Solutions to final exam in FYS4160

Problem 1

a) By definition of the inverse, $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu = const$. Therefore, $\nabla_\sigma(g^{\mu\rho}g_{\rho\nu}) = \nabla_\sigma(g^{\mu\rho})g_{\rho\nu} + g^{\mu\rho}\nabla_\sigma g_{\rho\nu} = 0$ (Leibniz rule) $\rightsquigarrow \nabla_\sigma(g^{\mu\rho})g_{\rho\nu} = 0 \rightsquigarrow \nabla_\sigma(g^{\mu\rho}) = 0$.

b) Using the result from a), $\nabla^\mu V_\mu = g^{\mu\rho}\nabla_\rho(g_{\mu\sigma}V^\sigma) = g^{\mu\rho}g_{\mu\sigma}\nabla_\rho(V^\sigma) = \delta^\rho_\sigma\nabla_\rho V^\sigma = \nabla_\rho V^\rho = 0$. Note that you cannot argue that $V_\mu V^\mu$ is covariantly constant which is the case for $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$ as in part a).

Problem 2

The stress-energy tensor of a perfect fluid in a frame moving with 4-velocity u^μ with respect to the fluid rest-frame is given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}\,.$$

Its trace is therefore given by $T^\mu_\mu = (\rho+p)u^\mu u_\mu + pg^\mu_\mu = -(\rho+p) + 4p = -\rho + 3p = 0$ (the last step being valid only for radiation). As should have been expected for a scalar quantity like the trace, this is independent of the frame. We would get the same result if we actually did the boost along the x-axis.

Marking: 0.5 pt for correct $T^{\mu\mu}$, 0.5 for $\rho = 3p$. 2 pts if T^μ_μ is correctly derived (-0.5 if not argued why it must be the same in all frames).

Problem 3

a) The Einstein equivalence principle (EEP) states that in small enough regions of spacetime, the laws of physics reduce to those of special relativity. It is therefore impossible to detect the existence of a gravitational field by means of local experiments. The principle implies that gravity is universal. We cannot escape gravity, i.e. there exists no gravitationally neutral object (which is the decisive difference to other forces). This implies that the notion of acceleration due to gravity is ambiguous and that we should define unaccelerated motion as motion in free fall. Since a force is something that leads to acceleration, by Newton's second law, we conclude that gravity is *not* a force but rather a manifestation of a fundamental feature of spacetime – its curvature.

Marking: 1 pt for principle. 2 for geometry connection. 1 for difference. ALWAYS: minus points for wrong statements

b) Riemannian manifolds are mathematical objects that capture the idea of being locally flat, such as Euclidian space or Minkowski space (the only freedom we have here is the signature of the metric!). Such as the EEP suggests that the laws of physics reduce to those of special relativity in small enough regions of spacetime, Riemannian manifolds with the correct signature can be described by (flat) Minkowski space in small patches of the manifold. Locally, i.e. by means of only one test particle, we cannot detect gravity which corresponds to the fact that we can always find locally inertial coordinates on a Riemannian manifold. We *can* however map out a gravitational field with at least *two* test particles (via the observed tidal forces), which corresponds to the fact the one cannot in general define a single inertial reference frame for two points on the manifolds that are separated by a finite distance (which

led us to the construction of the curvature tensor, and geodesic deviation corresponding to tidal forces). Therefore, the definition of a Riemannian manifold entails an atlas, i.e. a collection of patches that are locally flat and that can be sewed together smoothly even though the manifold can have non-vanishing (intrinsic) curvature. Also the *topology* (i.e. the global shape of the manifold) can be non-trivial – which is not a direct consequence of the equivalence principle, but certainly does not violate the requirement of locality.
Marking: 0.5 pt for locally flat = locally SR. 0.5 for Minkowski vs. Euclidian distinction. 1 for existence of local coordinates = impossible to test gravity with 1 testparticle. 1 for non-local properties.

c) The electromagnetic force does not have the same universality as gravity does. Even in such a toy universe we would be able to single out gravity: The curvature of spacetime effects the trajectory of *all* particles, i.e. every particle has an identical gravitational charge, but the electromagnetic force does not effect electrically neutral objects (to which elementary particles of opposite charge can combine). Therefore, electromagnetism cannot be represented as something as intrinsic/universal as the curvature of spacetime.
This conclusion changes if only particles with the same *sign* of the charge were to exist, because then there would be no neutral particles. There is *still* a difference, however, because charge is quantized. This implies that the charge of a bound system is the same as the sum of the charges of the components – while this is not true for the mass. This difference could for example be tested in a binary system of two compound objects that rotate around each other: increasing the size (mass/charge) of each of the objects by adding more particles would imply that *fraction* of energy radiated in gravitational and electromagnetic eaves changes.
Marking: 1 pt for correct conclusion + correct argument in each case.

