Lecture 21. 3. april 2018

Acceleration of gravity

A free particle has vanishing 4-acceleration and moves along a time-like geodesic curve. The i-component of the geodesic equation is

$$\ddot{x}^i + \Gamma^i_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$$

We define *the acceleration of gravity* as the 3-acceleration of a free particle instantaneously at rest. Since the spatial components of the 4-velocity of a particle at rest vanish, the acceleration of gravity is given by

$$\ddot{x}^i = -\Gamma^i_{tt}\dot{t}^2 .$$

Hence the acceleration of gravity is given by the Christoffel symbols Γ^i_{tt}. They vanish in a local inertial reference frame, i.e. in a freely falling non-rotating reference frame. *There is an acceleration of gravity in any non-inertial laboratory independent of the geometrical properties of spacetime.*

In the Newtonian limit $d\tau = dt$, $\dot{t} = 1$ and the components of the acceleration of gravity are written $\ddot{x}^i = g^i$. It follows that

$$g^i = -\Gamma^i_{tt} .$$

The Riemann curvature tensor

As a preparation for defining the Riemann curvature tensor we shall now consider parallel transport of vectors

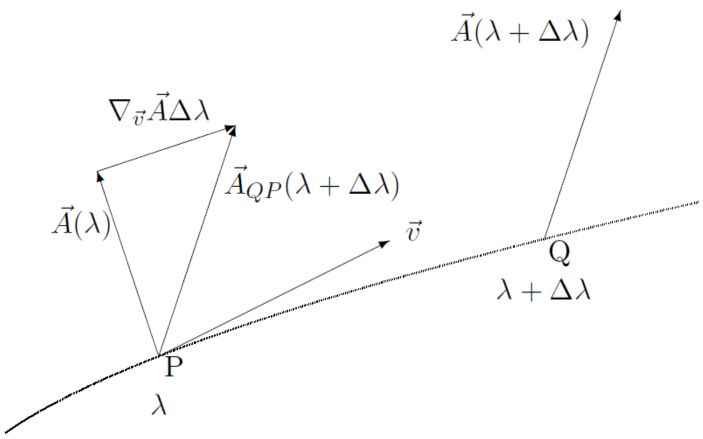


Figure 5.1: Parallel transport of $\vec{A}$

The covariant directional derivative of a vector field $\vec{A}$ along a vector $\vec{u}$ was defined and interpreted geometrically in section 4.2, as follows

$$\begin{aligned}\nabla_{\vec{v}}\vec{A} &= \frac{d\vec{A}}{d\lambda} = A^{\mu}{}_{;\nu}v^{\nu}\vec{e}_{\mu} \\ &= \lim_{\Delta\lambda\to 0}\frac{\vec{A}_{QP}(\lambda+\Delta\lambda)-\vec{A}(\lambda)}{\Delta\lambda}\end{aligned} \tag{5.1}$$

Let $\vec{A}_{QP}$ be the parallel transported of $\vec{A}$ from Q to P. Then to first order in $\Delta\lambda$ we have: $\vec{A}_{QP} = \vec{A}_P + (\nabla_{\vec{v}}\vec{A})_P\Delta\lambda$ and

$$\vec{A}_{PQ} = \vec{A}_Q - (\nabla_{\vec{v}}\vec{A})_Q\Delta\lambda \tag{5.2}$$

