Lecture 23. 10. April 2018

5.5 Bianchi's 2nd identity

Exterior differentiation of (5.56) $\Rightarrow$

$$\begin{aligned} \underline{d}\,\underline{R}^{\mu}{}_{\nu} &= \underline{R}^{\mu}{}_{\lambda} \wedge \underline{\Omega}^{\lambda}{}_{\nu} - \underline{\Omega}^{\mu}{}_{\rho} \wedge \underline{\Omega}^{\rho}{}_{\lambda} \wedge \underline{\Omega}^{\lambda}{}_{\nu} - \underline{\Omega}^{\mu}{}_{\lambda} \wedge \underline{R}^{\lambda}{}_{\nu} + \underline{\Omega}^{\mu}{}_{\lambda} \wedge \underline{\Omega}^{\lambda}{}_{\rho} \wedge \underline{\Omega}^{\rho}{}_{\nu} \\ &= \underline{R}^{\mu}{}_{\lambda} \wedge \underline{\Omega}^{\lambda}{}_{\nu} - \underline{\Omega}^{\mu}{}_{\lambda} \wedge \underline{R}^{\lambda}{}_{\nu} \end{aligned} \tag{5.64}$$

We now have **Bianchi's 2nd identity** as a form equation:

$$\boxed{\underline{d}\,\underline{R}^{\mu}{}_{\nu} + \underline{\Omega}^{\mu}{}_{\lambda} \wedge \underline{R}^{\lambda}{}_{\nu} - \underline{R}^{\mu}{}_{\lambda} \wedge \underline{\Omega}^{\lambda}{}_{\nu} = 0} \tag{5.65}$$

As a component equation Bianchi's 2nd identity is given by

$$R^{\mu}{}_{\nu[\alpha\beta;\gamma]} = 0 \tag{5.66}$$

Definition 5.5.1 (Contraction)
'Contraction' is a tensor operation defined by

$$R_{\nu\beta} \equiv R^{\mu}{}_{\nu\mu\beta} \tag{5.67}$$

We must here have summation over μ. What we do, then, is constructing a new tensor from another given tensor, with a rank 2 lower than the given one.

The tensor with components $R_{\nu\beta}$ is called **the Ricci curvature tensor**. Another contraction gives **the Ricci curvature scalar**, $R = R^{\mu}{}_{\mu}$.

Riemann curvature tensor has four symmetries. The definition of the Riemann tensor implies that $R^{\mu}{}_{\nu\alpha\beta} = -R^{\mu}{}_{\nu\beta\alpha}$
Bianchi's 1st identity: $R^{\mu}{}_{[\nu\alpha\beta]} = 0$
From Cartan's 2nd structure equation follows

$$\begin{aligned} \underline{R}_{\,\mu\nu} &= \underline{d\Omega}_{\,\mu\nu} + \underline{\Omega}_{\mu\lambda} \wedge \underline{\Omega}^{\lambda}{}_{\nu} \\ \Rightarrow R_{\,\mu\nu\alpha\beta} &= -R_{\,\nu\mu\alpha\beta} \end{aligned} \tag{5.68}$$

By choosing a locally Cartesian coordinate system in an inertial frame we get the following expression for the components of the Riemann curvature tensor:

$$R_{\,\mu\nu\alpha\beta} = \frac{1}{2}(g_{\mu\beta,\nu\alpha} - g_{\mu\alpha,\nu\beta} + g_{\nu\alpha,\mu\beta} - g_{\nu\beta,\mu\alpha}) \tag{5.69}$$

from which it follows that $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$. Contraction of μ and α leads to:

$$\begin{aligned} R^{\alpha}{}_{\nu\alpha\beta} &= R^{\alpha}{}_{\beta\alpha\nu} \\ \Rightarrow R_{\nu\beta} &= R_{\beta\nu} \end{aligned} \tag{5.70}$$

i.e. the Ricci tensor is symmetric. In 4-D the Ricci tensor has 10 independent components.

In 4-dimensional spacetime the four symmetries of the Riemann curvature tensor reduce the number of independent components from 256 to 20.

6.1 Energy-momentum conservation

6.1.1 Newtonian fluid

Energy-momentum conservation for a Newtonian fluid in terms of the divergence of the energy momentum tensor can be shown as follows. The total derivative of a velocity field is

$$\frac{D\vec{v}}{Dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \tag{6.1}$$

$\frac{\partial \vec{v}}{\partial t}$ is the local derivative which gives the change in $\vec{v}$ as a function of time at a given point in space. $(\vec{v} \cdot \vec{\nabla})\vec{v}$ is called the **convective** derivative of $\vec{v}$. It represents the change of $\vec{v}$ for a moving fluid particle due to the inhomogeneity of the fluid velocity field. In component notation the above become

$$\frac{Dv^i}{Dt} \equiv \frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} \tag{6.2}$$

The continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v^i)}{\partial x^i} = 0 \tag{6.3}$$

Euler's equation of motion (ignoring gravity)

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p \quad \text{or} \quad \rho \left(\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} \right) = -\frac{\partial p}{\partial x^i} \tag{6.4}$$

The **energy momentum tensor** is a symmetric tensor of rank 2 that describes material characteristics.