<u>Problem 4</u>
The four-velocity $U^\mu = dx^\mu/d\tau$ satisfies $U_\mu U^\mu = -1$. For a stationary observer ($U^i = 0$) in Schwarzschild coordinates, this implies

$$-1 = U_\mu U^\mu = g^{00}(U^0)^2 = -\left(1-\frac{2GM}{r}\right)^{-1} \quad \rightsquigarrow \quad U^0 = \left(1-\frac{2GM}{r}\right)^{-1/2} . \tag{1}$$

The acceleration a^μ is given by the covariant directional derivative of the 4-velocity:

$$\begin{aligned} a^\mu &= \frac{D}{d\tau}U^\mu = U^\nu \nabla_\nu U^\mu = U^\nu \partial_\nu U^\mu + \Gamma^\mu_{\nu\rho} U^\nu U^\rho && (2)\\ &\stackrel{(1)}{=} \Gamma^\mu_{00}\left(1-\frac{2GM}{r}\right)^{-1} && (3)\\ &= \frac{1}{2} g^{\mu\rho}\left(\partial_t g_{0\rho} + \partial_t g_{\rho 0} - \partial_\rho g_{00}\right)\left(1-\frac{2GM}{r}\right)^{-1} && (4)\\ &= -\frac{1}{2}\left(1-\frac{2GM}{r}\right)^{-1} g^{\mu\rho}\delta^r_\rho \partial_r g_{00} , && (5) \end{aligned}$$

where the last step follows because the Schwarzschild metric is static (and g_{00} only depends on r). With $g^{rr} = -g_{00}$ this simplifies to

$$a^\mu \;=\; \left(0, -\frac{1}{2}\partial_r\left(1-\frac{2GM}{r}\right) = \frac{GM}{r^2}, 0, 0\right) \tag{6}$$

because the Schwarzschild metric is diagonal. The magnitude of the actual acceleration felt by the observer is given by the magnitude of the spatial acceleration,

$$a = \sqrt{-a^i a_i} = \left(1-\frac{2GM}{r}\right)^{-\frac{1}{2}}\frac{GM}{r^2}\,. \tag{7}$$

For $r \gg 2GM$, this agrees as expected with the Newtonian acceleration (GM/r^2), but for smaller distances the actual acceleration is larger. Approaching the Schwarzschild radius, it would require an infinite amount of acceleration to escape.
Marking: 1 pt each for arriving at (1), (2), (6), (7). One for the physics discussion.

Problem 5

a) The stated geodesic equation takes the form of a conservation equation, $\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = const$. For a circular orbit we have $\dot{r} = 0$, so $V_{\text{eff}} = \frac{1}{2}L^2r^{-2} - GML^2r^{-3}$ must have an extremum and hence $0 = V'_{\text{eff}}(r_c) = -L^2r_c^{-3} + 3GML^2r_c^{-4} \rightsquigarrow r_c = 3GM$. We still need to check that the orbit is indeed *unstable* by finding the sign of the second derivative: $V''_{\text{eff}} = 3L^2r^{-4} - 12GML^2r^{-5} \rightsquigarrow V''_{\text{eff}}(r_c) = L^2(GM)^{-4}\,(3/3^4 - 12/3^5) = -L^2(3GM)^{-4} < 0$.
Marking: 1 pt for identifying circular orbits, 1 for r_c from first derivative, 1 for stability.

b) The four-velocity $u^\mu = dx^\mu/d\lambda \equiv \dot{x}^\mu$ is light-like. For the Schwarzschild metric:

$$0 = u_\mu u^\mu = g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2 \tag{8}$$

$$\overset{\theta=\frac{\pi}{2}}{=} -\left(1-\frac{2GM}{r}\right)\dot{t}^2 + \left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 \tag{9}$$

