Lecture 22. 9. April 2018

Due to the antisymmetry (5.10) we can define a matrix of *curvature-forms*

$$\underline{R}^{\mu}_{\ \nu} = \frac{1}{2} R^{\mu}_{\ \nu\alpha\beta} \underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} \tag{5.16}$$

Inserting the components of the Riemann tensor from eq. (5.14) gives

$$\underline{R}^{\mu}_{\ \nu} = (\vec{e}_{\alpha}(\Gamma^{\mu}_{\ \nu\beta}) + \Gamma^{\rho}_{\ \nu\beta}\Gamma^{\mu}_{\ \rho\alpha} - \frac{1}{2} c^{\rho}_{\ \alpha\beta}\Gamma^{\mu}_{\ \nu\rho}) \underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} \tag{5.17}$$

The connection forms:

$$\underline{\Omega}^{\mu}_{\ \nu} = \Gamma^{\mu}_{\ \nu\alpha} \underline{\omega}^{\alpha}$$

Exterior derivatives of basis forms:

$$\underline{d\omega}^{\rho} = -\frac{1}{2} c^{\rho}_{\ \alpha\beta} \underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta}$$

Exterior derivatives of connection forms (C1: $\underline{d\omega}^{\rho} = -\underline{\Omega}^{\rho}_{\ \alpha} \wedge \underline{\omega}^{\alpha}$) :

$$\begin{aligned} \underline{d\Omega}^{\mu}_{\ \nu} &= \underline{d}\Gamma^{\mu}_{\ \nu\beta} \wedge \underline{\omega}^{\beta} + \Gamma^{\mu}_{\ \nu\rho} \underline{d\omega}^{\rho} \\ &= \vec{e}_{\alpha}(\Gamma^{\mu}_{\ \nu\beta} \underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} - \frac{1}{2} c^{\rho}_{\ \alpha\beta}\Gamma^{\mu}_{\ \nu\rho} \underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} \end{aligned}$$

The curvature forms can now be written as:

$$\boxed{\underline{R}^{\mu}_{\ \nu} = \underline{d\Omega}^{\mu}_{\ \nu} + \underline{\Omega}^{\mu}_{\ \lambda} \wedge \underline{\Omega}^{\lambda}_{\ \nu}}$$

This is Cartans 2nd structure equation.

Example. The Riemann curvature tensor of a spherical surface calculated from Cartan's structure equations

Let $r = R$ be the radius of the spherical surface. The calculation is performed in 5 steps.

1. Write down the line element and introduce a form basis dual to an orthonormal vector basis.

$$dl^2 = \underline{\omega}^{\hat{\theta}} \otimes \underline{\omega}^{\hat{\theta}} + \underline{\omega}^{\hat{\varphi}} \otimes \underline{\omega}^{\hat{\varphi}} = R^2 \underline{d\theta} \otimes \underline{d\theta} + R^2 \sin^2\theta \underline{d\varphi} \otimes \underline{d\varphi}$$

giving

$$\underline{\omega}^{\hat{\theta}} = R\underline{d\theta} \quad , \quad \underline{\omega}^{\hat{\varphi}} = R\sin\theta \underline{d\varphi}$$

2. Use Cartan's 1. structure equation, $\underline{d\omega}^{\mu} = \underline{\omega}^{\nu} \wedge \underline{\Omega}^{\mu}{}_{\nu}$, to calculate the structure forms. Since R is constant and using Poincare's lemma and the antisymmetry of the connection forms we get

$$\underline{d\omega}^{\hat{\theta}} = 0 = \underline{\omega}^{\hat{\varphi}} \wedge \underline{\Omega}^{\hat{\theta}}{}_{\hat{\varphi}}, \text{ giving } \underline{\Omega}^{\hat{\theta}}{}_{\hat{\varphi}} = f(\theta,\varphi)\underline{\omega}^{\hat{\varphi}},$$

where the function $f(\theta,\varphi)$ is to be determined from the antisymmetry of the connection forms.