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} \tag{6.5}$$

$c \equiv 1$

T^{00} represents energy density.

T^{i0} represents momentum density.

T^{ii} represents pressure ($T^{ii} > 0$).

T^{ii} represents stress ($T^{ii} < 0$).

T^{ij} represents shear forces ($i \neq j$).

Example **6.1.1 (Energy momentum tensor for a Newtonian fluid)**

$$\begin{aligned} T^{00} &= \rho \qquad T^{i0} = \rho v^i \\ T^{ij} &= \rho v^i v^j + p\delta^{ij} \end{aligned} \tag{6.6}$$

where p is pressure, assumed isotropic here. We choose a locally Cartesian coordinate system in an inertial frame such that the covariant derivatives are reduced to partial derivatives. The divergence of the momentum energy tensor, $T^{\mu\nu}_{;\nu}$ has 4 components, one for each value of μ.

The zeroth component is

$$\begin{aligned} T^{0\nu}_{;\nu} = T^{0\nu}_{,\nu} &= T^{00}_{,0} + T^{0i}_{,i} \\ &= \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i} \end{aligned} \tag{6.7}$$

which by comparison to Newtonian hydrodynamics implies that $T^{0\nu}_{;\nu} = 0$ is the continuity equation. This equation represents the conservation of energy.

The ith component of the divergence is

$$\begin{aligned} T^{i\nu}_{,\nu} &= T^{i0}_{,0} + T^{ij}_{,j} \\ &= \frac{\partial(\rho v^i)}{\partial t} + \frac{\partial(\rho v^i v^j + p\delta^{ij})}{\partial x^j} \\ &= \rho\frac{\partial v^i}{\partial t} + v^i\frac{\partial \rho}{\partial t} + v^i\frac{\partial \rho v^j}{\partial x^j} + \rho v^j\frac{\partial v^i}{\partial x^j} + \frac{\partial p}{\partial x^i} \end{aligned} \tag{6.8}$$

now, according to the continuity equation

$$\begin{aligned} &\frac{\partial(\rho v^i)}{\partial x^i} = -\frac{\partial \rho}{\partial t} \\ \Rightarrow \quad &T^{i\nu}_{,\nu} = \rho\frac{\partial v^i}{\partial t} + v^i\frac{\partial \rho}{\partial t} - v^i\frac{\partial \rho}{\partial t} + \rho v^j\frac{\partial v^i}{\partial x^j} + \frac{\partial p}{\partial x^i} \\ &\qquad = \rho\frac{Dv^i}{Dt} + \frac{\partial p}{\partial x^i} \\ \therefore \quad &T^{i\nu}_{;\nu} = 0 \Rightarrow \rho\frac{Dv^i}{Dt} = -\frac{\partial p}{\partial x^i} \end{aligned} \tag{6.9}$$

which is Euler's equation of motion. It expresses the conservation of momentum.

The equations $T^{\mu\nu}_{;\nu} = 0$ are general expressions for energy and momentum conservation.

6.1.2 Perfect fluids

A perfect fluid is a fluid with no viscosity and is given by the energy-momentum tensor

$$T_{\mu\nu} = (\rho + \frac{p}{c^2})u_\mu u_\nu + p g_{\mu\nu} \tag{6.10}$$

where ρ and p are the mass density and the stress, respectively, measured in the fluids rest frame, u_μ are the components of the 4-velocity of the fluid.

In a comoving orthonormal basis the components of the 4-velocity are $u^{\hat{\mu}} = (c, 0, 0, 0)$. Then the energy-momentum tensor is given by

$$T_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{6.11}$$

where $p > 0$ is pressure and $p < 0$ is tension.

There are three different types of perfect fluids that are useful.

1. **dust** or non-relativistic gas is given by $p = 0$ and the energy-momemtum tensor $T_{\mu\nu} = \rho u_\mu u_\nu$.

2. **radiation** or ultra-relativistic gas is given by a traceless energy-momemtum tensor, i.e. $T^\mu{}_\mu = 0$. It follows that $p = \frac{1}{3}\rho c^2$.

3. **vacuum energy**: If we assume that no velocity can be measured relatively to vacuum, then all the components of the energy-momentum tensor must be Lorentz-invariant. It follows that $T_{\mu\nu} \propto g_{\mu\nu}$. If vacuum is defined as a perfect fluid we get $p = -\rho c^2$ so that $T_{\mu\nu} = p g_{\mu\nu} = -\rho c^2 g_{\mu\nu}$.