For circular motion, we have $\dot{r} = 0$. Hence,

$$\frac{dt}{d\phi} = \frac{\dot{t}}{\dot{\phi}} = \frac{r}{\sqrt{1-\frac{2GM}{r}}} \overset{r=r_c}{=} 3\sqrt{3}\,GM\,. \tag{10}$$

Integrating over one revolution yields

$$\Delta t = \int_0^{2\pi} d\phi \frac{dt}{d\phi} = 6\sqrt{3}\pi\,GM\,. \tag{11}$$

The proper time measured by a stationary observer ($dr = d\phi = d\theta = 0$) in the Schwarzschild geometry is given by

$$-d\tau^2 = -\left(1-\frac{2GM}{r}\right)dt^2 \tag{12}$$

At $r = r_c$, this becomes $d\tau/dt = 1/\sqrt{3}$, such that our stationary observer sees the emitted photon again after $\Delta T = 6\pi\, GM$.

For a distant observer, $r \gg 2GM$, the eigentime coincides with the coordinate time, so she measures that the photon takes $\Delta t = 6\sqrt{3}\pi\, GM$ to move once around the black hole.

Marking: 1 pt each for arriving at (9), 'circular' means $\dot{r} = 0$, (10), 1 each for the two results.

c) First we need to transform the equation for $r = r(\lambda)$ given in the problem into one for $r = r(\phi)$, by noting that $\dot{r} = (dr/d\phi)\dot{\phi} \equiv r'\dot{\phi}$:

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}r'^2\dot{\phi}^2 + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} && (13) \\ &= \frac{L^2}{2r^4}r'^2 + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}\,. && (14) \end{aligned}$$

We now simplify this equation by taking the derivative with respect to ϕ on both sides (as we did when deriving the perihelion precession):

$$\begin{aligned} 0 &= -\frac{2L^2}{r^5}r'^3 + \frac{L^2}{r^4}r'r'' + \frac{L^2}{r^3}r' - \frac{3GML^2}{r^4}r' && (15) \\ \rightsquigarrow \quad 0 &= -\frac{2}{r}r'^2 + r'' + r - 3GM\,. && (16) \end{aligned}$$

First, we note that $r = r_c = 3GM$ indeed solves this equation. Then we introduce, as indicated in the problem, a small perturbation η, by substituting $r = 3GM(1+\eta)$ and keeping only linear terms in η:

$$\begin{aligned} 0 &= -\frac{6GM}{1+\eta}\eta'^2 + 3GM\eta'' + 3GM(1+\eta) - 3GM && (17) \\ &\simeq 3GM(\eta'' + \eta) && (18) \\ \rightsquigarrow \quad 0 &= \frac{d^2\eta}{d\phi^2} + \eta\,. && (19) \end{aligned}$$

The solution to this equation is $\eta(\phi) = Ae^{\phi} + Be^{-\phi}$, which exhibits exponential growth in ϕ (because the solution proportional to B will exponentially decay). In other words, the size of the perturbation η grows without bound – instead of oscillating as it would in case of a *stable* orbit – so the circular orbit at $r = r_c$ is *unstable*.

Marking: 1 pt for re-writing $\mathcal{E}$ as function of r, r', 1 for simplifying (+ not that that r_c indeed OK), 1 for final equation for η, 1 for solution + intepretation

Problem 6

a) We first calculate the Christoffel symbols, keeping only terms linear in $h_{\mu\nu}$:

$$\begin{aligned} \Gamma^{\mu}_{\rho\sigma} &= \frac{1}{2}g^{\mu\lambda}\left(g_{\rho\lambda,\sigma} + g_{\sigma\lambda,\rho} - g_{\rho\sigma,\lambda}\right) && (20) \\ &= \frac{1}{2}g^{\mu\lambda}\left(h_{\rho\lambda,\sigma} + h_{\sigma\lambda,\rho} - h_{\rho\sigma,\lambda}\right) && (21) \\ &\simeq \frac{1}{2}\eta^{\mu\lambda}\left(h_{\rho\lambda,\sigma} + h_{\sigma\lambda,\rho} - h_{\rho\sigma,\lambda}\right)\,. && (22) \end{aligned}$$

