Lecture 12. 20.02.1018

In the following we shall need the Lorentz transformation expressed in terms of the velocity parameter, θ. The Lorentz transformation between two orthonormal basis sets with a relative velocity v is given by the matrix

$$\begin{pmatrix} \gamma & \gamma\frac{v}{c} & 0 & 0 \\ \gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

The velocity parameter is defined by

$$v = c\tanh\theta,$$

Giving

$$\gamma = \cosh\theta \quad , \quad \gamma\frac{v}{c} = \sinh\theta$$

Hence as expressed in terms of the velocity parameter the Lorentz transformation takes the form

$$\begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Consider an event P which is simultaneous with an event P_0 at the origin particle in the accelerated frame Σ (see Figure 3.9).

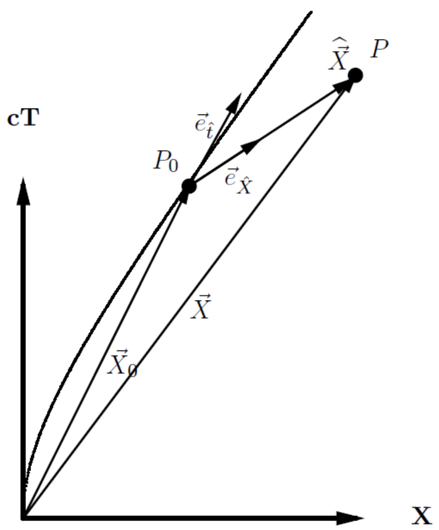


Figure 3.9: Simultaneity in hyperbolically accelerated reference frames. The vector $\widehat{\vec{X}}$ lies along the "simultaneity line" which makes the same angle with the X-axis as does $\vec{e}_{\hat{t}}$ with the cT-axis.

The components of the distance vector from P_0 to P as decomposed in an orthonormal basis co-moving with the origin particle is $\hat{\vec{X}} = (0, \hat{x}, \hat{y}, \hat{z})$, where $\hat{x}, \hat{y}$ and $\hat{z}$ are physical distances measured simultaneously in Σ. The space co-ordinates in Σ are defined by

$$x \equiv \hat{x} \quad , \quad y \equiv \hat{y} \quad , \quad z \equiv \hat{z} \, .$$

The position vector of P is $\vec{X} = \vec{X}_0 + \hat{\vec{X}}$. The relationship between basis vectors in IF and the comoving orthonormal basis is given by a Lorentz transformation in the x-direction.

$$\begin{aligned} \vec{e}_{\hat{\mu}} &= \vec{e}_{\mu} \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \\ &= (\vec{e}_T, \vec{e}_X, \vec{e}_Y, \vec{e}_Z,) \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{3.40}$$

where θ is the **rapidity** defined by

$$\tanh\theta \equiv \frac{U_0}{c} \tag{3.41}$$

U_0 being the velocity of the "origin particle".

$$\begin{aligned} U_0 &= \frac{dX_0}{dT_0} = c \tanh \frac{gt}{c} \\ \therefore \theta &= \frac{gt}{c} \end{aligned} \tag{3.42}$$

$$
\begin{aligned}
\vec{e}_{\hat{t}} &= \vec{e}_T \cosh \frac{gt}{c} + \vec{e}_X \sinh \frac{gt}{c} \\
\vec{e}_{\hat{x}} &= \vec{e}_T \sinh \frac{gt}{c} + \vec{e}_X \cosh \frac{gt}{c} \\
\vec{e}_{\hat{y}} &= \vec{e}_Y \\
\vec{e}_{\hat{z}} &= \vec{e}_Z
\end{aligned} \tag{3.43}
$$

The equation $\vec{X} = \vec{X}_0 + \hat{\vec{X}}$ can now be decomposed in IF:

$$
cT\vec{e}_T + X\vec{e}_X + Y\vec{e}_Y + Z\vec{e}_Z = \\
\frac{c}{g}\sinh\frac{gt}{c}\vec{e}_T + \frac{c^2}{g}\left(\cosh\frac{gt}{c} - 1\right)\vec{e}_X + \frac{x}{c}\sinh\frac{gt}{c}\vec{e}_T + x\cosh\frac{gt}{c}\vec{e}_X + y\vec{e}_Y + z\vec{e}_Z \tag{3.44}
$$

This then, gives the coordinate transformations

$$
\begin{aligned}
T &= \frac{c}{g}\sinh\frac{gt}{c} + \frac{x}{c}\sinh\frac{gt}{c} \\
X &= \frac{c^2}{g}\left(\cosh\frac{gt}{c} - 1\right) + x\cosh\frac{gt}{c} \\
Y &= y \\
Z &= z \\
\Rightarrow \quad \frac{gT}{c} &= \left(1 + \frac{gx}{c^2}\right)\sinh\frac{gt}{c} \\
1 + \frac{gX}{c^2} &= \left(1 + \frac{gx}{c^2}\right)\cosh\frac{gt}{c}
\end{aligned}
$$