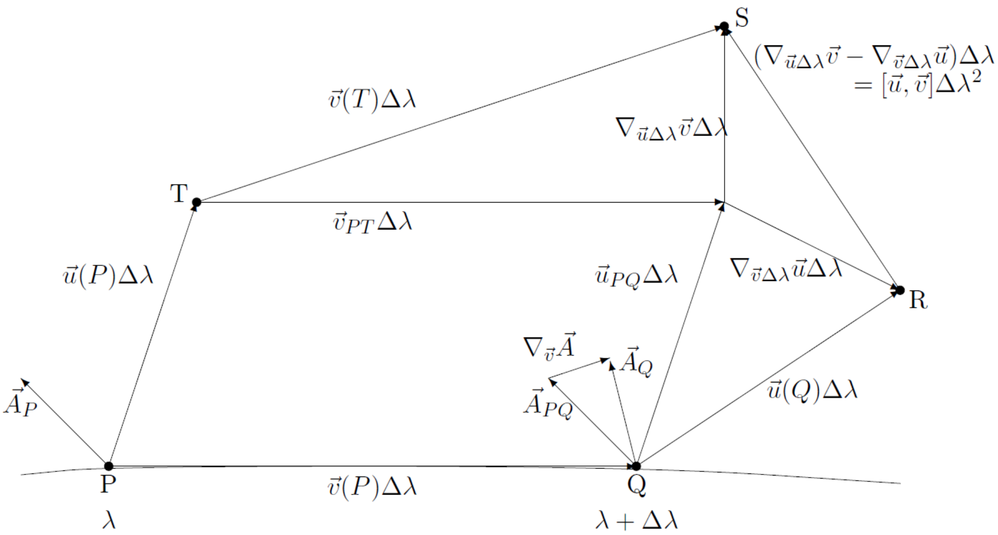
To second order in $\Delta\lambda$ we have:

$$\vec{A}_{PQ} = (1 - \nabla_{\vec{v}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2)\vec{A}_Q \tag{5.3}$$

If $\vec{A}_{PQ}$ is parallel transported further on to R we get

$$\begin{aligned}\vec{A}_{PQR} = &(1 - \nabla_{\vec{u}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{u}}\nabla_{\vec{u}}(\Delta\lambda)^2) \\ &\cdot(1 - \nabla_{\vec{v}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2)\vec{A}_R\end{aligned} \tag{5.4}$$

where $\vec{A}_Q$ is replaced by $\vec{A}_R$ because the differential operator always shall be applied to the vector in the first position. If we parallel transport $\vec{A}$ around the whole polygon in figure 5.3 we get:



$$\vec{A}_{PQRSTP} = (1 + \nabla_{\vec{u}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{u}}\nabla_{\vec{u}}(\Delta\lambda)^2)$$
$$\cdot (1 + \nabla_{\vec{v}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2)$$
$$\cdot (1 - \nabla_{[\vec{u},\vec{v}]}(\Delta\lambda)^2) \cdot (1 - \nabla_{\vec{u}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{u}}\nabla_{\vec{u}}(\Delta\lambda)^2)$$
$$\cdot (1 - \nabla_{\vec{v}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2)\vec{A}_P$$

Calculating to 2. order in $\Delta\lambda$ gives:

$$\vec{A}_{PQRSTP} = \vec{A}_P + ([\nabla_{\vec{u}}, \nabla_{\vec{v}}] - \nabla_{[\vec{u},\vec{v}]})(\Delta\lambda)^2\vec{A}_P \tag{5.6}$$

There is a variation of the vector under parallel transport around the closed polygon:

$$\delta\vec{A} = \vec{A}_{PQRSTP} - \vec{A}_P = ([\nabla_{\vec{u}}, \nabla_{\vec{v}}] - \nabla_{[\vec{u},\vec{v}]})\vec{A}_P(\Delta\lambda)^2 \tag{5.7}$$

We now introduce the Riemann's curvature tensor as:

$$R(\quad, \vec{A}, \vec{u}, \vec{v}) \equiv ([\nabla_{\vec{u}}, \nabla_{\vec{v}}] - \nabla_{[\vec{u},\vec{v}]})(\vec{A}) \tag{5.8}$$

The components of the Riemann curvature tensor is defined by applying the tensor on basis vectors,

$$R^{\mu}{}_{\nu\alpha\beta}\vec{e}_\mu \equiv ([\nabla_\alpha, \nabla_\beta] - \nabla_{[\vec{e}_\alpha,\vec{e}_\beta]})(\vec{e}_\nu) \tag{5.9}$$

