Lecture 10 13.02.18

3.1.3 Gravitational time dilation

$$ds^2 = -(1 - \frac{r^2\omega^2}{c^2})c^2dt^2 + dr^2 + r^2d\theta^2 + dz^2 + 2r^2\omega d\theta dt \tag{3.21}$$

We now look at *standard clocks* with constant r and z.

$$ds^2 = c^2dt^2[-(1 - \frac{r^2\omega^2}{c^2}) + \frac{r^2}{c^2}(\frac{d\theta}{dt})^2 + 2\frac{r^2\omega}{c^2}\frac{d\theta}{dt}] \tag{3.22}$$

Let $\frac{d\theta}{dt} \equiv \dot{\theta}$ be the angular velocity of the clock in RF. The proper time interval measured by the clock is then

$$ds^2 = -c^2d\tau^2 \tag{3.23}$$

From this we see that

$$\boxed{d\tau = dt\sqrt{1 - \frac{r^2\omega^2}{c^2} - \frac{r^2\dot{\theta}^2}{c^2} - 2\frac{r^2\omega\dot{\theta}}{c^2}}} \tag{3.24}$$

which may be written

$$d\tau = dt\sqrt{1 - r^2\left(\omega + \dot{\theta}\right)^2 / c^2}\,.$$

Here t represents proper time in R at the axis, $r = 0$, which is equal to the proper time in F, and τ represents proper time at an arbitrary point in R. We see that the rate of proper time in R decreases with increasing distance from the axis. Also it decreases with increasing angular velocity ω of R relative to F, and it depends upon the angular velocity $\dot{\theta}$ of the clock in R, both its magnitude and sign. The rate of proper time in R compared to that in F, $d\tau / dt$, is maximal for $\dot{\theta} = -\omega$. Such a clock is at rest in F which is non-rotating relative to the large scale cosmic masses. For this clock $d\tau = dt$. As considered in R such a clock moves together with the large scale cosmic masses. Hence *a clock at rest relative to the large scale cosmic masses goes at a maximal rate*. Such a twin ages maximally fast.

A non-moving standard clock in RF: $\dot{\theta} = 0 \Rightarrow$

$$d\tau = dt\sqrt{1 - \frac{r^2\omega^2}{c^2}} \tag{3.25}$$

Seen from IF, the non-rotating laboratory system, (3.25) represents the **velocity dependent time dilation** from the special theory of relativity.

But how is (3.25) interpreted in RF? The clock does not move relative to an observer in this system, hence what happens can not bee interpreted as a velocity dependent phenomenon. According to Einstein, the fact that standard clocks slow down the farther away from the axis of rotation they are, is due to a **gravitational effect**.

We will now find the gravitational potential at a distance r from the axis. The sentripetal acceleration is v^2/r, $v = r\omega$ so:

$$\Phi = -\int_0^r g(r)dr = -\int_0^r r\omega^2 dr = -\frac{1}{2}r^2\omega^2$$

We then get:

$$d\tau = dt\sqrt{1 - \frac{r^2\omega^2}{c^2}} = dt\sqrt{1 + \frac{2\Phi}{c^2}} \tag{3.26}$$

In RF the position dependent time dilation is interpreted as a **gravitational time dilation***: Time flows slower further down in a gravitational field.*

3.1.5 The Sagnac effect

IF description:

Here the velocity of light is isotropic, but the emitter/receiver moves due to the disc's rotation as shown in Figure 3.7. Photons are emitted/received in/from opposite directions. Let t_1 be the travel time of photons which move *with* the rotation.

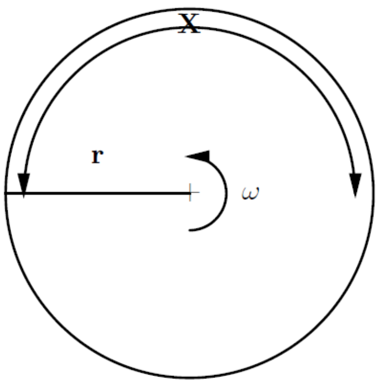

Figure 3.7: The Sagnac effect demonstrates the **anisotropy** of the speed of light when measured in a rotating reference frame.

Then

$$\begin{aligned} &\Rightarrow \quad 2\pi r + r\omega t_1 = ct_1 \\ &\Rightarrow \quad t_1 = \frac{2\pi r}{c - r\omega} \end{aligned} \tag{3.28}$$

Let t_2 be the travel time for photons moving against the rotation of the disc. The difference in travel time is

A is the area enclosed by the photon path or orbit.

$$\begin{aligned} \Delta t = t_1 - t_2 &= 2\pi r\left(\frac{1}{c - r\omega} - \frac{1}{c + r\omega}\right) \\ &= \frac{2\pi r 2r\omega}{c^2 - r^2\omega^2} \\ &= \gamma^2\frac{4A\omega}{c^2} \end{aligned} \tag{3.29}$$

RF description:

$ds^2 = 0$ along the world line of a photon

$$ds^2 = -\left(1 - \frac{r^2\omega^2}{c^2}\right)c^2dt^2 + r^2d\theta^2 + 2r^2\omega d\theta dt$$

$$\text{let} \quad \dot{\theta} = \frac{d\theta}{dt}$$

$$r^2\dot{\theta}^2 + 2r^2\omega\dot{\theta} - (c^2 - r^2\omega^2) = 0$$

$$\dot{\theta} = \frac{-r^2\omega \pm \sqrt{(r^4\omega^2 + r^2c^2 - r^4\omega^2)}}{r^2}$$

$$\begin{aligned}\dot{\theta} &= -\omega \pm \frac{rc}{r^2} \\ &= -\omega \pm \frac{c}{r}\end{aligned} \tag{3.30}$$

The speed of light: $v_\pm = r\dot{\theta} = -r\omega \pm c$. We see that in the rotating frame RF, the measured (coordinate) velocity of light is NOT isotropic. The difference in the travel time of the two beams is

$$\begin{aligned}\Delta t &= \frac{2\pi r}{c - r\omega} - \frac{2\pi r}{c + r\omega} \\ &= \gamma^2 \frac{4A\omega}{c^2}\end{aligned} \tag{3.31}$$

We had, during the lecture, a conversation concerning the question:

Does the general theory of relativity contain a general principle of relativity, valid for rotational motion?

If this is the case it should be a valid point of view for an observer at rest in the frame R to consider this frame as at rest while the outside world rotates.

The meaning of the term "valid point of view" is that the observer must be able to explain the results of his experiments and observations from this point of view.

One argument against the validity of the principle of relativity for rotational motion was raised. A star sufficiently far away would, from this point of view, move with superluminal velocity, $v > c$, and that is not allowed. The mass of a particle with rest mass m_0 and velocity v is according to the special theory of relativity $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$, so it must have a velocity $v < c$.

We shall investigate this question later when we have at our disposition general relativistic Lagrangian dynamics.