{a} &\Rightarrow a = \frac{a_0}{1+z} \\ da = -\frac{a_0}{(1+z)^2}dz &= -\frac{a}{1+z}dz \end{aligned} \tag{10.82}$$

Eq. (10.34) gives

$$\begin{aligned} (\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho &= \frac{8\pi G}{3}\frac{\rho_0 a_0^3}{a^3} \\ &= \frac{8\pi G}{3}\rho_0(1+z)^3 \end{aligned} \tag{10.83}$$

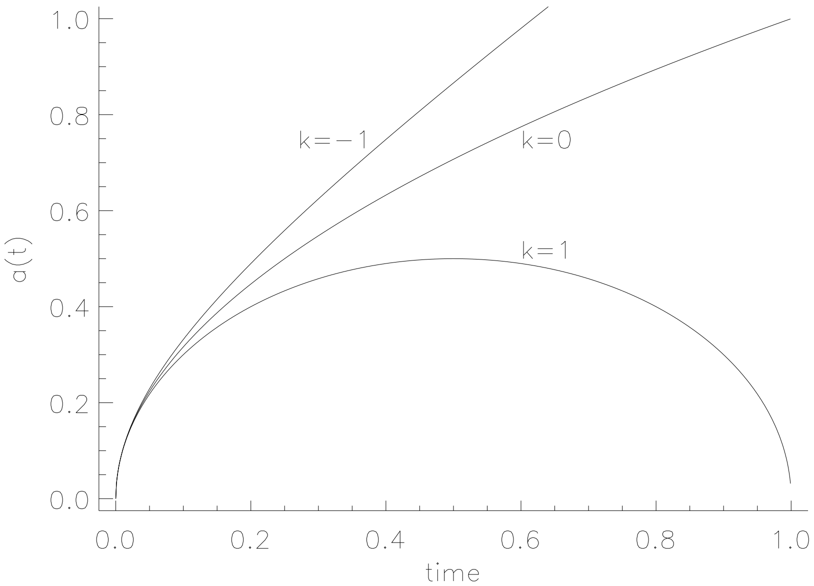


Figure 10.3: For $k = 1$ the density is larger than the critical density, and the universe is closed. For $k = 0$ we have $\rho = \rho_{\rm cr}$ and the expansion velocity of the universe will approach zero as $t \rightarrow \infty$. For $k = -1$ we have $\rho < \rho_{\rm cr}$. The universe is then open, and will continue expanding forever.

Using $H_0^2 = \frac{8\pi G}{3}\rho_0$ gives $\frac{\dot{a}}{a} = H_0(1+z)^{\frac{3}{2}}$. From $\dot{a} = \frac{da}{dt}$ we get:

$$dt = \frac{da}{\dot{a}} = \frac{da}{a\frac{\dot{a}}{a}} = -\frac{dz}{H_0(1+z)^{\frac{5}{2}}} \tag{10.84}$$

Integration gives the age of the universe:

$$t_0 = -\frac{1}{H_0}\int_{\infty}^{0} \frac{dz}{(1+z)^{\frac{5}{2}}} = \frac{2}{3}\frac{1}{H_0}\Big[\frac{1}{(1+z)^{\frac{3}{2}}}\Big]_{\infty}^{0} = \frac{2}{3}t_H \tag{10.85}$$

where the Hubble-time $t_H \equiv \frac{1}{H_0}$ is the age of the universe if the expansion rate had been constant. The 'Look-back-time' to a source with redshift z is:

$$\Delta t = t_H \int_0^z \frac{dz}{(1+z)^{\frac{5}{2}}} = \frac{2}{3}t_H\Big[1 - \frac{1}{(1+z)^{\frac{3}{2}}}\Big] \tag{10.86}$$

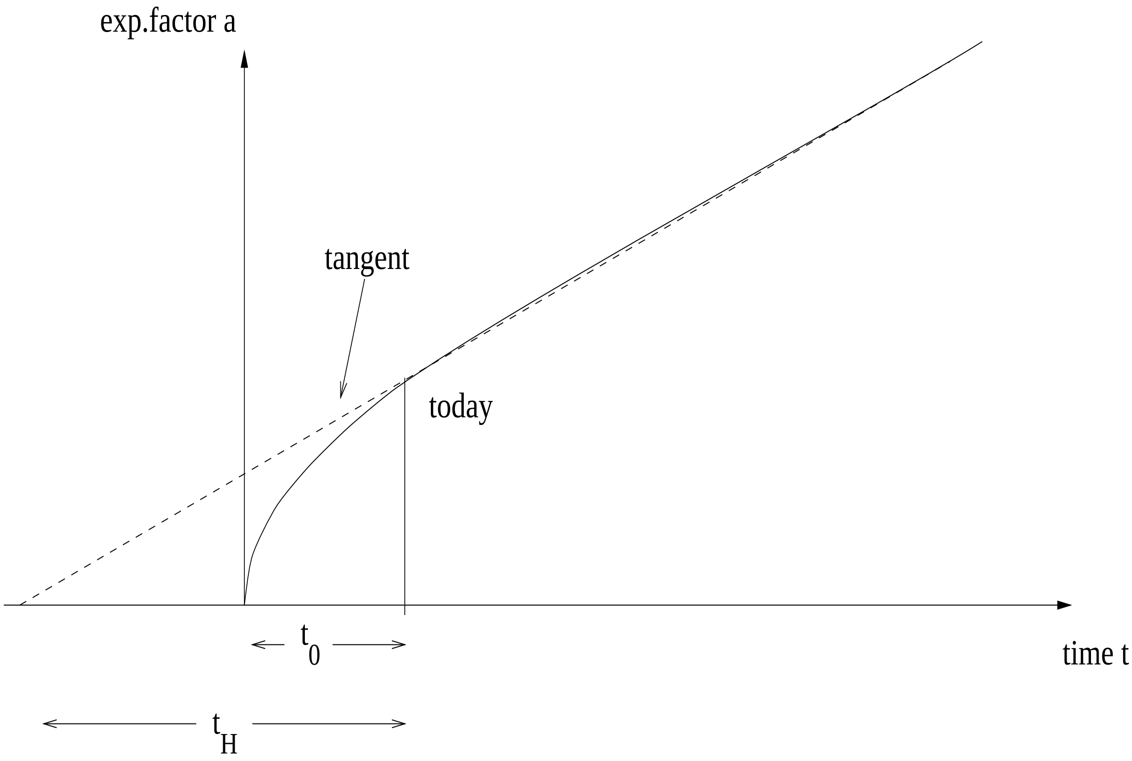


Figure 10.4: t_H is the age of the universe if the expansion had been constant, BUT:The exp.rate was faster closer to the Big Bang, so the age is lower.

$$\Delta t = t_0[1 - \frac{1}{(1+z)^{3/2}}] \tag{10.87}$$

Hence, the redshift of an object with lookback time Δt is

$$z = \frac{1}{(1 - \frac{\Delta t}{t_0})^{2/3}} - 1 \tag{10.88}$$

10.4.3 Transition from radiation- to matter dominated universe

We consider the early universe filled with radiation and matter, but where vacuum energy cam be neglected. The universe is assumed to be flat. Then Friedmann's 1. equation takes the form

$$\dot{a}^2 = \frac{8\pi G}{3}(\rho_M + \rho_R)\, a^2 \tag{10.89}$$

For matter,

$$\rho_M a^3 = \rho_{M0} \tag{10.90}$$

For radiation,

$$\rho_R a^4 = \rho_{R0} \tag{10.91}$$

Hence

$$a^2 \dot{a}^2 = \frac{8\pi G}{3} \left(\rho_{M0}\, a + \rho_{R0}\right) \tag{10.92}$$