It may be noted that in a coordinate basis this equation reduces to

$$R^{\mu}_{\nu\alpha\beta}\vec{e}_\mu = \left[\nabla_\alpha, \nabla_\beta\right](\vec{e}_\nu).$$

Anti-symmetry follows from the definition:

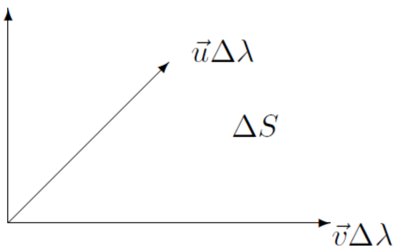
$$R^{\mu}{}_{\nu\beta\alpha} = -R^{\mu}{}_{\nu\alpha\beta} \tag{5.10}$$

The expression for the variation of $\vec{A}$ under parallel transport around the polygon, Eq. (5.7), can now be written as:

$$\begin{aligned}\delta\vec{A} &= R(\quad, \vec{A}, \vec{u}, \vec{v})(\Delta\lambda)^2 \\ &= R(\quad, A^\nu \vec{e}_\nu, u^\alpha \vec{e}_\alpha, v^\beta \vec{e}_\beta)(\Delta\lambda)^2 \\ &= \vec{e}_\mu R^\mu{}_{\nu\alpha\beta} A^\nu u^\alpha v^\beta \cdot (\Delta\lambda)^2 \\ &= \frac{1}{2}\vec{e}_\mu R^\mu{}_{\nu\alpha\beta} A^\nu (u^\alpha v^\beta - u^\beta v^\alpha)(\Delta\lambda)^2\end{aligned}$$

The area of the parallellogram defined by the vectors $\vec{u}\Delta\lambda$ and $\vec{v}\Delta\lambda$ is

$$\Delta\vec{S} = \vec{u} \times \vec{v}(\Delta\lambda)^2$$



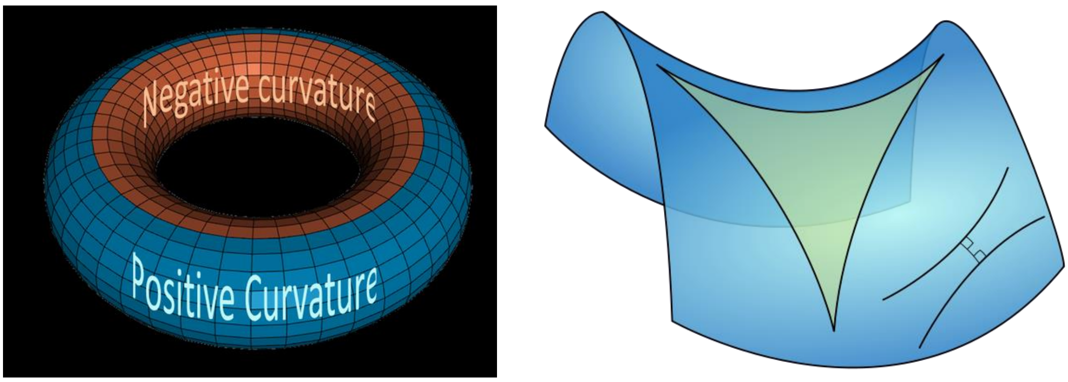
Using that

$$(\vec{u} \times \vec{v})^{\alpha\beta} = u^\alpha v^\beta - u^\beta v^\alpha \ .$$

we can write Eq. (5.11) as:

$$\boxed{\delta\vec{A} = \frac{1}{2} A^\nu R^\mu{}_{\nu\alpha\beta} \Delta S^{\alpha\beta} \vec{e}_\mu} \ . \qquad (5.12)$$

The components of the Riemann tensor expressed by the connection- and structure-coefficients are given below:



The sum of the angles of a triangle on a surface of negative curvature is less than that of a plane triangle, i.e. less than π , and geodesics bend away from each other.

