Lecture 16. 06.0503.2018

> The equation of a time-like geodesic curve is deduced from a variational principle which says that if there are two fixed points P_1 and P_2 in space-time, then there exists an open subset of space-time containing these two points such that among all curves contained in this open subset, the geodesic will be the curve of longest length between these two points.

So, *the variational principle says that timelike geodesics maximizes length among all curves in space-time nearby the geodesic.*

There exist non-geodesic curves between two events far away from the geodesic curve between the events, along which a particle following the curve may have larger proper time between the events that a particle following the geodesic curve. Consider for example a clock at rest outside the Earth compared to a clock moving freely along a circular path, and calculate the proper time between two meetings of the clocks. (An exact calculation requires the Schwarzshild spacetime.)

I have not seen a general theorem telling which time-like curve represent maximal proper time between two events in general.

The canonical momentum p_μ conjugated to a coordinate x^μ is defined as

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} \tag{4.43}$$

The Lagrange-equations can now be written as

$$\boxed{\frac{dp_\mu}{d\lambda} = \frac{\partial L}{\partial x^\mu} \quad \text{or} \quad \dot{p}_\mu = \frac{\partial L}{\partial x^\mu}}\,. \tag{4.44}$$

A coordinate which the Lagrange-function does not depend on is known as a **cyclic coordinate**. Hence, $\frac{\partial L}{\partial x^\mu} = 0$ for a cyclic coordinate. From this follows:

> The canonical momentum conjugated to a cyclic coordinate is a **constant of motion**

ie. $p_\mu = C$ (constant) if x^μ is cyclic.

4.5.1 Equation of motion from Lagrange's equation

The Lagrange function for a free particle is:

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{4.48}$$

where $g_{\mu\nu} = g_{\mu\nu}(x^\lambda)$. And the Lagrange equations are

$$\begin{aligned} \frac{\partial L}{\partial x^\beta} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\beta}\right) &= 0, \\ \frac{\partial L}{\partial \dot{x}^\beta} &= g_{\beta\nu} \dot{x}^\nu, \\ \frac{\partial L}{\partial x^\beta} &= \frac{1}{2} g_{\mu\nu,\beta} \dot{x}^\mu \dot{x}^\nu. \end{aligned} \tag{4.49}$$

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\beta}\right) \equiv \left(\frac{\partial L}{\partial \dot{x}^\beta}\right)^{\bullet} &= \dot{g}_{\beta\nu} \dot{x}^\nu + g_{\beta\nu} \ddot{x}^\nu \\ &= g_{\beta\nu,\mu} \dot{x}^\mu \dot{x}^\nu + g_{\beta\nu} \ddot{x}^\nu \ . \end{aligned} \tag{4.50}$$

Now, (4.50) and (4.49) together give:

$$\frac{1}{2} g_{\mu\nu,\beta} \dot{x}^\mu \dot{x}^\nu - g_{\beta\nu,\mu} \dot{x}^\mu \dot{x}^\nu - g_{\beta\nu} \ddot{x}^\nu = 0. \tag{4.51}$$

The second term on the left hand side of (4.51) may be rewritten making use of the fact that $\dot{x}^\mu \dot{x}^\nu$ is symmetric in $\mu\nu$, as as follows

$$\begin{aligned} g_{\beta\nu,\mu} \dot{x}^\mu \dot{x}^\nu &= \frac{1}{2}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu}) \dot{x}^\mu \dot{x}^\nu \\ \Rightarrow\ g_{\beta\nu} \ddot{x}^\nu + \frac{1}{2}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \dot{x}^\mu \dot{x}^\nu &= 0. \end{aligned} \tag{4.52}$$

Finally, since we are free to multiply (4.52) through by $g^{\alpha\beta}$, we can isolate $\ddot{x}^\alpha$ to get the equation of motion in a particularly elegant and simple form:

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \tag{4.53}$$

where the **Christoffel symbols** $\Gamma^\alpha_{\mu\nu}$ in (4.53) are defined by

$$\Gamma^\alpha_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}). \tag{4.54}$$

Equation(4.53) describes a **geodesic** curve .

Example. Vertical free fall in a uniformly accelerated reference frame

The Lagrange function of the particle is

$$L = -\frac{1}{2}\left(1+\frac{gx}{c^2}\right)\dot{t}^2 + \frac{1}{2}\frac{\dot{x}^2}{c^2},$$

where the dot denotes differentiation with respect to the proper time τ of the freely falling particle. This gives

$$\frac{\partial L}{\partial x} = -\frac{g}{c^2}\left(1+\frac{gx}{c^2}\right)\dot{t}^2 \quad , \quad \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{c^2} .$$

Hence the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} - \left(\frac{\partial L}{\partial \dot{x}}\right)^{\bullet} = 0$$

takes the form

$$\ddot{x} + g\left(1+\frac{gx}{c^2}\right)\dot{t}^2 = 0 .$$

A first integral of this equation is the 4-velocity identity which in the present case takes the form

$$-\left(1+\frac{gx}{c^2}\right)^2 \dot{t}^2 + \frac{\dot{x}^2}{c^2} = -1 .$$

Since the metric is static the momentum

$$p_t = \partial L / \partial \dot{t} = -\left(1+\frac{gx}{c^2}\right)^2 \dot{t}$$

is a constant of motion. Its value is determined by the initial condition. Inserting the expression for $\dot{t}$ into the 4-velocity identity gives

$$-\frac{p_t^2}{\left(1+\frac{gx}{c^2}\right)^2} + \frac{\dot{x}^2}{c^2} = -1$$

Assume that the particle is falling from rest at an initial position $x = x_0$, i.e. $\dot{x}(x_0) = 0$. Hence

$$p_t = -\left(1+\frac{gx_o}{c^2}\right) .$$

Inserting this into the 4-velocity identity gives after a short calculation

$$\int_{x_0}^{x} \frac{1+\dfrac{gx}{c^2}}{\sqrt{\left(1+\dfrac{gx_0}{c^2}\right)^2-\left(1+\dfrac{gx}{c^2}\right)^2}}\,dx = \int_0^{\tau} c\,d\tau$$

which gives

$$\sqrt{\left(1+\frac{gx_0}{c^2}\right)^2-\left(1+\frac{gx}{c^2}\right)^2} = \frac{g\tau}{c}$$

or

$$1+\frac{gx}{c} = \sqrt{\left(1+\frac{gx_0}{c^2}\right)^2-\left(\frac{g\tau}{c}\right)^2}$$

Note that we did not use the Euler-Lagrange equation to find the position of the particle as a function of its proper time. It was sufficient to use the 4-velocity identity and a constant of motion. This is often the case.