The present values of the critical density and the density parameters are

$$\rho_{cr0} = \frac{3H_0^2}{8\pi G} \tag{10.93}$$

$$\Omega_{M0} = \frac{\rho_{M0}}{\rho_{cr0}} \tag{10.94}$$

$$\Omega_{R0} = \frac{\rho_{R0}}{\rho_{cr0}} \tag{10.95}$$

giving

$$a\dot{a} = H_0 \left(\rho_{M0}\, a + \rho_{R0}\right)^{1/2} \tag{10.96}$$

Integration with $a(0) = 0$ leads to

$$H_0 t = \frac{4}{3} \frac{\Omega_{R0}^{3/2}}{\Omega_{M0}^2} + \frac{2}{3} \frac{\left(\rho_{M0}\, a - 2\rho_{R0}\right)\left(\rho_{M0}\, a + \rho_{R0}\right)^{1/2}}{\Omega_{M0}^2} \tag{10.97}$$

From eqs. (10.90) and (10.91) follows that at the transition time t_{eq} when $\rho_M = \rho_R$, the scale factor has the value

$$a_{eq} = \frac{\rho_{R0}}{\rho_{M0}} = \frac{\Omega_{R0}}{\Omega_{M0}} \tag{10.98}$$

Inserting this into eq. (10.97) gives

$$t_{eq} = \frac{2}{3}\left(2 - \sqrt{2}\right) \frac{\Omega_{R0}^{3/2}}{\Omega_{M0}^2} t_H \tag{10.99}$$

The microwave background radiation has a temperature 2,73 K corresponding to a density parameter $\Omega_{R0} = 8,4 \cdot 10^{-5}$. In a flat universe without vacuum energy $\Omega_{M0} = 1 - \Omega_{R0}$. From the value of H_0 as determined by measurements we have $t_H \approx 14 \cdot 10^9$ years. This leads to $t_{eq} = 47 \cdot 10^3$ years.

10.4.4 Friedmann-Lemaître model

The dynamics of galaxies and clusters of galaxies has made it clear that far stronger gravitational fields are needed to explain the observed motions than those produced by visible matter (McGaugh 2001). At the same time it has become clear that the density of this dark matter is only about 30% of the critical density, although it is a prediction by the usual versions of the inflationary universe models that the density ought to be equal to the critical density (Linde 2001). Also the recent observations of the temperature fluctuations of the cosmic microwave radiation have shown that space is either flat or very close to flat (Bernadis et.al 2001, Stompor et al. 2001, Pryke et al. 2001). The energy that fills up to the critical density must be evenly distributed in order not to affect the dynamics of the galaxies and the clusters.

Furthermore, about two years ago observations of supernovae of type Ia with high cosmic red shifts indicated that the expansion of the universe is accelerating (Riess et al. 1998, Perlmutter et al. 1999). This was explained as a result of repulsive gravitation due to some sort of vacuum energy. Thereby the missing energy needed to make space flat, was identified as vacuum energy. Hence, it seems that we live in a flat universe with vacuum energy having a density around 70% of the critical density and with matter having a density around 30% of the critical density.

Until the discovery of the accelerated expansion of the universe the standard model of the universe was assumed to be the Einstein-DeSitter model, which is a flat universe model dominated by cold matter. This universe model is thoroughly presented in nearly every text book on general relativity and cosmology. Now it seems that we must replace this model with a new "standard model" containing both dark matter and vacuum energy.

Recently several types of vacuum energy or so called quintessence energy have been discussed (Zlatev, Wang and Steinhardt 1999, Carroll 1998). However, the most simple type of vacuum energy is the Lorentz invariant vacuum energy (LIVE), which has constant energy density during the expansion of the universe (Zeldovich 1968, Grøn 1986). This type of energy can be mathematically represented by including a cosmological constant in Einstein's gravitational field equations. The flat universe model with cold dark matter and this type of vacuum energy is the Friedmann-Lemaître model.

The field equations for the flat Friedmann-Lemaître is found by putting $k = p = 0$ in equation (10.35). This gives

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \Lambda \tag{10.100}$$

Integration leads to

$$a\dot{a}^2 = \frac{\Lambda}{3}a^3 + K \tag{10.101}$$

where K is a constant of integration. Since the amount of matter in a volume comoving with the cosmic expansion is constant, $\rho_M a^3 = \rho_{M0} a_0^3$, where the index 0 refers to measured values at the present time. Normalizing the expansion factor so that $a_0 = 1$ and comparing eqs.(10.42) and(10.101) then gives $K = (8\pi G/3)\rho_{M0}$. Introducing a new variable x by $a^3 = x^2$ and integrating once more with the initial condition $a(0) = 0$ we obtain

$$a^3 = \frac{3K}{\Lambda}\sinh^2\left(\frac{t}{t_\Lambda}\right), \qquad t_\Lambda = \frac{2}{\sqrt{3\Lambda}} \tag{10.102}$$

The vacuum energy has a constant density ρ_Λ given by

$$\Lambda = 8\pi G\rho_\Lambda \tag{10.103}$$

The critical density, which is the density making the 3-space of the universe flat, is

$$\rho_{cr} = \frac{3H^2}{8\pi G} \tag{10.104}$$

The relative density, i.e. the density measured in units of the critical density, of the matter and the vacuum energy, are respectively

$$\Omega_M = \frac{\rho}{\rho_{cr}} = \frac{8\pi G\rho_M}{3H^2} \tag{10.105}$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}} = \frac{\Lambda}{3H^2} \tag{10.106}$$

Since the present universe model has flat space, the total density is equal to the critical density, i.e. $\Omega_M + \Omega_\Lambda = 1$. Eq. (10.101) with the normalization $a(t_0) = 1$, where t_0 is the present age of the universe, gives $3H_0^2 = 3K + \Lambda$. Eq. (10.34) with $k = 0$ gives $8\pi G\rho_0 = 3H_0^2 - \Lambda$. Hence $K = 8\pi G\rho_0/3$ and $\frac{3K}{\Lambda} = \frac{8\pi G\rho_0}{\Lambda} = \frac{\rho_0}{\rho_\Lambda} = \frac{\Omega_{M0}}{\Omega_{\Lambda 0}}$. In terms of the values of the relative densities at the present time the expression for the expansion factor then takes the form

$$a = A^{1/3}\sinh^{2/3}\left(\frac{t}{t_\Lambda}\right), \qquad A = \frac{\Omega_{M0}}{\Omega_{\Lambda 0}} = \frac{1-\Omega_{\Lambda 0}}{\Omega_{\Lambda 0}} \tag{10.107}$$

Using the identity $\sinh(x/2) = \sqrt{(\cosh x - 1)/2}$ this expression may be written

$$a^3 = \frac{A}{2}\left[\cosh\left(\frac{2t}{t_\Lambda}\right) - 1\right] \tag{10.108}$$