$$
\begin{aligned}
\vec{e}_\mu R^\mu{}_{\nu\alpha\beta} &= [\nabla_\alpha, \nabla_\beta]\vec{e}_\nu - \nabla_{[\vec{e}_\alpha, \vec{e}_\beta]}\vec{e}_\nu \\
&= (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha - c^\rho{}_{\alpha\beta}\nabla_\rho)\vec{e}_\nu \\
&= \nabla_\alpha \nabla_\beta \vec{e}_\nu - \nabla_\beta \nabla_\alpha \vec{e}_\nu - c^\rho{}_{\alpha\beta}\nabla_\rho \vec{e}_\nu \\
\text{(Kozul-connection)} \quad &= \nabla_\alpha \Gamma^\mu{}_{\nu\beta}\vec{e}_\mu - \nabla_\beta \Gamma^\mu{}_{\nu\alpha}\vec{e}_\mu - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho}\vec{e}_\mu \\
&= (\nabla_\alpha \Gamma^\mu{}_{\nu\beta})\vec{e}_\mu + \Gamma^\mu{}_{\nu\beta}\nabla_\alpha \vec{e}_\mu \\
&\quad - (\nabla_\beta \Gamma^\mu{}_{\nu\alpha})\vec{e}_\mu - \Gamma^\mu{}_{\nu\alpha}\nabla_\beta \vec{e}_\mu - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho}\vec{e}_\mu \\
&= \vec{e}_\alpha(\Gamma^\mu{}_{\nu\beta})\vec{e}_\mu + \Gamma^\rho{}_{\nu\beta}\Gamma^\mu{}_{\rho\alpha}\vec{e}_\mu \\
&\quad - \vec{e}_\beta(\Gamma^\mu{}_{\nu\alpha})\vec{e}_\mu - \Gamma^\rho{}_{\nu\alpha}\Gamma^\mu{}_{\rho\beta}\vec{e}_\mu - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho}\vec{e}_\mu \; .
\end{aligned}
$$

This gives (in arbitrary basis):

$$
\begin{aligned}
R^\mu{}_{\nu\alpha\beta} &= \vec{e}_\alpha(\Gamma^\mu{}_{\nu\beta}) - \vec{e}_\beta(\Gamma^\mu{}_{\nu\alpha}) \\
&\quad + \Gamma^\rho{}_{\nu\beta}\Gamma^\mu{}_{\rho\alpha} - \Gamma^\rho{}_{\nu\alpha}\Gamma^\mu{}_{\rho\beta} - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho} \; .
\end{aligned}
$$

In coordinate basis eq. (5.14) is reduced to:

$$
R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu{}_{\nu\beta,\alpha} - \Gamma^\mu{}_{\nu\alpha,\beta} + \Gamma^\rho{}_{\nu\beta}\Gamma^\mu{}_{\rho\alpha} - \Gamma^\rho{}_{\nu\alpha}\Gamma^\mu{}_{\rho\beta} \; ,
$$

where $\Gamma^\mu{}_{\nu\beta} = \Gamma^\mu{}_{\beta\nu}$ are the Christoffel symbols.

Since the basis vectors are derivative operators the first two terms is a linear combination of derivatives of the connection coefficients. In a local Cartesian coordinate system co-moving with a local inertial reference frame all the connection coefficients vanish, and only the first two terms in the expression of the components of the Riemann curvature tensor remain. As we have seen above this means that in such a system there is no acceleration of gravity. But in general the derivatives of the connections coefficients will not vanish. Hence in general spacetime is curved. This shows that the acceleration of gravity does not depend upon the curvature of spacetime. It depends instead upon the motion of the reference frame. The curvature of spacetime is given by a tensor and is an invariant property of spacetime at the considered position. The acceleration of gravity is, however, not an invariant property of spacetime since it is given by certain connection coefficients which are not tensor components. They can be transformed away.