$$\underline{d\omega}^{\hat{\varphi}} = R\cos\theta \underline{d\theta} \wedge \underline{d\varphi} = \underline{\omega}^{\hat{\theta}} \wedge \frac{1}{R}\frac{\cos\theta}{\sin\theta}\underline{\omega}^{\hat{\varphi}} = \underline{\omega}^{\hat{\theta}} \wedge \underline{\Omega}^{\hat{\varphi}}{}_{\hat{\theta}}, \text{ giving } \underline{\Omega}^{\hat{\varphi}}{}_{\hat{\theta}} = g(\theta,\varphi)\underline{\omega}^{\hat{\theta}} + \frac{1}{R}\frac{\cos\theta}{\sin\theta}\underline{\omega}^{\hat{\varphi}}.$$

Using that $\underline{\Omega}^{\hat{\theta}}{}_{\hat{\varphi}} = \underline{\Omega}_{\hat{\theta}\hat{\varphi}} = -\underline{\Omega}_{\hat{\varphi}\hat{\theta}} = -\underline{\Omega}^{\hat{\varphi}}{}_{\hat{\theta}}$ we get $f(\theta,\varphi) = \frac{1}{R}\frac{\cos\theta}{\sin\theta}, \quad g(\theta,\varphi) = 0$. Hence

$$\underline{\Omega}^{\hat{\varphi}}{}_{\hat{\theta}} = -\underline{\Omega}^{\hat{\theta}}{}_{\hat{\varphi}} = \frac{1}{R}\frac{\cos\theta}{\sin\theta}\underline{\omega}^{\hat{\varphi}} = \cos\theta \underline{d\varphi}.$$

The reason for going back to coordinate basis here is that then it is easier to calculate the exterior derivative $\underline{d\Omega}^{\hat{\varphi}}{}_{\hat{\theta}}$.

3. Calculate the Riemann curvature forms from Cartan's 2. structure equation.

$$\underline{R}^{\hat{\theta}}{}_{\hat{\varphi}} = \underline{d\Omega}^{\hat{\theta}}{}_{\hat{\varphi}} + \underline{\Omega}^{\hat{\theta}}{}_{\hat{\varphi}} \wedge \underline{\Omega}^{\hat{\varphi}}{}_{\hat{\varphi}} = \underline{d\Omega}^{\hat{\theta}}{}_{\hat{\varphi}} = \underline{d}(\cos\theta \underline{d\varphi}) = -\sin\theta \underline{d\theta} \wedge \underline{d\varphi} = -\frac{1}{R^2}\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\varphi}} = -\underline{R}^{\hat{\varphi}}{}_{\hat{\theta}}.$$

4. Calculate the non-vanishing components of the Riemann tensor from $\underline{R}^{\hat{\mu}}{}_{\hat{\nu}} = (1/2)R^{\mu}{}_{\nu\alpha\beta}\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta}$. This gives $R^{\hat{\theta}}{}_{\hat{\varphi}\hat{\theta}\hat{\varphi}} = R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = -R^{\hat{\theta}}{}_{\hat{\varphi}\hat{\varphi}\hat{\theta}} = -R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\theta}\hat{\varphi}} = \frac{1}{R^2}$.

5. Calculate the components of the Ricci curvature tensor and the Ricci curvature scalar.

$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{\varphi}\hat{\varphi}} = \frac{1}{R^2} \quad , \quad R = R^{\hat{\theta}}{}_{\hat{\theta}} + R^{\hat{\varphi}}{}_{\hat{\phi}} = \frac{2}{R^2}.$$

Torsion

The *torsion* operator $\mathbf{T}$ is defined by

$$\boxed{\mathbf{T}(\mathbf{u}\wedge\mathbf{v}) \equiv \nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} - [\mathbf{u},\mathbf{v}]\,.} \tag{6.131}$$

The operator $\mathbf{T}$ is a two-form with vector components

$$\mathbf{T}(\mathbf{u}\wedge\mathbf{v}) = -\left(\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} + c^{\rho}{}_{\mu\nu}\right) u^{\mu} v^{\nu} \mathbf{e}_{\rho}. \tag{6.132}$$