The age t_0 of the universe is found from $a(t_0) = 1$, which by use of the formula $arc\tanh x = arc\sinh(x/\sqrt{1-x^2})$, leads to the expression

$$t_0 = t_\Lambda arc\tanh\sqrt{\Omega_{\Lambda 0}} \tag{10.109}$$

Inserting typical values $t_0 = 15 \cdot 10^9$years, $\Omega_{\Lambda 0} = 0.7$ we get $A = 0.43$, $t_\Lambda = 12 \cdot 10^9$years. With these values the expansion factor is $a = 0.75\sinh^{2/3}(1.2t/t_0)$. This function is plotted in fig. 10.5. The Hubble parameter as a function of time is

$$H = (2/3t_\Lambda)\coth(t/t_\Lambda) \tag{10.110}$$

Inserting $t_0 = 1.2t_\Lambda$ we get $Ht_0 = 0.8\coth(1.2t/t_0)$, which is plotted in fig. 10.6 The Hubble parameter decreases all the time and approaches a constant value $H_\infty = 2/3t_\Lambda$ in the infinite future. The present value of the Hubble parameter is

$$H_0 = \frac{2}{3t_\Lambda\sqrt{\Omega_{\Lambda 0}}} \tag{10.111}$$

The corresponding Hubble age is $t_{H0} = (3/2)t_\Lambda\sqrt{\Omega_{\Lambda 0}}$. Inserting our numerical values gives $H_0 = 64\mathrm{km/secMpc}^{-1}$ and $t_{H0} = 15.7 \cdot 10^9$years. In this universe model the age of the universe is nearly as large as the Hubble age, while in the Einstein-DeSitter model the corresponding age is $t_{0ED} = (2/3)t_{H0} = 10.5 \cdot 10^9 years$. The reason for this difference is that in the Einstein-DeSitter model the expansion is decelerated all the time, while in the Friedmann-Lemaître model the repulsive gravitation due to the vacuum energy have made the expansion accelerate lately (see below). Hence, for a given value of the Hubble parameter

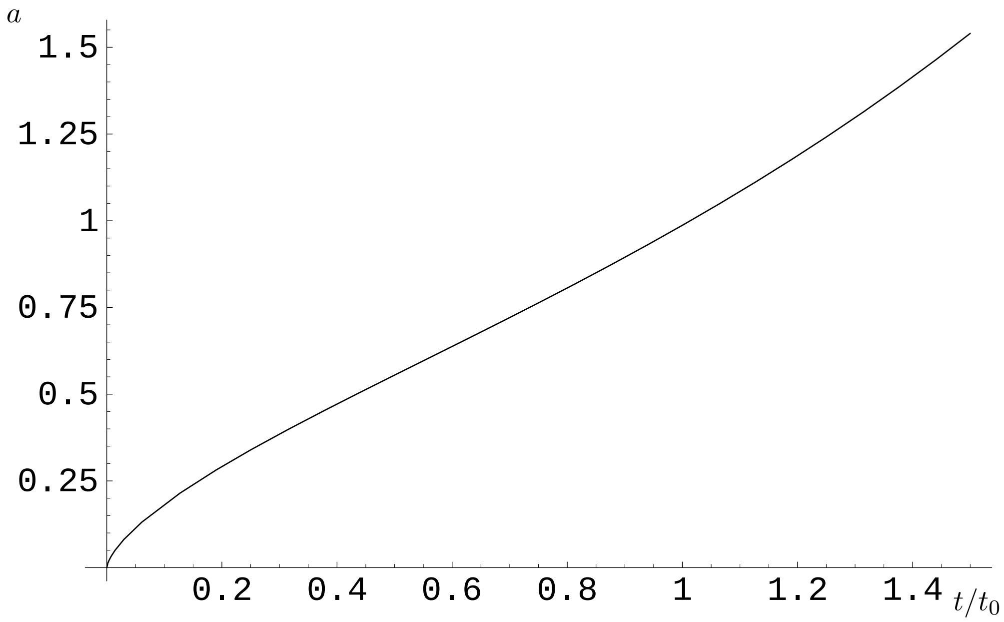


Figure 10.5: The expansion factor as function of cosmic time in units of the age of the universe.

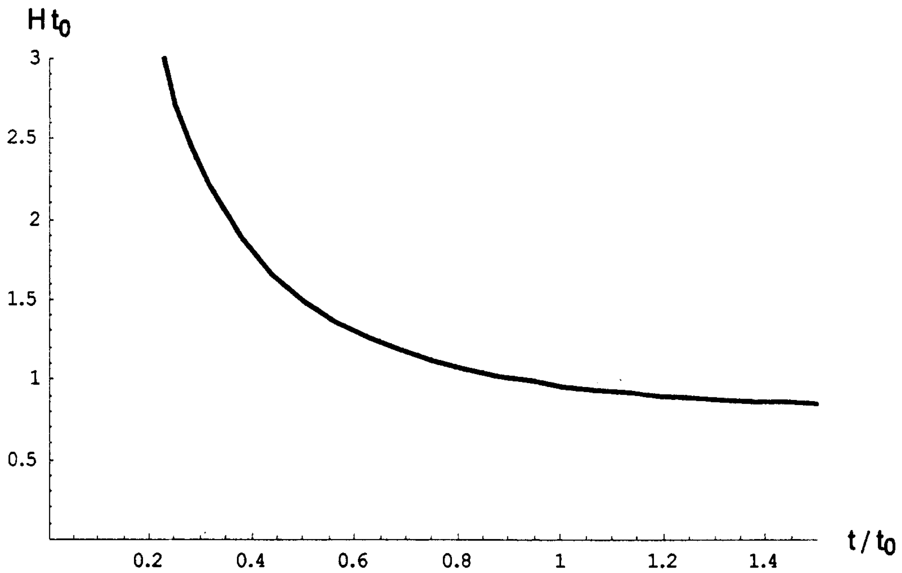


Figure 10.6: The Hubble parameter as function of cosmic time.

the previous velocity was larger in the Einstein-DeSitter model than in the Friedmann-Lemaître model.

The ratio of the age of the universe and its Hubble age depends upon the present relative density of the vacuum energy as follows,

$$\frac{t_0}{t_{H0}} = H_0 t_0 = \frac{2}{3} \frac{arc \tanh \sqrt{\Omega_{\Lambda 0}}}{\sqrt{\Omega_{\Lambda 0}}} \tag{10.112}$$

This function is depicted graphically in fig. 10.7 The age of the universe increases

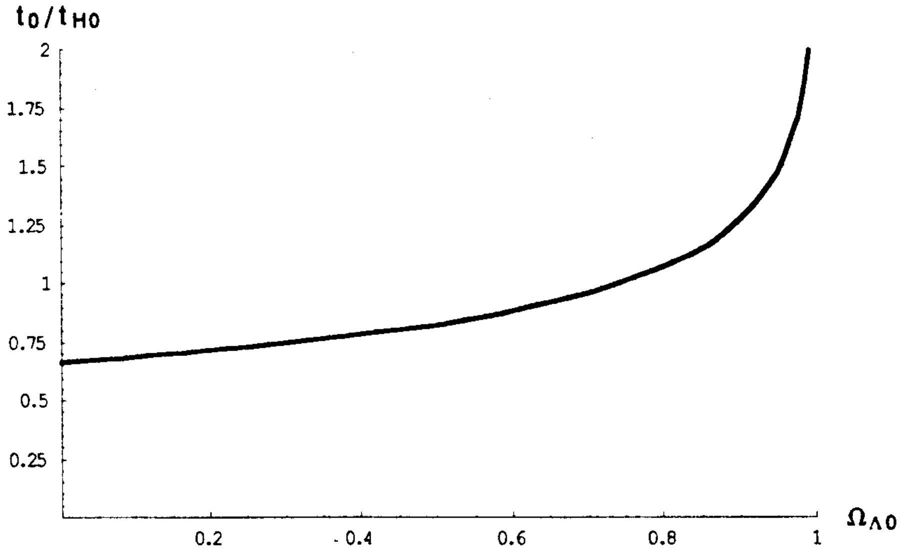


Figure 10.7: The ratio of the age of the universe and the Hubble age as function of the present relative density of the vacuum energy.

with increasing density of vacuum energy. In the limit that the density of the vacuum approaches the critical density, thereis no dark matter, and the universe model approaches the DeSitter model with exponential expansion and no Big Bang. This model behaves in the same way as the Steady State cosmological model and is infinitely old.