The Christoffel symbols are thus linear in $h_{\mu\nu}$, and the Ricci tensor to linear order becomes

$$
\begin{aligned}
R_{\mu\nu} &= R^{\rho}{}_{\mu\rho\nu} \simeq \partial_\rho \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} && (23)\\
&\simeq \frac{1}{2}\eta^{\rho\lambda}\left(h_{\nu\lambda,\mu\rho} + h_{\mu\lambda,\nu\rho} - h_{\nu\mu,\lambda\rho}\right) - \frac{1}{2}\eta^{\rho\lambda}\left(h_{\rho\lambda,\mu\nu} + h_{\mu\lambda,\rho\nu} - h_{\rho\mu,\lambda\nu}\right) && (24)\\
&\simeq \frac{1}{2}\left(h_{\nu}{}^{\rho}{}_{,\mu\rho} + \underline{h_{\mu}{}^{\rho}{}_{,\nu\rho}} - h_{\nu\mu,}{}^{\rho}{}_{\rho} - h_{,\mu\nu} - \underline{h_{\mu}{}^{\rho}{}_{,\rho\nu}} + h^{\rho}{}_{\mu,\rho\nu}\right) && (25)
\end{aligned}
$$

The raising of indices in the last step was done with respect to the Minkowski metric, which is consistent because all terms are already first order in $h^{\mu\nu}$ and the neglected terms hence second order. The underlined terms cancel, because partial derivatives commute, which leaves us with the result stated in the problem once we use Einstein's equations in vacuum, $R_{\mu\nu} = 0$.
Marking: 1 pt each for Christoffels, correct use of first order in $h_{\mu\nu}$, result for $R_{\mu\nu}$, application to vacuum equations

b) The equation is linear in $h_{\mu\nu}$, so it sufficed to replace $h_{\mu\nu} \to \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ in the expression for $R_{\mu\nu}$, and check that the result vanishes for any 4-vector field ξ_μ:

$$
\frac{1}{2}\left(\xi_{\nu,}{}^{\rho}{}_{\mu\rho} - \xi_{\nu,\mu}{}^{\rho}{}_{\rho} - \xi_{\rho,}{}^{\rho}{}_{\mu\nu} + \xi_{\rho,\mu}{}^{\rho}{}_{\nu}\right) + \frac{1}{2}\left(\xi_{\rho,\nu\mu}{}^{\rho} - \xi_{\mu,\nu}{}^{\rho}{}_{\rho} - \xi_{\rho,}{}^{\rho}{}_{\mu\nu} + \xi_{\mu,\rho}{}^{\rho}{}_{\nu}\right) \qquad (26)
$$

Now, because partial derivatives commute, the 1$^{\text{st}}$ and 2$^{\text{nd}}$ terms cancel. The same goes for the 3$^{\text{rd}}$ and 4$^{\text{th}}$ terms, as well as for the 5$^{\text{th}}$ and 6$^{\text{th}}$, and 7$^{\text{th}}$ and 8$^{\text{th}}$ terms. The significance of this observation is that not all 10 components of $h_{\mu\nu}$ (recall that the metric is symmetric!) are *physical* degrees of freedom. This is in principle the same as in the full theory, which is invariant under general coordinate transformations. In the linearized version of the theory, only a subgroup of these symmetries remains, which takes the form of *gauge transformations* (in analogy to the case of electrodynamics, which is a theory for a 4-vector field A_μ that stays invariant under the replacement $A_\mu \to A_\mu + \partial_\mu \alpha$ for any *scalar* spacetime function α).
Marking: 2 pts for correct proof, 1 for interpretation

c) One of the ways to fix the gauge freedom derived in b) is known as the Lorenz (no 't'!) gauge, $\partial_\sigma \partial_\nu h^{\sigma}{}_{\mu} + \partial_\sigma \partial_\mu h^{\sigma}{}_{\nu} - \partial_\mu \partial_\nu h = 0$, from which the statement in the problem immediately follows. To see that this is indeed always possible to achieve, start from a given $h_{\mu\nu}$ and define $\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}$. Then,