Introducing the scalar torsion components $T^{\rho}_{\mu\nu}$ by (using the sign convention of [MTW73])

$$\mathbf{T}(\mathbf{u}\wedge\mathbf{v}) = T^{\rho}_{\mu\nu} u^{\mu} v^{\nu} \mathbf{e}_{\rho}, \tag{6.133}$$

we have

$$T^{\rho}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} - c^{\rho}{}_{\mu\nu}. \tag{6.134}$$

In a coordinate basis $c^{\rho}{}_{\mu\nu} = 0$, so that

$$T^{\rho}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu}. \tag{6.135}$$

The spacetime of the general theory of relativity is assumed to be torsion free (we will later see that in this case the connection is compatible with the metric). Then the connection coefficients are related to the structure constants by

$$\boxed{c^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu},} \tag{6.136}$$

and in the special case where we are in a coordinate basis, the structure coefficients vanish, and the Christoffel symbols are symmetric in their lower indices.

From eqs. (6.133) and (6.134) follow that the torsion operator has the component form

$$\mathbf{T} = \frac{1}{2}\left(\Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} - c^{\rho}{}_{\mu\nu}\right) \mathbf{e}_{\rho} \otimes \boldsymbol{\omega}^{\mu} \wedge \boldsymbol{\omega}^{\nu}. \tag{6.177}$$

Inserting eqs. (6.165) and (6.176) we get

$$\mathbf{T} = \mathbf{e}_\rho \otimes \left(\mathbf{d}\boldsymbol{\omega}^\rho + \boldsymbol{\Omega}^\rho{}_\nu \wedge \boldsymbol{\omega}^\nu \right). \tag{6.178}$$

The torsion two-forms $\mathbf{T}^\rho$ are defined by

$$\mathbf{T} \equiv \mathbf{e}_\rho \otimes \mathbf{T}^\rho. \tag{6.179}$$

Hence,

$$\mathbf{T}^\rho = \mathbf{d}\boldsymbol{\omega}^\rho + \boldsymbol{\Omega}^\rho{}_\nu \wedge \boldsymbol{\omega}^\nu. \tag{6.180}$$

A torsion-free space is called *Riemannian*. In general relativity spacetime is assumed to be Riemannian.

7.5 The equation of geodesic deviation

Consider two nearby geodesic curves, both parametrised by a parameter λ. Let $\mathbf{s}$ be a vector connecting points on the two curves with the same value of λ. The connecting vector $\mathbf{s}$ is said to measure the *geodesic deviation* of the curves.

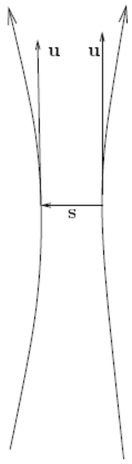

Figure 7.5: The two solid lines are neighbouring geodesics. They are connected by an infinitesimal vector $\mathbf{s}$ that obeys the equation of geodesic deviation

In order to deduce an equation describing how the geodesic deviation varies along the curves, we consider the covariant directional derivative of $\mathbf{s}$ along the curve $\nabla_{\mathbf{u}}\mathbf{s}$ where $\mathbf{u}$ is the tangent vector to the curve.

Let $\mathbf{u}$ and $\mathbf{s}$ be coordinate basis vectors of a coordinate system. Then $[\mathbf{s}, \mathbf{u}] = 0$, so that

$$\nabla_{\mathbf{u}}\mathbf{s} = \nabla_{\mathbf{s}}\mathbf{u} \tag{7.100}$$

giving

$$\nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\mathbf{s} = \nabla_{\mathbf{u}}\nabla_{\mathbf{s}}\mathbf{u}. \tag{7.101}$$

Furthermore

$$\begin{aligned} \mathbf{R}(\mathbf{u}, \mathbf{s})\mathbf{u} &= \left([\nabla_{\mathbf{u}}, \nabla_{\mathbf{s}}] - \nabla_{[\mathbf{u},\mathbf{s}]}\right)\mathbf{u} \\ &= [\nabla_{\mathbf{u}}, \nabla_{\mathbf{s}}]\,\mathbf{u}. \end{aligned} \tag{7.102}$$

Thus

$$\nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\mathbf{s} = \nabla_{\mathbf{s}}\nabla_{\mathbf{u}}\mathbf{u} + \mathbf{R}(\mathbf{u}, \mathbf{s})\mathbf{u}. \tag{7.103}$$