A dimensioness quantity representing the rate of change of the cosmic expansion velocity is the deceleration parameter, which is defined as $q = -\ddot{a}/aH^2$. For the present universe model the deceleration parameter as a function of time is

$$q = \frac{1}{2}[1 - 3 \tanh^2(t/t_\Lambda)] \tag{10.113}$$

which is shown graphically in fig. 10.8 The inflection point of time t_1 when deceleration turned into acceleration is given by $q = 0$. This leads to

$$t_1 = t_\Lambda arc \tanh(1/\sqrt{3}) \tag{10.114}$$

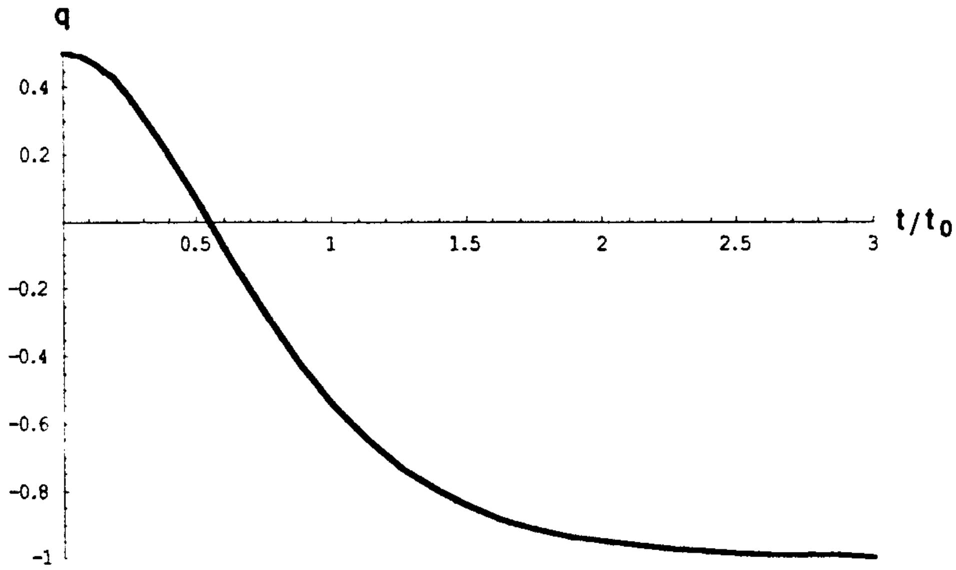


Figure 10.8: The deceleration parameter as function of cosmic time.

or expressed in terms of the age of the universe

$$t_1 = \frac{arc\tanh(1/\sqrt{3})}{arc\tanh\sqrt{\Omega_{\Lambda 0}}} t_0 \tag{10.115}$$

The corresponding cosmic red shift is

$$z(t_1) = \frac{a_0}{a(t_1)} - 1 = \left(\frac{2\Omega_{\Lambda 0}}{1 - \Omega_{\Lambda 0}}\right)^{1/3} - 1 \tag{10.116}$$

Inserting $\Omega_{\Lambda 0} = 0.7$ gives $t_1 = 0.54 t_0$ and $z(t_1 = 0.67$.

The results of analysing the observations of supernova SN 1997 at $z = 1.7$, corresponding to an emission time $t_e = 0.30 t_0 = 4.5 \cdot 10^9 years$, have provided evidence that the universe was decelerated at that time (Riess n.d.). M.Turner and A.G.Riess (Turner and Riess 2001) have recently argued that the other supernova data favour a transition from deceleration to acceleration for a red shift around $z = 0.5$.

Note that the expansion velocity given by Hubble's law, $v = Hd$, always decreases as seen from fig. 10.6. This is the velocity away from the Earth of the cosmic fluid at a fixed physical distance d from the Earth. The quantity $\dot{a}$ on the other hand, is the velocity of a fixed fluid particle comoving with the expansion of the universe. If such a particle accelerates, the expansion of the universe is said to accelerate. While $\dot{H}$ tells how fast the expansion velocity changes at a fixed distance from the Earth, the quantity $\ddot{a}$ represents the acceleration of a free particle comoving with the expanding universe. The connection between these two quantities are $\ddot{a} = a(\dot{H} + H^2)$.

The ratio of the inflection point of time and the age of the universe, as given in eq.(10.115), is depicted graphically as function of the present relative density of vacuum energy in fig. 10.9 The turnover point of time happens earlier the

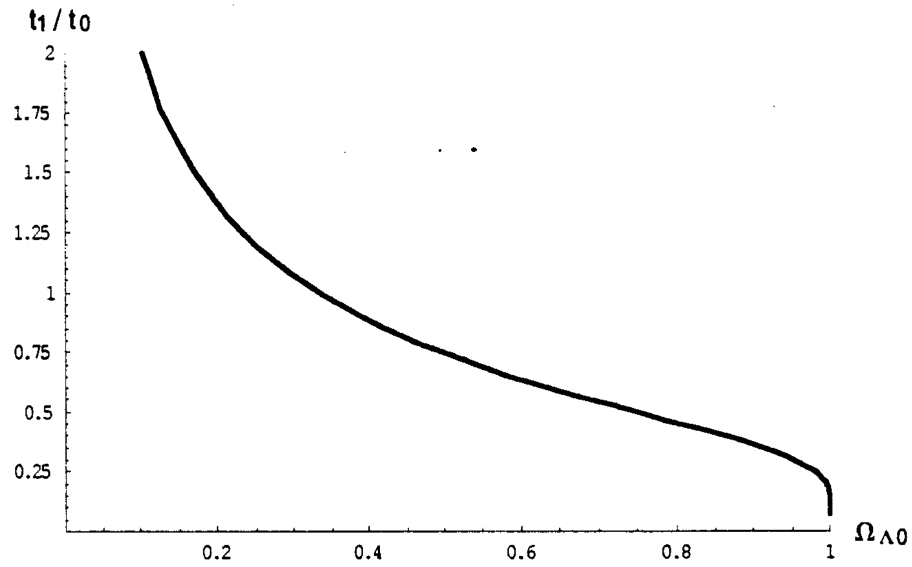


Figure 10.9: The ratio of the point of time when cosmic decelerations turn over to acceleration to the age of the universe.

greater the vacuum density is. The change from deceleration to acceleration would happen at the present time if $\Omega_{\Lambda 0} = 1/3$.

The red shift of the inflection point given in eq.(10.116) as a function of vacuum energy density, is plotted in fig. 10.10 Note that the red shift of future points of time is negative, since then $a > a_0$. If $\Omega_{\Lambda 0} < 1/3$ the transition to acceleration will happen in the future.

The critical density is

$$\rho_{cr} = \rho_\Lambda \tanh^{-2}(t/t_\Lambda) \tag{10.117}$$

This is plotted in fig. 10.11. The critical density decreases with time.

