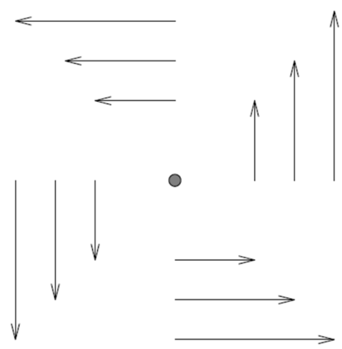
Lecture 15. 05.03.2018

Example 4.1.12. The acceleration of a velocity field representing rigid rotation

The velocity field is

$$\vec{v} = r\omega \vec{e}_{\hat{\theta}} = \omega \vec{e}_{\theta}$$

Note that the co-ordinate component of the velocity is not equal to the physical velocity component. *The physical velocity components are those in an orthonormal basis.*



We shall calculate the acceleration field, $\vec{a} = \frac{d\vec{v}}{dt}$. Using the chain rule of differentiation we get

$$\vec{a} = \frac{\partial \vec{v}}{\partial x^{\mu}} \frac{dx^{\mu}}{dt} = v^{\mu} \frac{\partial \vec{v}}{\partial x^{\mu}}$$

Since the only non-vanishing velocity component is $v^{\theta} = \omega$ we get

$$\vec{a} = \omega \frac{\partial \vec{v}}{\partial \theta}.$$

This gives

$$\vec{a} = \omega \frac{\partial \left(\omega \vec{e}_{\theta} \right)}{\partial \theta} = \omega^2 \frac{\partial \vec{e}_{\theta}}{\partial \theta}.$$

We have found earlier that

$$\vec{e}_r = \cos\theta \vec{e}_x + \sin\theta \vec{e}_y \quad , \quad \vec{e}_{\theta} = -r\sin\theta \vec{e}_x + r\cos\theta \vec{e}_y .$$

Differentiation gives

$$\frac{\partial \vec{e}_{\theta}}{\partial \theta} = -r\cos\theta \vec{e}_x - r\sin\theta \vec{e}_y = -r\vec{e}_r = -r\vec{e}_{\hat{r}} .$$

Hence the acceleration is

$$\vec{a} = -r\omega^2 \vec{e}_{\hat{r}} .$$

This is the centripetal acceleration for circular motion.

Letting $\{x^{\hat{i}}\} = x, y$ be Cartesian coordinates with orthonormal basis vectors we shall here use the formula

$$\Gamma^{\mu}_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x^{\hat{\nu}}} \frac{\partial^2 x^{\hat{\nu}}}{\partial x^{\alpha} \partial x^{\beta}}$$

to calculate the Christoffel symbols in a polar co-ordinate system, with $\{x^{i}\} = r, \theta$.

Example 4.2.1 (The Christoffel symbols in plane polar coordinates)

$$x = r\cos\theta, \qquad y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\frac{y}{x}$$

$$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \sin\theta$$
$$\frac{\partial y}{\partial r} = \sin\theta, \qquad \frac{\partial y}{\partial \theta} = r\cos\theta \qquad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\begin{aligned}
\Gamma^{r}{}_{\theta\theta} &= \frac{\partial r}{\partial x}\frac{\partial^2 x}{\partial \theta^2} + \frac{\partial r}{\partial y}\frac{\partial^2 y}{\partial \theta^2} \\
&= \cos\theta(-r\cos\theta) + \sin\theta(-r\sin\theta) \\
&= -r(\cos\theta^2 + \sin\theta^2) = -r
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} &= \frac{\partial \theta}{\partial x}\frac{\partial^2 x}{\partial \theta \partial r} + \frac{\partial \theta}{\partial y}\frac{\partial^2 y}{\partial \theta \partial r} \\
&= -\frac{\sin\theta}{r}(-\sin\theta) + \frac{\cos\theta}{r}(\cos\theta) \\
&= \frac{1}{r}
\end{aligned}$$

Parallel transport

The geometrical interpretation of the covariant derivative was given by Levi-Civita.

Consider a curve S in any (eg. curved) space. It is parameterized by λ, ie. $x^\mu = x^\mu(\lambda)$. λ is invariant and chosen to be the curve length.

The tangent vector field of the curve is $\vec{u} = (dx^\mu/d\lambda)\vec{e}_\mu$. The curve passes through a vector field $\vec{A}$. The covariant directional derivative of the vector field along the curve is defined as:

$$\nabla_{\vec{u}}\vec{A} = \frac{d\vec{A}}{d\lambda} \equiv A^\mu{}_{;\nu}\frac{dx^\nu}{d\lambda}\vec{e}_\mu = A^\mu{}_{;\nu}u^\nu\vec{e}_\mu \tag{4.27}$$

The vectors in the vector field are said to be connected by parallel transport along the curve if

$$A^\mu{}_{;\nu}u^\nu = 0$$

Definition 4.3.1 (Geodesic curves)
A geodesic curve is defined in such a way that,the vectors of the tangent vector field of the curve is connected by parallell transport.