$$
\begin{aligned}
0 &\overset{!}{=} \partial_\sigma \partial_\nu \tilde{h}^{\sigma}{}_{\mu} + \partial_\sigma \partial_\mu \tilde{h}^{\sigma}{}_{\nu} - \partial_\mu \partial_\nu \tilde{h} && (27)\\
&= 2\partial_\sigma \partial_{(\nu} h^{\sigma}{}_{\mu)} - \partial_\mu \partial_\nu h + \left(\partial_\sigma \partial_\nu \partial^\sigma \xi_\mu + \partial_\sigma \partial_\nu \partial_\mu \xi^\sigma\right) \\
&\quad + \left(\partial_\sigma \partial_\mu \partial^\sigma \xi_\nu + \partial_\sigma \partial_\mu \partial_\nu \xi^\sigma\right) - 2\partial_\mu \partial_\nu \partial^\sigma \xi_\sigma && (28)\\
\rightsquigarrow \quad \Box \partial_{(\mu}\xi_{\nu)} &= \frac{1}{2}\partial_\mu \partial_\nu h - \partial_\sigma \partial_{(\mu} h^{\sigma}{}_{\nu)} && (29)
\end{aligned}
$$

This is a set of inhomogenous partial differential equations for ξ_μ, with a uniquely determined (though typically hard to find in practice) solution for any set of functions

$h_{\mu\nu}$.
The wave equation $\Box f(t, \mathbf{x}) = (-\partial_t^2 + \nabla^2) f = 0$ is satisfied for $f(t, \mathbf{x}) = g(\mathbf{x} - \mathbf{e}t)$, where $\mathbf{e}$ is a unit 3-vector and g is an arbitrary function from R^3 to R. This describes a wave package of the form $g(\mathbf{x})$ moving with speed $c = 1$ in the direction of $\mathbf{e}$ (for a different speed, $\mathbf{e}$ in the argument would need to be multiplied with $v \neq 1$). In our case $f = h_{\mu\nu}$ are the metric components – so this must describe a gravitational wave.
Marking: 2 pts each for a fully correct argument: i) why $\Box$ and ii) why grav. waves

d) A monopole describes a spherically symmetric solution. The wave propagates in vacuum, but by Birkhoff's theorem any spherically symmetric vacuum solution must be static, i.e. cannot describe propagation. A dipole would imply a mass-center that moves back and forth – which is impossible for an isolated system due to 4-momentum conservation (unlike the case of electromagnetism, where the total charge of a system can move back and forth if tied e.g. to a spring). Therefore, the first contribution in the multipole expansion comes from the quadrupole term.
Marking: 1 pt each

Problem 7

a) The particle horizon is the maximal distance that any particle (or piece of information) can have propagated since $t = 0$ (which, as we recall, is defined by $a(t \to 0) \to 0$), and hence equal to the maximal radius of the past light cone. It is defined by

$$d_H(t_0) \equiv a(t_0) \int_0^{t_0} \frac{dt}{a(t)} \,. \tag{30}$$

The Hubble horizon, or better just the inverse 'Hubble rate' is defined by

$$H^{-1} \equiv a/\dot{a} \,. \tag{31}$$

While its significance is less fundamental than that of d_H, H^{-1} can be thought of as a 'local' version of d_H. For typical matter contents, for example, we have to a good approximation $H^{-1}(t) \sim d_H(t)$ – see problems b) and c). The Hubble rate H is also the factor of proportionality with which (not too!) distant objects appear redshifted the further they are away.
Marking: 1 pt each for def and significance

b) For $a(t) \propto t^n$, $n < 1$, we have

$$\frac{d_H(t)}{a(t)} = \int_0^t \frac{dt'}{a(t)(t'/t)^n} = \frac{t}{(1-n)a} \propto t^{1-n} \propto a^{1/n-1} \tag{32}$$

$$\frac{1}{aH} = \frac{1}{\dot{a}} = \frac{t}{na} = \frac{1-n}{n}\frac{d_H}{a} \tag{33}$$