Eq. (10.116) shows that the relative density of the vacuum energy is

$$\Omega_\Lambda = \tanh^2(t/t_\Lambda) \tag{10.118}$$

which is plotted in fig. 10.12. The density of the vacuum energy approaches the critical density. Since the density of the vacuum energy is constant, this is better expressed by saying that the critical density approaches the density of the vacuum energy. Furthermore, since the total energy density is equal to the critical density all the time, this also means that the density of matter decreases faster than the critical density. The density of matter as function of time is

$$\rho_M = \rho_\Lambda \sinh^{-2}(t/t_\Lambda) \tag{10.119}$$

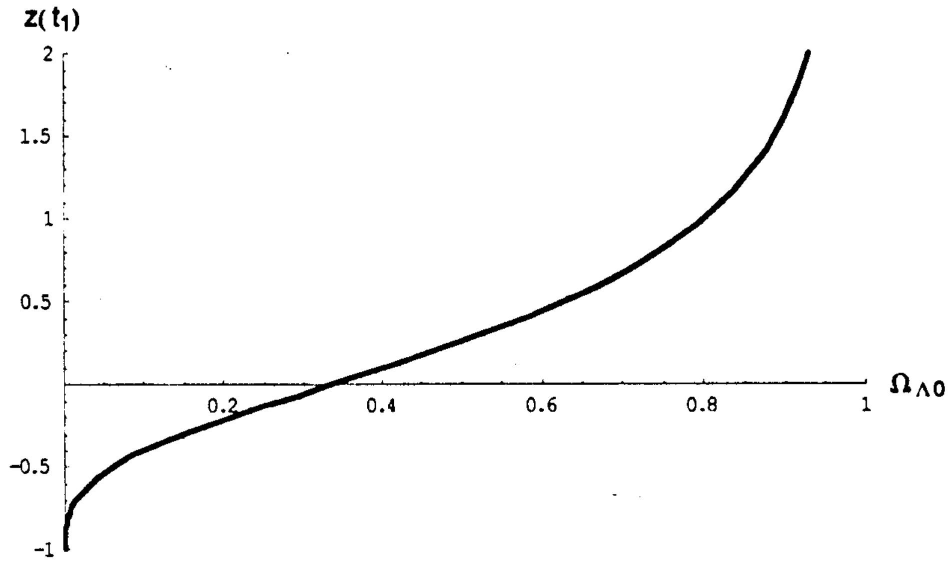


Figure 10.10: The cosmic red shift of light emitted at the turnover time from deceleration to acceleration as function of the present relative density of vacuum energy.

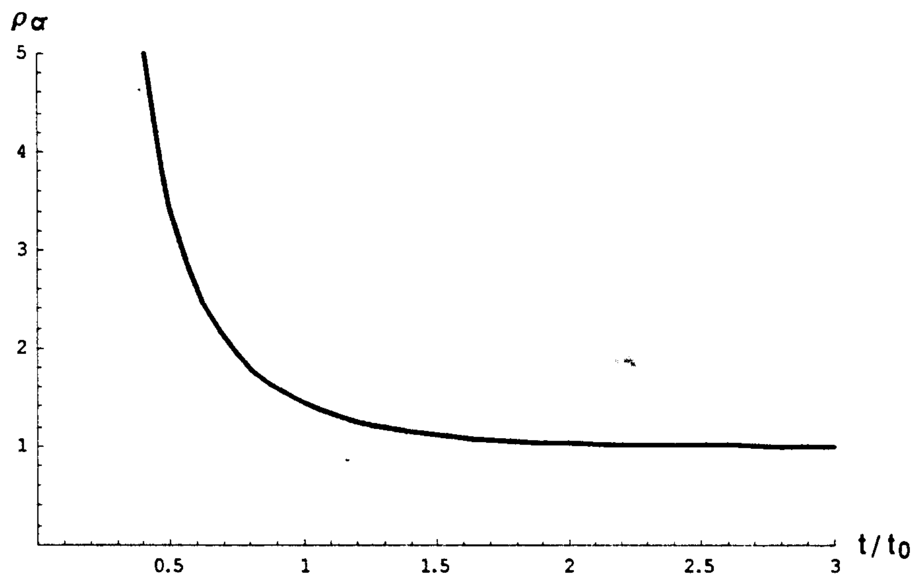


Figure 10.11: The critical density in units of the constant density of the vacuum energy as function of time.

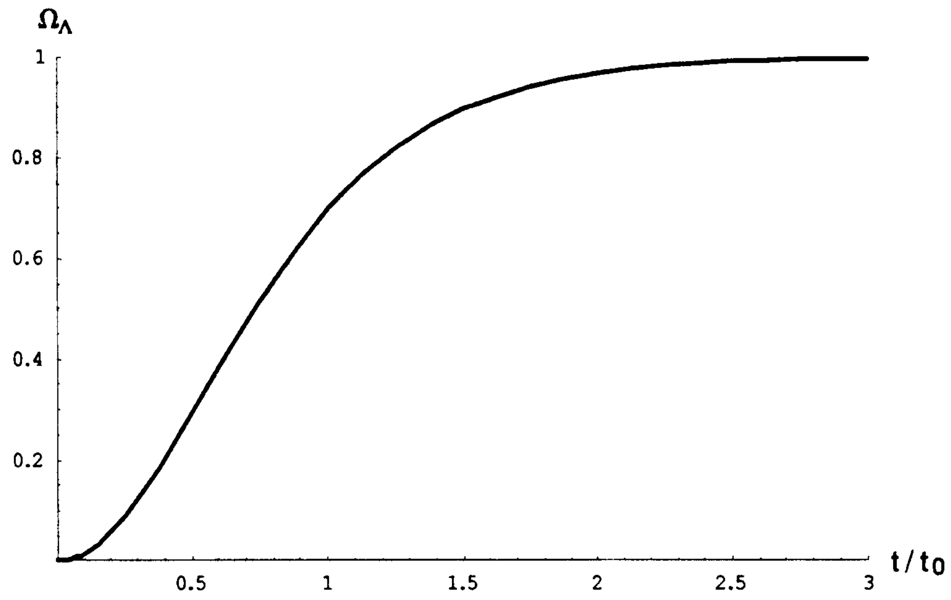


Figure 10.12: The relative density of the vacuum energy density as function of time.

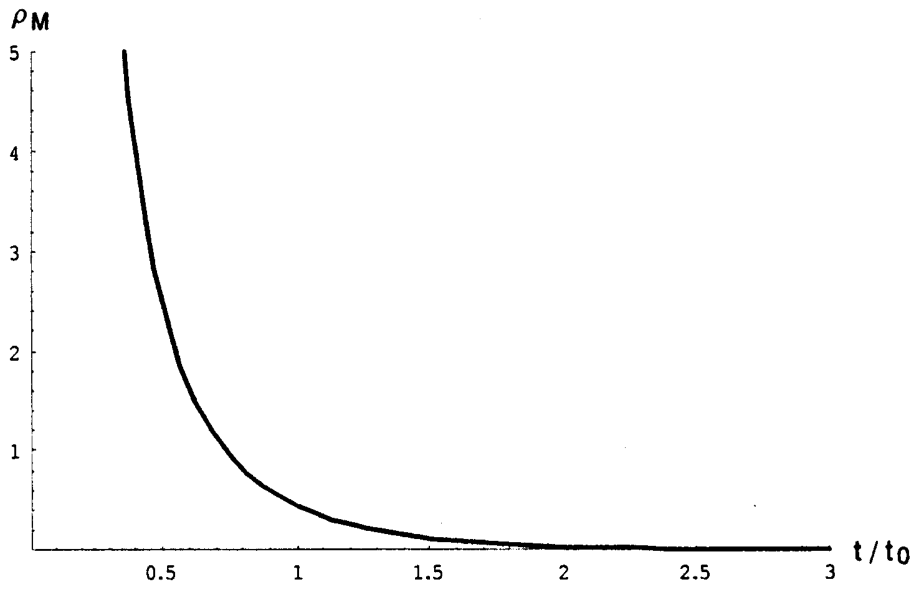


Figure 10.13: The density of matter in units of the density of vacuum energy as function of time.

