Lecture 13 260218

The line element (the metric) gives :

ds^2 is an invariant quantity

$$\begin{aligned} ds^2 &= -c^2dT^2 + dX^2 + dY^2 + dZ^2 \\ &= -(1+\frac{gx}{c^2})^2c^2dt^2 + dx^2 + dy^2 + dz^2 \end{aligned} \qquad (3.47)$$

Note: When the metric is diagonal the unit vectors are orthogonal.

Clocks as rest in the accelerated system:

$$dx = dy = dz = 0, \quad ds^2 = -c^2d\tau^2$$

$$\Downarrow$$

$$-c^2d\tau^2 = -(1+\frac{gx}{c^2})^2c^2dt^2$$

$$\Downarrow$$

$$\boxed{d\tau = (1+\frac{gx}{c^2})dt} \qquad (3.48)$$

Here $d\tau$ is the **proper time** and dt the **coordinate time**.

An observer in the accelerated system Σ experiences a gravitational field in the negative x-direction. When $x < 0$ then $d\tau < dt$. The coordinate clocks tick equally fast independently of their position. This implies that time passes slower further down in a gravitational field.

Consider a standard clock moving in the x-direction with velocity $v = dx/dt$. Then

$$\begin{aligned} -c^2d\tau^2 &= -\left(1+\frac{gx}{c^2}\right)^2c^2dt^2 + dx^2 \\ &= -\left[\left(1+\frac{gx}{c^2}\right)^2 - \frac{v^2}{c^2}\right]c^2dt^2 \end{aligned} \qquad (3.49)$$

Hence

$$d\tau = \sqrt{\left(1+\frac{gx}{c^2}\right)^2 - \frac{v^2}{c^2}}\,dt \qquad (3.50)$$

This expresses the combined effect of the gravitational- and the kinematic time dilation.

Let us consider how light moves in the uniformly accelerated reference frame. As a simple example we consider light moving in the y-direction in the laboratory frame,

$$X = X_0 \quad , \quad Y = Y_0 + cT \quad , \quad Z = 0$$

Inserting this in the coordinate transformation above we obtain

$$\frac{gT}{c}=\frac{g}{c^2}(y-y_0)\quad,\quad\left(1+\frac{gx}{c^2}\right)^2+\left[\frac{g}{c^2}(y-y_0)\right]^2=\left(1+\frac{gX_0}{c^2}\right)^2.$$

This shows that the light moves along a circular path. Differentiating the equation of the trajectory with respect to *x* we obtain

$$\frac{dy}{dx}=\frac{x+c^2/g}{y-y_0}.$$

Hence $dy/dx=0$ at the horizon. In other word the light moves in the vertical direction at the horizon. At that position the light has no motion in the y-direction. The reason is that the time does not progress at the horizon.

Note also from the line-element that

$$\frac{dx}{dt}=\left(1+\frac{gx}{c^2}\right)c$$

For light moving in the x-direction. Light moves slower the further down it is in the gravitational field, and the velocity of the light approaches zero as the light approaches the horizon. Light moves neither in the horizontal nor the vertical direction at the horizon.

4.1 Differentiation of forms

We must have a method of differentiation that maintains the anti symmetry, thus making sure that what we end up with after differentiation is still a form.

4.1.1 Exterior differentiation

The exterior derivative of a 0-form, i.e. a scalar function, f, is given by:

$$\underline{d}f=\frac{\partial f}{\partial x^\mu}\,\underline{\omega}^\mu=f_{,\mu}\underline{\omega}^\mu \tag{4.1}$$

where $\underline{\omega}^\mu$ are coordinate basis forms:

$$\underline{\omega}^\mu\left(\frac{\partial}{\partial x^\nu}\right)=\delta^\mu_\nu \tag{4.2}$$

We then (in general) get:

$$\underline{\omega}^\mu=\delta^\mu_\nu\underline{\omega}^\nu=\frac{\partial x^\mu}{\partial x^\nu}\underline{\omega}^\nu=\underline{d}x^\mu \tag{4.3}$$

In coordinate basis we can always write the basis forms as exterior derivatives of the coordinates. The differential dx^μ is given by

$$\underline{d}x^\mu(d\vec{r})=dx^\mu \tag{4.4}$$

where $d\vec{r}$ is an infinitesimal position vector. $\underline{d}x^\mu$ are *not* infinitesimal quantities. In coordinate basis the exterior derivative of a p-form

$$\underline{\alpha} = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}\underline{d}x^{\mu_1} \wedge \cdots \wedge \underline{d}x^{\mu_p} \tag{4.5}$$

will have the following component form:

$$\underline{d}\,\underline{\alpha} = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p,\mu_0}\underline{d}x^{\mu_0} \wedge \underline{d}x^{\mu_1} \wedge \cdots \wedge \underline{d}x^{\mu_p} \tag{4.6}$$

where $,\mu_0 \equiv \frac{\partial}{\partial x^{\mu_0}}$. **The exterior derivative of a p-form is a $(p+1)$-form.**

Consider the exterior derivative of a p-form $\underline{\alpha}$.

$$\underline{d\alpha} = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p,\mu_0}\underline{d}x^{\mu_0} \wedge \cdots \wedge \underline{d}x^{\mu_p}. \tag{4.7}$$

Let $(\underline{d\alpha})_{\mu_0\cdots\mu_p}$ be the form components of $\underline{d\alpha}$. They must, by definition, be antisymmetric under an arbitrary interchange of indices.

$$\underline{d\alpha} = \frac{1}{(p+1)!}(\underline{d\alpha})_{\mu_0\cdots\mu_p}\underline{d}x^{\mu_0} \wedge \cdots \wedge \underline{d}x^{\mu_p}$$

which, by (4.7) $\Rightarrow$

$$= \frac{1}{p!}\alpha_{[\alpha_{\mu_1\cdots\mu_p,\mu_0}]}\underline{d}x^{\mu_0} \wedge \cdots \wedge \underline{d}x^{\mu_p}$$

$$\boxed{\therefore (\underline{d\alpha})_{\mu_0\cdots\mu_p} = (p+1)\alpha_{[\mu_1\cdots\mu_p,\mu_0]}} \tag{4.8}$$

The form equation $\underline{d\alpha} = 0$ in component form is

$$\alpha_{[\mu_1\cdots\mu_p,\mu_0]} = 0 \tag{4.9}$$