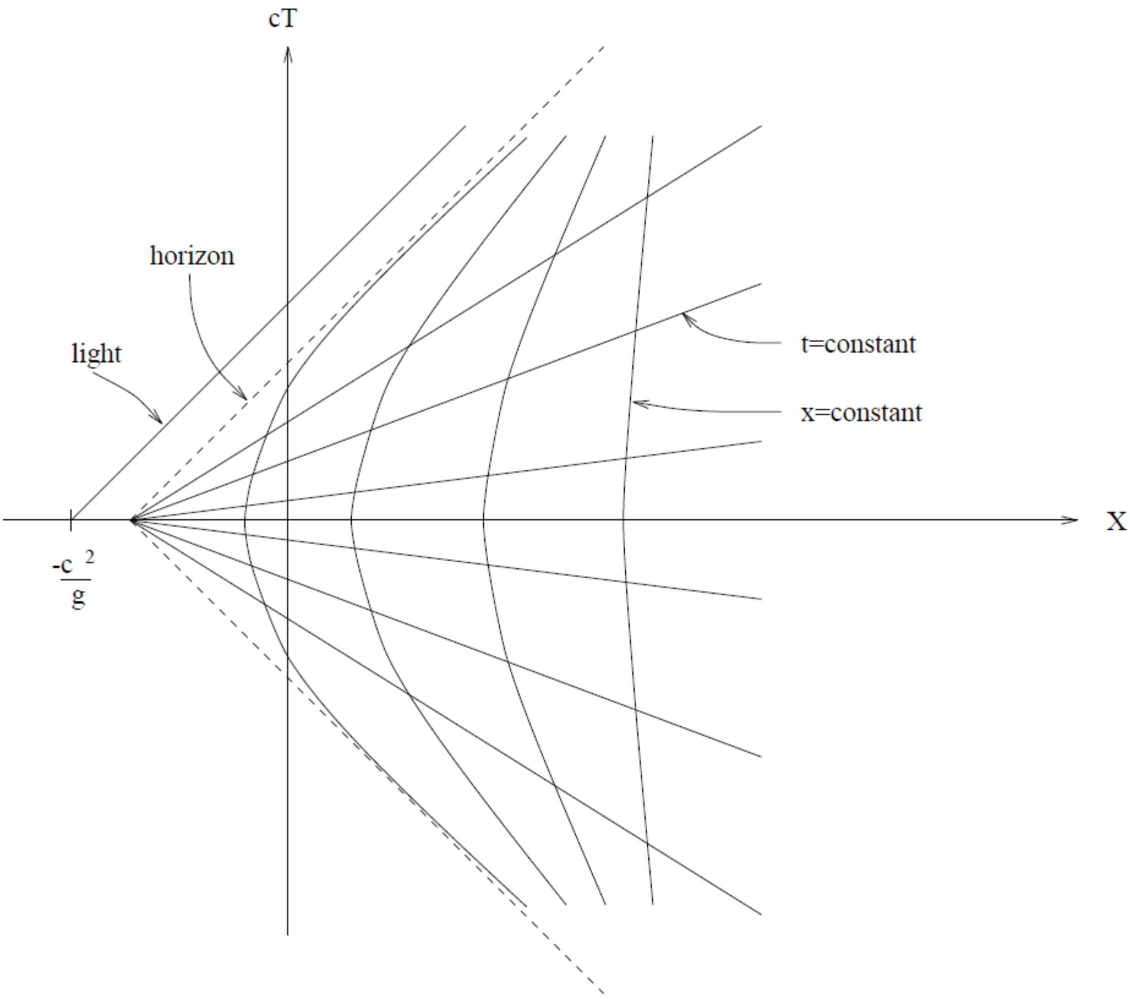
Now dividing the last two of the above equations we get

$$
\frac{gT}{c} = \left(1 + \frac{gX}{c^2}\right)\tanh\frac{gt}{c} \tag{3.45}
$$

showing that the coordinate curves t = constant are straight lines in the T,X-frame passing through the point T = 0, X = $-\frac{c^2}{g}$. Using the identity $\cosh^2\theta - \sinh^2\theta = 1$ we get

$$
\left(1 + \frac{gX}{c^2}\right)^2 - \left(\frac{gT}{c}\right)^2 = \left(1 + \frac{gx}{c^2}\right)^2 \tag{3.46}
$$

showing that the coordinate curves x = constant are hypebolae in the T,X-diagram.


cT
horizon
light
t=constant
x=constant
X
-c 2
g