which is shown graphically in fig. 10.13 The relative density of matter as function of time is

$$\Omega_M = \cosh^{-2}(t/t_\Lambda) \tag{10.120}$$

which is depicted in fig. 10.14 Adding the relative densities of fig. 10.13 and

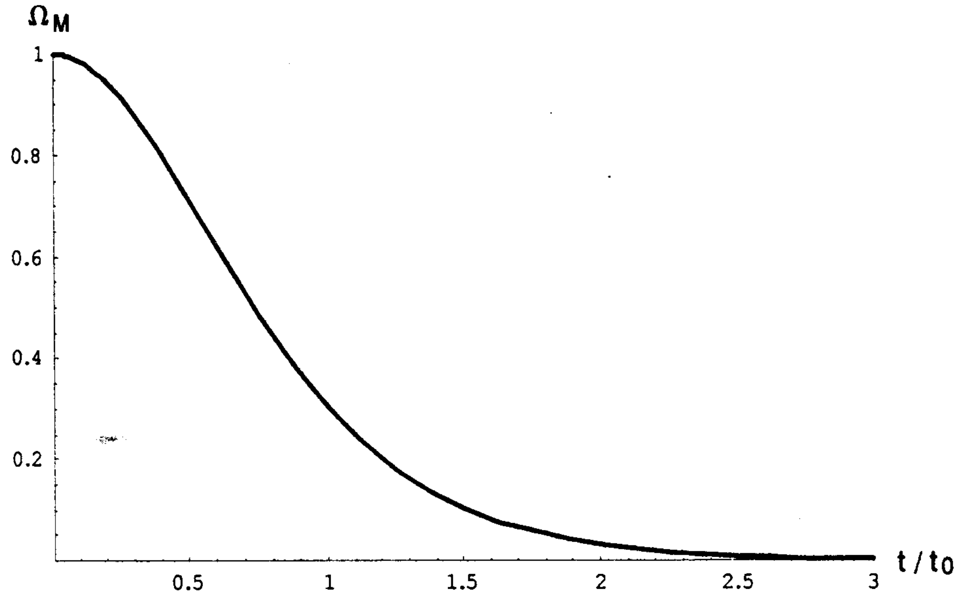


Figure 10.14: The relative density of matter as function of time.

fig. 10.14 or the expressions (10.117) and (10.119) we get the total relative density $\Omega_{TOT} = \Omega_M + \Omega_\Lambda = 1$.

The universe became vacuum dominated at a point of time t_2 when $\rho_\Lambda(t_2) = \rho_M(t_2)$. From eq.(10.119) follows that this point of time is given by $\sinh(t_2/t_\Lambda) = 1$. According to eq.(10.109) we get

$$t_2 = \frac{arc\sinh(1)}{arc\tanh(\sqrt{\Omega_{\Lambda 0}})} t_0 \tag{10.121}$$

From eq.(10.107) follows that the corresponding red shift is

$$z(t_2) = A^{-1/3} - 1 \tag{10.122}$$

Inserting $\Omega_{\Lambda 0} = 0.7$ gives $t_2 = 0.73 t_0$ and $z(t_2) = 0.32$. The transition to accelerated expansion happens before the universe becomes vacuum dominated.

Note from eqs.(10.113) and (10.118) that in the case of the flat Friedmann-Lemaître universe model, the deceleration parameter may be expressed in terms of the relative density of vacuum only, $q = (1/2)(1 - 3\Omega_\Lambda)$. The supernova Ia observations have shown that the expansion is now accelerating. Hence if the universe is flat, this alone means that $\Omega_{\Lambda 0} > 1/3$.

As mentioned above, many different observations indicate that we live in a universe with critical density, where cold matter contributes with about 30%

of the density and vacuum energy with about 70%. Such a universe is well described by the Friedmann-Lemaître universe model that have been presented above.

However, this model is not quite without problems in explaining the observed properties of the universe. In particular there is now much research directed at solving the so called *coincidence problem.* As we have seen, the density of the vacuum energy is constant during the expansion, while the density of the matter decreases inversely proportional to a volume comoving with the expanding matter. Yet, one observes that the density of matter and the density of the vacuum energy are of the same order of magnitude at the present time. This seems to be a strange and unexplained coincidence in the model. Also just at the present time the critical density is approaching the density of the vacuum energy. At earlier times the relative density was close to zero, and now it changes approaching the constant value 1 in the future. S. M. Carroll (Carroll 2001) has illustrated this aspect of the coincidence problem by plotting $\dot{\Omega}_\Lambda$ as a function of $\ln(t/t_0)$. Differentiating the expression (10.118) we get

$$\frac{t_\Lambda}{2}\frac{d\Omega_\Lambda}{dt} = \frac{\sinh(t/t_\Lambda)}{\cosh^3(t/t_\Lambda)} \tag{10.123}$$

which is plotted in fig. 10.15

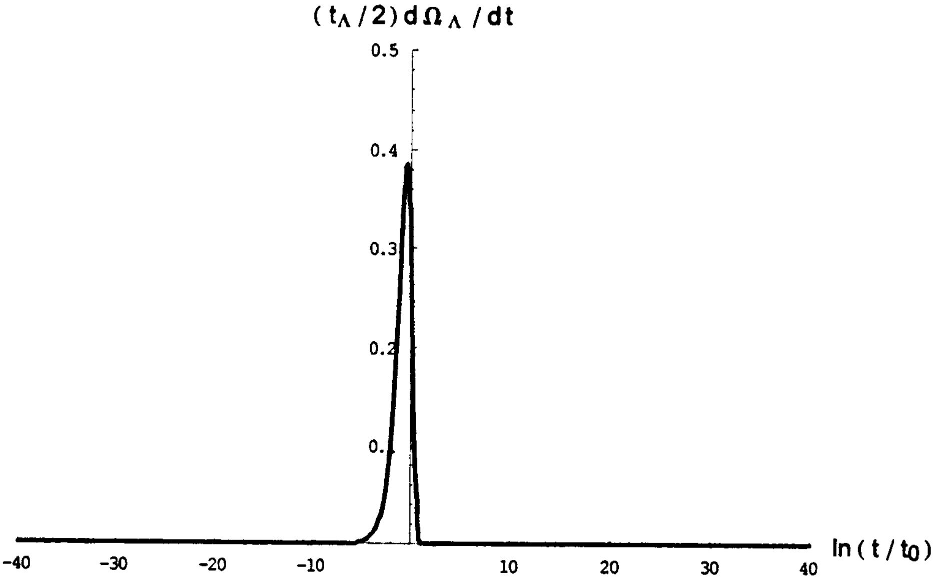


Figure 10.15: Rate of change of Ω_Λ as function of $\ln(\frac{t}{t_0})$. The value $\ln(\frac{t}{t_0}) = -40$ corresponds to the cosmic point of time $t_0 \sim 1s$.

Putting $\ddot{\Omega}_\Lambda = 0$ we find that the rate of change of Ω_Λ was maximal at the point of time t_1 when the deceleration of the cosmic expansion turned into acceleration. There is now a great activity in order to try to explain these coincidences by introducing more general forms of vacuum energy called quintessence, and with a density determined dynamically bythe evolution of a scalar field (Turner 2001).

However, the simplest type of vacuum energy is the LIVE. One may hope that a future theory of quantum gravity may settle the matter and let us understand the vacuum energy. In the meantime we can learn much about the dynamics of a vacuum dominated universe by studying simple and beautiful universe models such as the Friedmann-Lemaître model.

