Lecture 11 19.02.18

Non-integrability of a simultaneity curve in a rotating frame

We have made a separation of the spacetime line-element, ds^2, in a spatial part, dl^2, and a temporal part, $c^2 d\hat{t}^2$, according to $ds^2 = dl^2 - c^2 d\hat{t}^2$, where

$$dl^2 = \left(g_{ij} - \frac{g_{i0} g_{j0}}{g_{00}} \right) dx^i dx^j \quad , \quad d\hat{t} = \sqrt{-g_{00}} \left(dx_0 + \frac{g_{i0}}{g_{00}} dx^i \right) \quad , \quad x_0 = ct \, .$$

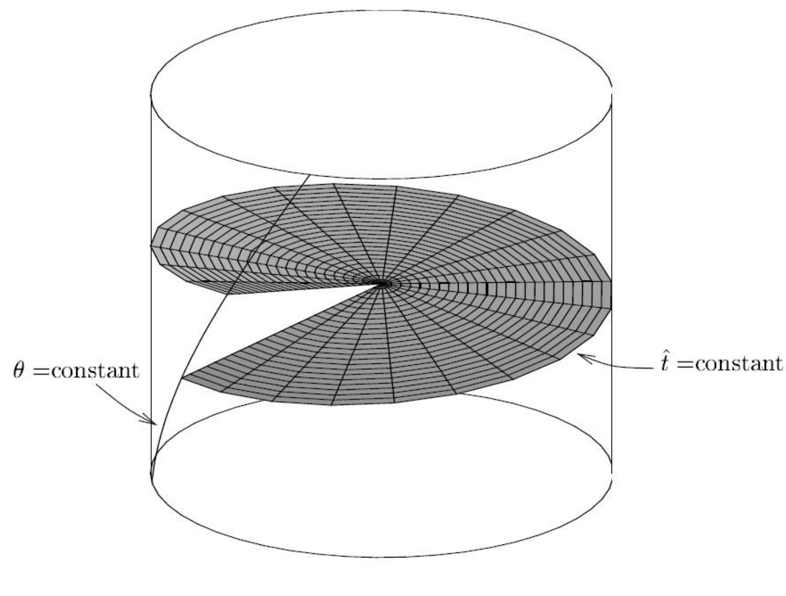
As applied to the rotating reference frame R this gives

$$dl^2 = dr^2 + \frac{r^2}{1 - r^2\omega^2 / c^2} d\theta^2 + dz^2 \quad , \quad d\hat{t} = \sqrt{1 - \frac{r^2\omega}{c^2}} \left(dt - \frac{r^2\omega}{1 - r^2\omega^2 / c^2} d\theta \right) .$$

Here $dt = 0$ means simultaneity in the non-rotating laboratory system, F, and $d\hat{t} = 0$ simultaneity in the rotating frame, R. The simultaneity of the laboratory frame is defined globally, but simultaneity in the rotating frame, R, is only defined locally. With $d\hat{t} = 0$ we get

$$dt = \frac{r^2\omega}{1 - r^2\omega^2 / c^2} d\theta$$

which is not a total differential. This means that simultaneity in the rotating frame R cannot be defined around a closed curve about the axis. If define simultaneous events in R along a circle about the axis, we come to progressively later events in F as given by the formula for dt above. Going around the circle we arrive at the point of departure at a later event than the one we started from. This means that the 3-space defined by simultaneity in R does not represent a simultaneity space in F. In a Minkowski diagram with reference to F the 3-space is shaped as shown in the figure below. It has a discontinuity.



Orthonormal basis field in the rotating frame

We saw in Lecture 9 how the spatial metric representing a simultaneity space of an observer with 4-velocity $\vec{u}$ was defined in terms of orthogonal basis vectors, where the time-like basis-vector was chosen to be the 4-velocity of the observer. It has a magnitude *c*.

