UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in Fys 4160 The general theory of relativity
Day of exam: May 31. Kl. 14.30. Sal 3D Silurveien 2
Exam hours: 4 hours
This examination paper consists of 9 page(s) including 7 pages with formulae.
Appendices: 0
Permitted materials: 7 pages with formulae attached to the problems.

Make sure that your copy of this examination paper is complete before answering.

Problem 1

Let u^α be the components of the tangent vector field of a time-like curve.

a) Show that $g_{\alpha\beta}u^\alpha u^\beta$ is constant along the curve. What is the value of the constant?

b) Show that the geodesic equation can be written in the following form: $\frac{du_\alpha}{ds}-\frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}u^\beta u^\gamma=0$.

c) Assume that the metric is static and the space is cylindrically symmetric with cylindrical coordinates $(r,\ \theta,\ z)$. What constants of motion are there then for a free particle?

Problem 2

Let $T^{\alpha\beta}=p\eta^{\alpha\beta}+\left(\rho+p/c^2\right)u^\alpha u^\beta$ be the components of the energy momentum tensor of an perfect fluid in flat space time with Minkowski metric $\eta_{\mu\nu}$. Here p is the pressure and ρ the mass density of the fluid, and u^α the components of its 4-velocity.

a) Explain why the conservation law $T^{\alpha\beta}{}_{;\beta}=0$ in this case reduces to $T^{\alpha\beta}{}_{,\beta}=0$.

b) We shall consider the Newtonian limit where p/c^2 can be neglected compared to ρ in the second term of $T^{\alpha\beta}$, and the components of the 4-velocity of the fluid are $u^\alpha\approx\left(c,\vec{v}\right)$ where $\vec{v}$ is the ordinary velocity of a fluid element. Show that in this case the conservations law a) implies mass conservation as represented by the equation of continuity, $\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\vec{v}\right)=0$,

c) and momentum conservation as represented by the Euler equation of motion,
$\rho\left(\frac{\partial\vec{v}}{\partial t}+\left(\vec{v}\cdot\nabla\right)\vec{v}\right)=-\nabla p$.

Problem 3

Consider a De Sitter spacetime with coordinates (t, r) and line element

$$ds^2 = -c^2dt^2 + e^{2Ht}\left(dr^2 + r^2 d\Omega^2\right),$$

where the Hubble parameter H is constant.

a) Find the redshift of light emitter from a coordinate r as measured at a point of time t_0 by an observer at the origin. The Hubble parameter H is assumed to be known.

b) What is the 4-acceleration of a reference particle at rest in the coordinate system? What does your result tell about the reference frame in which these coordinates are co-moving? Will an observer with constant radial coordinate r experience an acceleration of gravity?

Introducing coordinates (T, R) by the transformation

$$R = re^{Ht} \;,\; T = t - \ln\left(1 - H^2r^2e^{2Ht}\right) \text{ or } r = \frac{R}{e^{HT}\sqrt{1 - H^2R^2/c^2}} \;,\; e^{Ht} = e^{HT}\sqrt{1 - H^2R^2/c^2} \,,$$

the line element takes the form (you need not show this)

$$ds^2 = -\left(c^2 - H^2R^2\right)dT^2 + \frac{dR^2}{1 - H^2R^2/c^2} + R^2 d\Omega^2 \,.$$

c) Find the redshift of light emitted from a coordinate R as measured by an observer at the origin. Why is your result different to the one in a)?

d) What is the 4-acceleration of a reference particle at rest in the coordinate system? What does your result tell about the reference frame in which these coordinates are co-moving? Will an observer with constant radial coordinate r experience an acceleration of gravity?

e) How does a reference particle with $r = r_0$ =constant move in the (T, R)-coordinate system.

f) How is the redshift of light explained in the (T, R)-coordinate system? How is it explained in the (t, r)-system?