10.5 Inflationary Cosmology

10.5.1 Problems with the Big Bang Models

The Horizon Problem

The Cosmic Microwave Background (CMB) radiation from two points A and B in opposite directions has the same temperature. This means that it has been radiated by sources of the same temperature in these points. Thus, the universe must have been in thermic equilibrium at the decoupling time, $t_d = 3 \cdot 10^5$years. This implies that points A and B, "at opposite sides of the universe", had been in causal contact already at that time. I.e., a light signal must have had time to move from A to B during the time from $t = 0$ to $t = 3 \cdot 10^5$ years. The points A and B must have been within each other's horizons at the decoupling.

Consider a photon moving radially in space descibed by the Robertson-Walker metric (10.14) with $k = 0$. Light follows a null geodesic curve, i.e. the curve is defined by $ds^2 = 0$. We get

$$dr = \frac{dt}{a(t)}. \tag{10.124}$$

The coordinate distance the photon has moved during the time t is

$$\Delta r = \int_0^t \frac{dt}{a(t)}. \tag{10.125}$$

The physical distance the light has moved at the time t is called the *horizon distance*, and is

$$l_h = a(t)\Delta t = a(t)\int_0^t \frac{dt}{a(t)}. \tag{10.126}$$

To find a quantitative expression for the "horizon problem", we may consider a model with critical mass density (Euclidian spacelike geometry.) Using $p = w\rho$ and $\Omega = 1$, integration of equation (10.36) gives

$$a \propto t^{\frac{2}{3+3w}}. \tag{10.127}$$

Inserting this into the expression for l_h and integrating gives

$$l_h = \frac{3w+3}{3w+1}t. \tag{10.128}$$

Let us call the volume inside the horizon the "horizon volume" and denote it by V_H. From equation (10.128) follows that $V_H \propto t^3$. At the decoupling time, the horizon volume may therefore be written

$$(V_H)_d = \left(\frac{t_d}{t_0}\right)^3 V_0, \tag{10.129}$$

where V_0 is the size of the present horizon volume. Events within this volume are causally connected, and a volume of this size may be in thermal equilibrium at the decoupling time.

Let $(V_0)_d$ be the size, at the decoupling, of the part of the universe that corresponds to the present horizon volume, i.e. the observable universe. For our Euclidean universe, the equation (10.127) holds, giving

$$(V_0)_d = \frac{a^3(t_d)}{a^3(t_0)} V_0 = \left(\frac{t_d}{t_0}\right)^{\frac{2}{w+1}} V_0. \tag{10.130}$$

From equations (10.129) and (10.130), we get

$$\frac{(V_0)_d}{(V_H)_d} = \left(\frac{t_d}{t_0}\right)^{\frac{3w+1}{w+1}}. \tag{10.131}$$

Using that $t_d = 10^{-4} t_0$ and inserting $w = 0$ for dust, we find $\frac{V_0)_d}{(V_H)_d} = 10^4$. Thus, there was room for 10^4 causally connected areas at the decoupling time within what presently represents our observable universe. Points at opposite sides of our observable universe were therefore not causally connected at the decoupling, according to the Friedmann models of the universe. These models can therefore not explain that the temperature of the radiation from such points is the same.

The Flatness Problem

According to eq. (10.42), the total mass parameter $\Omega = \frac{\rho}{\rho_{cr}}$ is given by

$$\Omega - 1 = \frac{k}{\dot{a}^2}. \tag{10.132}$$

By using the expansion factor (10.127) for a universe near critical mass density, we get

$$\frac{\Omega - 1}{\Omega_0 - 1} = \left(\frac{t}{t_0}\right)^{2\left(\frac{3w+1}{3w+3}\right)}. \tag{10.133}$$

For a radiation dominated universe, we get

$$\frac{\Omega - 1}{\Omega_0 - 1} = \frac{t}{t_0}. \tag{10.134}$$

Measurements indicate that $\Omega_0 - 1$ is of order of magnitude 1. The age of the universe is about $t_0 = 10^{17} s$. When we stipulate initial conditions for the universe, it is natural to consider the Planck time, $t_P = 10^{-43}s$, since this is the limit to the validity of general relativity. At earlier time, quantum effects will be important, and one can not give a reliable description without using quantum gravitation. The stipulated initial condition for the mass parameter then becomes that $\Omega - 1$ is of order 10^{-60} at the Planck time. Such an extreme fine tuning of the initial value of the universe's mass density can not be explained within standard Big Bang cosmology.

Other Problems

The Friedman models can not explain questions about why the universe is nearly homogeneous and has an isotropic expansion, nor say anything about why the universe is expanding.

10.5.2 Cosmic Inflation

Spontaneous Symmetry Breaking and the Higgs Mechanism

The particles responsible for the electroweak force, $W^{\pm}$ and Z^0 are massive (causing the weak force to only have short distance effects). This was originally a problem for the quantum field theory describing this force, since it made it difficult to create a renormalisable theory[1]. This was solved by Higgs and Kibble in 1964 by introducing the so-called Higgs mechanism.

The main idea is that the massive bosons $W^{\pm}$ and Z^0 are given a mass by interacting with a *Higgs field* ϕ. The effect causes the mass of the particles to be proportional to the value of the Higgs field in vacuum. It is therefore necessary for the mechanism that the Higgs field has a value different from zero in the vacuum (the *vacuum expectation value* must be non-zero).

Let us see how the Higgs field can get a non-zero vacuum expectation value. The important thing for our purpose is that the potential for the Higgs field may be temperature dependent. Let us assume that the potential for the Higgs field is described by the function

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4, \tag{10.135}$$

where the sign of μ^2 depends on whether the temperature is above or below a critical temperature T_c. This sign has an important consequence for the shape of the potential V. The potential is shown in figure 10.16 for two different temperatures. For $T > T_c$, $\mu^2 > 0$, and the shape is like in fig. 10.16(a), and there is a stable minimum for $\phi = 0$. However, for $T < T_c$, $\mu^2 < 0$, and the shape is like in fig. 10.16(b). In this case the potential has stable minima for $\phi = \pm\phi_0 = \pm\frac{|\mu|}{\sqrt{\lambda}}$ and an unstable maximum at $\phi = 0$. For both cases, the potential $V(\phi)$ is invariant under the symmetry transformation $\phi \mapsto -\phi$ (i.e. $V(\phi) = V(-\phi)$).

[1]The problem is that the Lagrangian for the gauge bosons can not include terms like $m^2W_\mu^2$, which are not gauge invariant

The "real" vacuum state of the system is at a stable minimum of the potential. For $T > T_c$, the minimum is in the "symmetric" state $\phi = 0$. On the other hand, for $T < T_c$ this state is unstable. It is therefore called a "false vacuum". The system will move into one of the stable minimas at $\phi = \pm\phi_0$. When the system is in one of these states, it is no longer symmetric under the change of sign of ϕ. Such a symmetry, which is not reflected in the vacuum state, is called *spontaneously broken.* Note that from figure 10.16(b) we see that the energy of the false vacuum is larger than for the real vacuum.

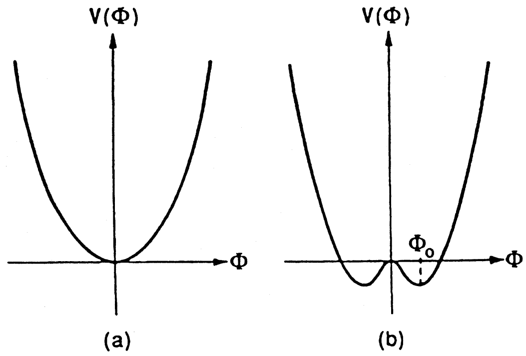


Figure 10.16: The shape of the potential depends on the sign of μ^2.
(a): Higher temperature than the critical, with $\mu^2 > 0$.
(b): Lower temperature than the critical, with $\mu^2 < 0$.