Let us define an orthonormal basis vector field co-moving with an observer at rest in an arbitrary reference frame. The 4-velocity of the observer is $\vec{u}$. We choose as time-like unit basis vector

$$\vec{e}_{\hat{0}} = (1/c)\vec{u}\,.$$

We shall express the orthonormal basis vectors in terms of the co-ordinate basis vectors in a c-ordinate basis $\{\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ where $\vec{e}_0$ is parallel to $\vec{u}$, and the spatial vectors need not be orthogonal to the time-like basis vector.

As shown in Lecture 9 a spacelike basis vector $\vec{e}_i$ may be separated in one component

$$\vec{e}_{i\parallel} = \frac{g_{i0}}{g_{00}}\vec{e}_0$$

along $\vec{e}_0$ and one $\vec{e}_{i\perp} = \vec{e}_i - \vec{e}_{i\parallel}$ orthogonal to $\vec{e}_0$, i.e.

$$\vec{e}_{i\perp} = \vec{e}_i - \frac{g_{i0}}{g_{00}}\vec{e}_0\,.$$

Since this vector has a magnitude $|\vec{e}_{i\perp}| = \sqrt{\vec{e}_{i\perp}\cdot\vec{e}_{i\perp}} = \sqrt{\gamma_{ii}}$, the corresponding unit vector is

$$\vec{e}_{\hat{i}} = (\gamma_{ii})^{-1/2}\left(\vec{e}_i - \frac{g_{i0}}{g_{00}}\vec{e}_0\right).$$

The second and third space-like vectors in the orthonormal basis are then given by

$$\vec{e}_{\hat{j}}\cdot\vec{e}_{\hat{i}} = \vec{e}_{\hat{j}}\cdot\vec{e}_{\hat{0}} = 0 \quad , \quad \vec{e}_{\hat{k}} = \vec{e}_{\hat{i}}\times\vec{e}_{\hat{j}}\,.$$

Let us now consider the rotating reference frame, R. The coordinate transformation is

$$T = t \quad , \quad R = r \quad , \quad \Theta = \theta + \omega t \quad , \quad Z = z$$

Hence the transformation from the coordinate basis vectors in F to those in R are

$$\vec{e}_t = \frac{\partial T}{\partial t}\vec{e}_T + \frac{\partial\Theta}{\partial t}\vec{e}_\Theta = \vec{e}_T + \omega\vec{e}_\Theta \quad , \quad \vec{e}_r = \vec{e}_R \quad , \quad \vec{e}_\theta = \vec{e}_\Theta \quad , \quad \vec{e}_z = \vec{e}_Z\,.$$

Note that even if $T = t$ the basis vectors $\vec{e}_T$ and $\vec{e}_t$ have different directions. The vector field $\vec{e}_T$ is directed along the world lines of the particles in F that are parallel to the cylinder axis in the figure above while the vector field $\vec{e}_t$ is directed along the world lines of the particles in R which has the spiral shape given by $\theta = \text{constant}$ shown in the Figure. The simultaneity space in F are the horizontal planes orthogonal to $\vec{e}_T$, and the simultaneity space in R is a succession of simultaneity spaces locally orthogonal to $\vec{e}_t$.

In order to find the orthonormal basis carried by an observer in R by means of the formulae above, we must first find the components of the 4-velocity in the co-moving coordinate system in R. Since the observer is at rest in R, the time component is the only non-vanishing component. It follows from the line element in R as applied to a clock at rest that the 4-velocity is

$$\vec{u} = c\frac{dt}{d\tau}\vec{e}_t = \frac{c}{\sqrt{1 - r^2\omega^2 / c^2}}\vec{e}_t .$$

Inserting the expressions for the components of the metric tensor and the spatial metric tensor in R then gives the orthonormal basis carried by an observer in R

$$\vec{e}_{\hat{t}} = \frac{1}{\sqrt{c^2 - r^2\omega^2}}\vec{e}_t \quad , \quad \vec{e}_{\hat{r}} = \vec{e}_r \quad , \quad \vec{e}_{\hat{\theta}} = \frac{\sqrt{1 - r^2\omega^2 / c^2}}{r}\vec{e}_\theta + \frac{r\omega / c^2}{\sqrt{1 - r^2\omega^2 / c^2}}\vec{e}_t .$$