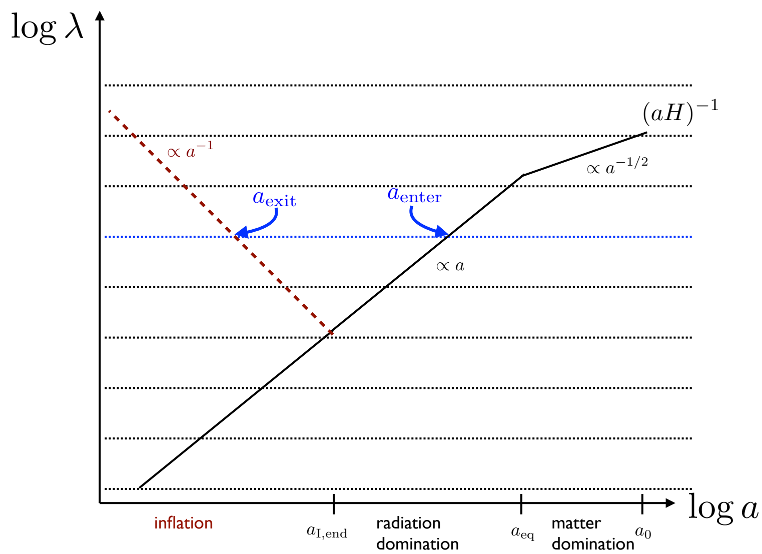
Marking: 1 pt each

c) Exponential growth, $a \propto \exp ct$, simply implies $H = c$ – so the Hubble rate is constant in this case. Thus,

$$\frac{d_H(t)}{a(t)} = \int_0^t \frac{dt'}{a(t)\exp[H(t'-t)]} = \frac{e^{Ht}}{aH}\left[e^{-Ht'}\right]_t^0 = \frac{1}{aH}\left(1-e^{-Ht}\right) \tag{34}$$

$$\frac{1}{aH} = \frac{1}{\dot{a}} = \frac{e^{Ht}}{e^{Ht}-1}\frac{d_H}{a} \tag{35}$$

Marking: 0.5 pt each, 1 for correct use of relation to H



d) The solid black line shows how the comoving horizon – calculated in b) – evolves with the scalefactor a: everything above the line is outside, everything below the line inside the Horizon. A given comoving (or coordinate) distance, on the other hand, does not change with the expansion of the universe, as indicated by the dotted lines. *Perturbations* to a perfectly homogeneous universe can be described by their spatial extent and hence a given scale λ – which corresponds to the *physical* extent of that region *today* (in an exact NFW spacetime, it does not make sense to distinguish different physical scales!). As apparent from the figure, *any* such perturbation that we can observed today (at $a_0 = 1$) inside the horizon has been outside the horizon at some earlier time – which violates causality because there is no possible causal mechanism that could have *created* such perturbations at such early times / super-horizon scales. This conclusion can be evaded if the evolution of the Horizon at early time is changed such that those scales actually *have* been inside the horizon nevertheless, at even earlier times. As an example, this is shown for the case of exponential growth as calculated in c) (dashed red line): the specific scale λ indicated with blue is then inside the horizon for $a < a_{\rm exit}$ (where some causal mechanism could produce a perturbation at that scale), outside for $a_{\rm exit} < a < a_{\rm enter}$ (where no perturnations can be causally affected) and then inside again for $a > a_{\rm enter}$ (where we eventually can observe it). Note that $a_{\rm exit}$ and $a_{\rm enter}$ depend on λ here! From this discussion, the issue of causality violation could in general be resolved if only the comoving Horizon were to *decrease* in time:

$$0 > \frac{d}{dt}\frac{1}{aH} = \frac{d}{dt}\frac{1}{\dot{a}} = -\frac{\ddot{a}}{\dot{a}^2} \quad \Leftrightarrow \quad \ddot{a} > 0\,. \tag{36}$$

Using the second Friedman equation, and $a > 0$, this is equivalent to requiring an equation of state with sufficiently negative pressure

$$p < -\rho/3\,. \tag{37}$$

The simplest example is a cosmological constant, with $p = -\rho$, which leads to the exponential growth studied in c).

Marking: 2pt for all aspects in the figure (in particular comoving scales rather than physical – which we have seen the lecture), 1 for explanation of causality violation, 0.5 for the connection to perturbations, 1 pt for inflation condition leading to $\ddot{a} > 0$, 0.5 for eq. of state.