The central idea, which originated the "inflationary cosmology", was to take into consideration the consequences of the unified quantum field theories, the gauge theories, at the construction of relativistic models for the early universe. According to the Friedmann models, the temperature was extremely high in the early history of the universe. If one considers Higgs fields associated with GUT models (grand unified theories), one finds a critical temperature T_c corresponding to the energy $kT_c = 10^{14} GeV$, where k is Boltzmann's constant. Before the universe was about $t_1 = 10^{-35} s$ old, the temperature was larger than this. Thus, the Higgs field was in the symmetric ground state. According to most of the inflation models, the universe was dominated by radiation at this time.

When the temperature decreases, the Higgs potential changes. This could happen as shown in figure 10.17. Here, there is a potential barrier at the critcal temperature, which means that there can not be a classical phase transition. The transition to the stable minimum must happen by quantum tunneling. This is called a first order phase transition.

Guth's Inflation Model

Alan Guth's original inflation model (Guth 1981) was based on a first order phase transition.

According to most of the inflationary models, the universe was dominated

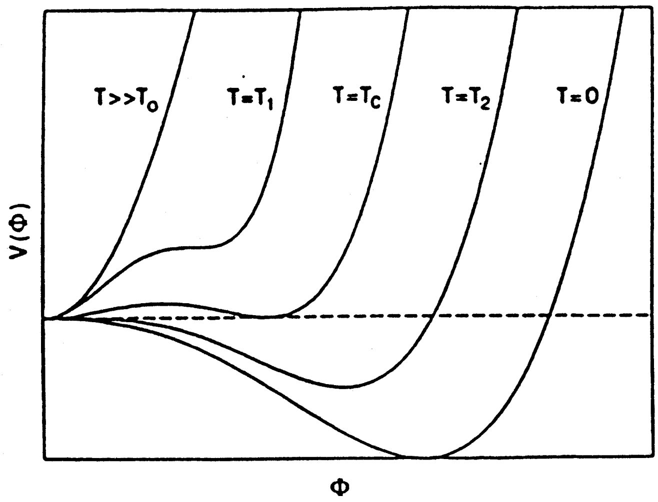


Figure 10.17: The temperature dependence of a Higgs potential with a first order phase transition.

by radiation during the time before $10^{-35}s$. The universe was then expanding so fast that there was no causal contact between the different parts of the universe that became our observable universe. Probably, the universe was rather homogeneous, with considerable spacelike variations in temperature. There was also areas of false vacuum, with energy densities characteristic of the GUT energy scale, which also controls it's critical temperature. While the energy density of the radiation decreased quickly, as a^{-4}, the energy density of vacuum was constant. At the time $t = 10^{-35}s$, the energy density of the radiation became less than that of the vacuum.

At the same time, the potential started to change, such that the vacuum went from being stable to being an unstable false vacuum. Thus, there was a first order phase transition to the real vacuum. Because of the inhomogeneouty of the universe's initial condition, this happened with different speed at differing places. The potential barrier slowed down the process, which happened by tunneling, and the universe was at several places considerably undercooled, and there appeared "bubbles" dominated by the energy of the false vacuum. These areas acted on themselves with repulsive gravity.

By integrating the equation of motion for the expansion factor in such a vacuum dominated bubble, one gets

$$a = e^{Ht}, \qquad H = \sqrt{\frac{8\pi G\rho_c}{3}}. \tag{10.136}$$

By inserting the GUT value above, we get $H = 6.6 \cdot 10^{34}s^{-1}$, i.e. $H^{-1} = 1.5 \cdot 10^{-35}s$. With reference to field theoretical works by Sindney Coleman and others, Guth argumented that a realistic duration of the nucleation process happening during the phase transition is $10^{-33}s$. During this time, the expansion factor increases by a factor of 10^{28}. This vacuum dominated epoch is called the *inflation era.*

Let us look closer at what happens with the energy of the universe in the course of it's development, according to the inflationary models. To understand this we first have to consider what happens at the end of the inflationary era. When the Higgs field reaches the minimum corresponding to the real vacuum, it starts to oscillate. According to the quantum description of the oscillating field, the energy of the false vacuum is converted into radiation and particles. In this way the equation of state for the energy dominating the development of the expansion factor changes from $p = -\rho$, characteristic for vacuum, to $p = \frac{1}{3}\rho$, characteristic of radiation.

The energy density and the temperature of the radiation is then increased enourmously. Before and after this short period around the time $t = 10^{-33}s$ the radiation energy increases adiabatically, such that $\rho a^4 =$ constant. According to Stefan-Boltzmanns law of radiadion, $\rho \propto T^4$. Therefore, $aT =$ constant during adiabatic expansion. This means that during the inflationary era, while the expansion factor increases exponentially, the energy density and temperature of radiation decreases exponentially. At the end of the inflationary era, the radiation is reheated so that it returns to the energy it had when the inflationary era started.

It may be interesting to note that the Newtonian theory of gravitation does not allow an inflationary era, since stress has no gravitational effect according to it.

The Inflation Models' Answers to the Problems of the Friedmann Models

The horizon problem will here be investigated in the light of this model. The problem was that there was room for about 10000 causally connected areas inside the area spanned by our presently observable universe at the time. Let us calculate the horizon radius l_h and the radius a of the region presently within the horizon, $l_h = 15 \cdot 10^9 ly = 1.5 \cdot 10^{26} cm$, at the time $t_1 = 10^{-35}s$ when the inflation started. From equation (10.128) for the radiation dominated period before the inflatinary era, one gets

$$l_h = 2t_1 = 6 \cdot 10^{-25} cm. \tag{10.137}$$

The radius, at time t_1, of the region corresponding to our observable universe, is found by using that $a \propto e^{Ht}$ during the inflation era from $t_1 = 10^{-35}s$ to $t_2 = 10^{-33}s$, $a \propto t^{\frac{1}{2}}$ in the radiation dominated period from t_2 to $t_3 = 10^{11}s$, and $a \propto t^{\frac{2}{3}}$ in the matter dominated period from t_3 until now, $t_0 = 10^{17}s$. This gives

$$a_1 = \frac{e^{Ht_1}}{e^{Ht_2}} \left(\frac{t_2}{t_3}\right)^{\frac{1}{2}} \left(\frac{t_3}{t_0}\right)^{\frac{2}{3}} l_h(t_0) = 1.5 \cdot 10^{-28} cm. \tag{10.138}$$

We see that at the beginning of the inflationary era the horizon radius, l_h, was larger than the radius a of the region corresponding to our observable universe. The whole of this region was then causally connected, and thermic equilibrium was established. This equilibrium has been kept since then, and explains the observed isotropy of the cosmic background radiation.

We will now consider the flatness problem. This problem was the necessity, in the Friedmann models, of fine tuning the initial density in order to obtain the closeness of the observed mass density to the critical density. Again, the inflationary models give another result. Inserting the expansion factor (10.136) into equation (10.132), we get

$$\Omega - 1 = \frac{k}{H^2} e^{-2Ht}, \tag{10.139}$$

where H is constant and given in eq. (10.136). The ratio between $\Omega - 1$ at the end of and the beginning of the inflationary era becomes

$$\frac{\Omega_2 - 1}{\Omega_1 - 1} = e^{-2H(t_2 - t_1)} = 10^{-56}. \tag{10.140}$$

Contrary to in the Friedmann models, where the mass density moves *away* from the critical density as time is increasing, the density approaches the critical density exponentially during the inflationary era. Within a large range of initial conditions, this means that according to the inflation models the universe should still have almost critical mass density.

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