Uniformly accelerated reference frames

Consider a particle moving along a straight line with velocity u and acceleration $a = \frac{du}{dT}$. Rest acceleration is $\hat{a}$.

$$\Rightarrow \quad a = \left(1 - u^2/c^2\right)^{3/2} \hat{a}. \tag{3.32}$$

Assume that the particle has constant rest acceleration $\hat{a} = g$. That is

$$\frac{du}{dT} = \left(1 - u^2/c^2\right)^{3/2} g. \tag{3.33}$$

Which on integration with $u(0) = 0$ gives

$$u = \frac{gT}{\left(1 + \frac{g^2T^2}{c^2}\right)^{1/2}} = \frac{dX}{dT}$$

$$\Rightarrow \quad X = \frac{c^2}{g}\left(1 + \frac{g^2T^2}{c^2}\right)^{1/2} + k$$

$$\Rightarrow \quad \frac{c^4}{g^2} = (X - k)^2 - c^2T^2 \tag{3.34}$$

This equation describes a hyperbola in the Minkowski diagram.

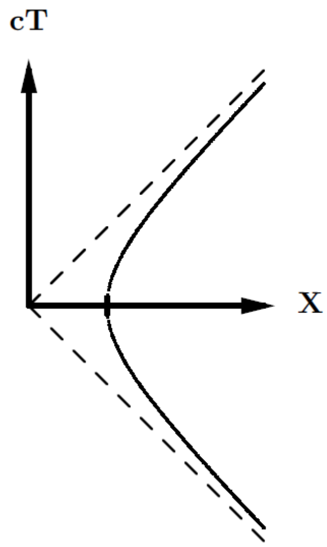


Figure 3.8: Hyperbolically accelerated reference frames are so called because the loci of particle trajectories in space-time are hyperbolae.

The proper time interval as measured by a clock which follows the particle:

$$d\tau = \left(1 - \frac{u^2}{c^2}\right)^{1/2} dT \tag{3.35}$$

Substitution for $u(T)$ and integration with $\tau(0) = 0$ gives

$$\begin{aligned} \tau &= \frac{c}{g}\operatorname{arcsinh}\left(\frac{gT}{c}\right) \\ \text{or} \quad T &= \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right) \\ \text{and} \quad X &= \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right) + k \end{aligned} \tag{3.36}$$

We now use this particle as the origin of space in an hyperbolically accelerated reference frame.

Definition 3.2.1 (Born-stiff motion)
Born-stiff motion of a system is motion such that every element of the system has constant rest length. We demand that our accelerated reference frame is Born-stiff.

Let the inertial frame have coordinates (T, X, Y, Z) and the accelerated frame have coordinates (t, x, y, z). We now denote the X-coordinate of the "origin particle" by X_0.

$$1 + \frac{gX_0}{c^2} = \cosh\frac{g\tau_0}{c} \tag{3.37}$$

where τ_0 is the proper time for this particle and k is set to $\frac{-c^2}{g}$. (These are Møller coordinates. Setting $k = 0$ gives Rindler coordinates).

Let us denote the accelerated frame by Σ. The coordinate time at an arbitrary point in Σ is defined by $t = \tau_0$. That is coordinate clocks in Σ run identically with the standard clock at the "origin particle". Let $\vec{X}_0$ be the position 4-vector of the "origin particle". Decomposed in the laboratory frame, this becomes

$$\vec{X}_0 = \left\{ \frac{c^2}{g} \sinh \frac{gt}{c}, \frac{c^2}{g} \left(\cosh \frac{gt}{c} - 1 \right), 0, 0 \right\} \tag{3.38}$$