This definition says that geodesic curves are 'as straight as possible'.
If vectors in a vector field $\vec{A}(\lambda)$ are connected by parallell transport by a displacement along a vector $\vec{u}$, we have $A^\mu{}_{;\nu}u^\nu = 0$. For geodesic curves, we then have:

$$\boxed{u^\mu{}_{;\nu}u^\nu = 0} \tag{4.30}$$

which is the *geodesic equation.*

$$(u^\mu{}_{,\nu} + \Gamma^\mu{}_{\alpha\nu}u^\alpha)u^\nu = 0 \tag{4.31}$$

Then we are using that $\frac{d}{d\lambda} \equiv \frac{dx^\nu}{d\lambda}\frac{\partial}{\partial x^\nu} = u^\nu\frac{\partial}{\partial x^\nu}$:

$$\frac{du^\mu}{d\lambda} = u^\nu\frac{\partial u^\mu}{\partial x^\nu} = u^\nu u^\mu{}_{,\nu} \tag{4.32}$$

The geodesic equation can also be written as:

$$\frac{du^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\nu} u^\alpha u^\nu = 0 \tag{4.33}$$

Usual notation: $\dot{} = \frac{d}{d\lambda}$

$$u^\mu = \frac{dx^\mu}{d\lambda} = \dot{x}^\mu \tag{4.34}$$

$$\boxed{\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\nu} \dot{x}^\alpha \dot{x}^\nu = 0} \tag{4.35}$$

4.4 The covariant Euler-Lagrange equations

Geodesic curves can also be defined as curves with an extremal distance between two points. Let a particle have a world-line (in space-time) between two points (events) P_1 and P_2. Let the curves be described by an invariant parameter λ (proper time τ is used for particles with a rest mass).

The covariant Euler-Lagrange equations

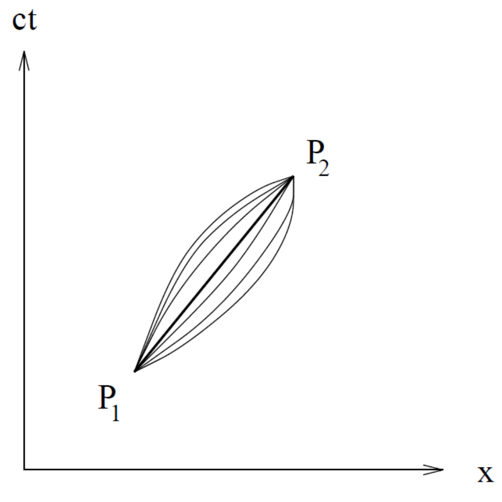


Figure 4.2: Different world-lines connecting P_1 and P_2 in a Minkowski diagram

The Lagrange-function is a function of coordinates and their derivatives,

$$L = L(x^\mu, \dot{x}^\mu), \qquad \dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}. \tag{4.36}$$

(Note: if $\lambda = \tau$ then $\dot{x}^\mu$ are the 4-velocity components)

The action-integral is $S = \int L(x^\mu, \dot{x}^\mu) d\lambda$. The principle of extremal action (Hamiltons-principle): The world-line of a particle is determined by the condition that S shall be extremal for all infinitesimal variations of curves which keep P_1 and P_2 rigid, ie.

$$\delta \int_{\lambda_1}^{\lambda_2} L(x^\mu, \dot{x}^\mu) d\lambda = 0, \tag{4.37}$$

where λ_1 and λ_2 are the parameter-values at P_1 and P_2. For all the variations the following condition applies:

$$\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0 \tag{4.38}$$

We write Eq. (4.37) as

$$\delta \int_{\lambda_1}^{\lambda_2} L d\lambda = \int_{\lambda_1}^{\lambda_2} \left[\frac{\partial L}{\partial x^\mu} \delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu} \delta \dot{x}^\mu \right] d\lambda \tag{4.39}$$

Partial integration of the last term

$$\int_{\lambda_1}^{\lambda_2} \frac{\partial L}{\partial \dot{x}^\mu} \delta \dot{x}^\mu d\lambda = \left[\frac{\partial L}{\partial \dot{x}^\mu} \delta x^\mu \right]_{\lambda_1}^{\lambda_2} - \int_{\lambda_1}^{\lambda_2} \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) \delta x^\mu d\lambda \tag{4.40}$$

Due to the conditions $\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0$ the first term becomes zero. Then we have :

$$\delta S = \int_{\lambda_1}^{\lambda_2} \left[\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) \right] \delta x^\mu d\lambda \tag{4.41}$$

The world-line the particle follows is determined by the condition $\delta S = 0$ for any variation δx^μ. Hence, the world-line of the particle must be given by

$$\boxed{\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) = 0} \tag{4.42}$$

These are the covariant **Euler-Lagrange** equations.

We shall later show that it is a consequence of Einstein's field equations that free particles (i.e. particles acted upon only by gravity in Newtonian terms) follow geodesics curves in spacetime.

A free particle in space-time (curved space-time includes gravitation) has the Lagrange function

$$L = \frac{1}{2}\vec{u} \cdot \vec{u} = \frac{1}{2}\dot{x}_\mu \dot{x}^\mu = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{4.45}$$

An integral of the Lagrange-equations is obtained readily from the 4-velocity identity:

$$\begin{cases} \dot{x}_\mu \dot{x}^\mu = -c^2 & \text{for a particle with rest-mass} \\ \dot{x}_\mu \dot{x}^\mu = 0 & \text{for light} \end{cases} \tag{4.46}$$

The line-element is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu d\lambda^2 = 2L d\lambda^2 \ . \tag{4.47}$$

Thus the Lagrange function of a free particle is obtained from the line-element.