Since the curves are geodesics $\nabla_{\mathbf{u}}\mathbf{u} = 0$, and the equation reduces to

$$\boxed{\nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\mathbf{s} + \mathbf{R}(\mathbf{s}, \mathbf{u})\mathbf{u} = 0} \tag{7.104}$$

where we have used the antisymmetry of the Riemann tensor. Equation (7.104) is called the *equation of geodesic deviation*. The component form of the equation is

$$\left(\frac{d^2\mathbf{s}}{d\lambda^2}\right)^{\mu} + R^{\mu}{}_{\alpha\nu\beta}u^{\alpha}s^{\nu}u^{\beta} = 0. \tag{7.105}$$

This equation shows that the Riemann tensor can be determined entirely from measurements of geodesic deviation.

In comoving geodesic normal coordinates with $\mathbf{u} = (1, 0, 0, 0)$ the equation reduces to

$$\left(\frac{d^2\mathbf{s}}{d\lambda^2}\right)^{i} + R^{i}{}_{0j0}s^{j} = 0. \tag{7.106}$$

In lecture 2 we found the equation for the *tidal acceleration*, i.e. the relative acceleration between two nearby particles

$$\frac{d^2\zeta^k}{dt^2} = -\zeta^i \frac{\partial^2\phi}{\partial x_i \partial x_k}\,.$$

where ζ^j is the the j-component of the separation vector, and ϕ is the Newtonian gravitational potential. Comparing these equations we see that in the Newtonian limit the non-vanishing components of the Riemann curvature tensor of spacetime are

$$R^i_{0j0} = \frac{\partial^2 \phi}{\partial x_i \partial x^j} .$$

In Newtonian physics the acceleration of gravity is given by

$$\vec{g} = -\nabla \phi \quad or \quad g^i = -\frac{\partial \phi}{\partial x_i} .$$

Comparing with the third equation in Lecture 21 we see that with a locally Cartesian coordinate system the non-vanishing Christoffel symbols are

$$\Gamma^i_{00} = \frac{\partial \phi}{\partial x_i} .$$

The Christoffel symbols are the first derivatives of the Newtonian gravitational potential and the components of the Riemann curvature tensor the second derivatives. Hence in the Newtonian approximation the non-vanishing components of the curvature tensor are

$$R^i_{0j0} = \frac{\partial \Gamma^i_{00}}{\partial x^j} .$$

The Newtonian tidal tensor

There are several definitions of the Newtonian tidal tensor which are mathematically equivalent. One is as follows. It is a tensor of rank 2 with components

$$E_{ij} = -\frac{\partial g_i}{\partial x^j} ,$$

i.e. E_{ij} is minus the change of the *i*-component of the acceleration of gravity due to a displacement in the *j*-direction. Since

$$g_i = -\frac{\partial \phi}{\partial x^i}$$

The components of the Newtonian tidal tensor may be written

$$E_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} .$$

It follows that the Newtonian tidal tensor is symmetrical.

The Newtonian gravitational field equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x_i \partial x^i} = 4\pi G\rho$$

can now be written

$$E_i^{\ i} = 4\pi G\rho \ .$$

Also it follows that the equation of tidal acceleration can be written

$$\frac{d^2\zeta^k}{dt^2} = -E_i^{\ k}\zeta^i \ ,$$

and that in the Newtonian limit the tidal tensor is related to the Riemann curvature tensor of spacetime by

$$R^i_{0j0} = E^i_{\ j} \ .$$

5.3 The Ricci identity

$$\vec{e_\mu} R^\mu_{\ \nu\alpha\beta} A^\nu = (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha - \nabla_{[\vec{e_\alpha},\vec{e_\beta}]})(\vec{A}) \qquad (5.50)$$

In coordinate basis this is reduced to

$$\vec{e_\mu} R^\mu_{\ \nu\alpha\beta} A^\nu = (A^\mu_{\ ;\beta\alpha} - A^\mu_{\ ;\alpha\beta})\vec{e_\mu} \ , \qquad (5.51)$$

where

$$A^\mu_{\ ;\alpha\beta} \equiv (A^\mu_{\ ;\beta})_{;\alpha} \qquad (5.52)$$

The **Ricci identity** on component form is:

$$A^\nu R^\mu_{\ \nu\alpha\beta} = A^\mu_{\ ;\beta\alpha} - A^\mu_{\ ;\alpha\beta} \qquad (5.53)$$

We can write this as:

$$\underline{d}^2\vec{A} = \frac{1}{2} R^\mu_{\ \nu\alpha\beta} A^\nu \vec{e_\mu} \otimes \underline{\omega}^\alpha \wedge \underline{\omega}^\beta \qquad (5.54)$$

This shows us that the 2nd exterior derivative of a vector is equal to zero only in a *flat* space. Equations (5.53) and (5.54) *both* represents the Ricci identity.

In a space with torsion the Ricci identity (5.53) takes the form

$$A^\nu R^\mu_{\ \nu\alpha\beta} = A^\mu_{\ ;\beta\alpha} - A^\mu_{\ ;\alpha\beta} + T^\lambda_{\ \alpha\beta} A^\mu_{\ ;\lambda} \ .$$

5.4 Bianchi's 1st identity

Cartan's 1st structure equation:

$$\underline{d}\,\underline{\omega}^\mu = -\underline{\Omega}^\mu{}_\nu \wedge \underline{\omega}^\nu \tag{5.55}$$

Cartan's 2nd structure equation:

$$\underline{R}^\mu{}_\nu = \underline{d}\,\underline{\Omega}^\mu{}_\nu + \underline{\Omega}^\mu{}_\lambda \wedge \underline{\Omega}^\lambda{}_\nu \tag{5.56}$$

Exterior differentiation of (5.55) and use of $Poincaré's$ lemma (4.16) gives: $(\underline{d}^2\,\underline{\omega}^\mu = 0)$

$$0 = \underline{d}\,\underline{\Omega}^\mu{}_\nu \wedge \underline{\omega}^\nu - \underline{\Omega}^\mu{}_\lambda \wedge \underline{d}\,\underline{\omega}^\lambda \tag{5.57}$$

Use of (5.55) gives:

$$\underline{d}\,\underline{\Omega}^\mu{}_\nu \wedge \underline{\omega}^\nu + \underline{\Omega}^\mu{}_\lambda \wedge \underline{\Omega}^\lambda{}_\nu \wedge \underline{\omega}^\nu = 0 \tag{5.58}$$

From this we see that

$$(\underline{d}\,\underline{\Omega}^\mu{}_\nu + \underline{\Omega}^\mu{}_\lambda \wedge \underline{\Omega}^\lambda{}_\nu) \wedge \underline{\omega}^\nu = 0 \tag{5.59}$$

We now get **Bianchi's 1st identity**:

$$\boxed{\underline{R}^\mu{}_\nu \wedge \underline{\omega}^\nu = 0} \tag{5.60}$$

On component form Bianchi's 1st identity is

$$\underbrace{\frac{1}{2} R^\mu{}_{\nu\alpha\beta}\underline{\omega}^\alpha \wedge \underline{\omega}^\beta}_{\underline{R}^\mu{}_\nu} \wedge \underline{\omega}^\nu = 0 \tag{5.61}$$

The component equation is: (remember the anti symmetry in α and β)

$$R^\mu{}_{[\nu\alpha\beta]} = 0 \tag{5.62}$$

or

$$R^\mu{}_{\nu\alpha\beta} + R^\mu{}_{\alpha\beta\nu} + R^\mu{}_{\beta\nu\alpha} + = 0 \tag{5.63}$$

where the anti symmetry $R^\mu{}_{\nu\alpha\beta} = -R^\mu{}_{\nu\beta\alpha}$ has been used. Without this anti symmetry we would have gotten six, and not three, terms in this equation.

It may be noted that in a space with torsion Bianchi's 1. identity takes the form

$$\underline{R}^\mu{}_\nu \wedge \underline{\omega}^\nu = \underline{dT}^\mu + \underline{\Omega}^\mu{}_\nu \wedge \underline{T}^\nu\,.$$