Lecture 27. 24. april 2018

7.3 Light cones in Schwarzschild spacetime

The Schwarzschild line-element (with $c = 1$) is

$$ds^2 = -(1 - \frac{R_S}{r})dt^2 + \frac{dr^2}{(1 - \frac{R_S}{r})} + r^2 d\Omega^2 \tag{7.38}$$

We will look at **radially moving photons** ($ds^2 = d\Omega^2 = 0$). We then get

$$\begin{aligned} \frac{dr}{\sqrt{1 - \frac{R_S}{r}}} = \pm\sqrt{1 - \frac{R_S}{r}}dt \Leftrightarrow \frac{r^{\frac{1}{2}}dr}{\sqrt{r - R_S}} = \pm\frac{\sqrt{r - R_S}}{r^{\frac{1}{2}}}dt \\ \frac{r\,dr}{r - R_S} = \pm dt \end{aligned} \tag{7.39}$$

with $+$ for outward motion and $-$ for inward motion. For inwardly moving photons, integration yields

$$r + t + R_S \ln|\frac{r}{R_S} - 1| = k = constant \tag{7.40}$$

We now introduce a new time coordinate t' such that the equation of motion for photons moving **inwards** takes the following form

$$r + t' = k \Rightarrow \frac{dr}{dt'}_{r} = -1 \tag{7.41}$$

The coordinate t' is called an ingoing Eddington-Finkelstein coordinate. The photons here always move with the *local* velocity of light, c. For photons moving **outwards** we have

$$r + R_S \ln|\frac{r}{R_S} - 1| = t + k \tag{7.42}$$

Making use of $t = t' - R_S \ln|\frac{r}{R_S} - 1|$ we get

$$\begin{aligned} & r + 2R_S \ln|\frac{r}{R_S} - 1| = t' + k \\ \Rightarrow\ & \frac{dr}{dt'} + \frac{2R_S}{r - R_S}\frac{dr}{dt'} = 1 \Leftrightarrow \frac{r + R_S}{r - R_S}\frac{dr}{dt'} = 1 \\ \Leftrightarrow\ & \frac{dr}{dt'} = \frac{r - R_S}{r + R_S} \end{aligned} \tag{7.43}$$

Making use of ordinary Schwarzschild coordinates we would have gotten the following coordinate velocities for inn- and outwardly moving photons:

$$\frac{dr}{dt} = \pm(1 - \frac{R_S}{r}) \tag{7.44}$$

which shows us how light is decelerated in a gravitational field. Figure 7.1 shows how this is viewed by a non-moving observer located far away from the mass. In Figure 7.2 we have instead used the alternative time coordinate t'. The special theory of relativity is valid locally, and all material particles thus have to remain inside the light cone.

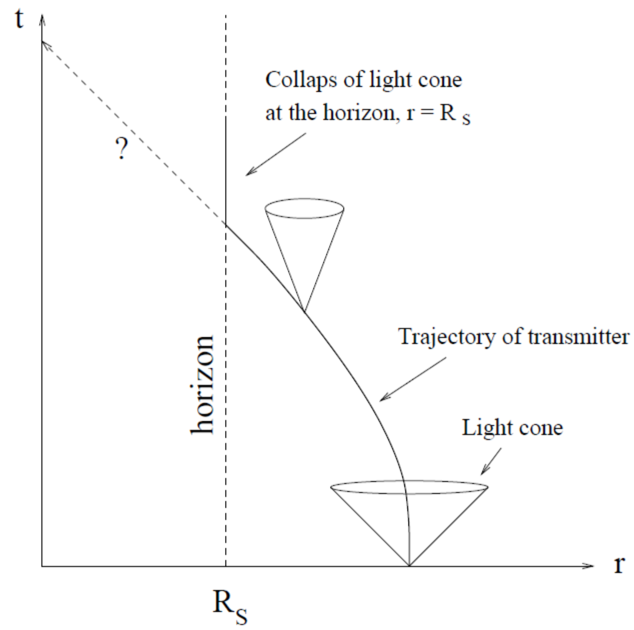

Figure 7.1: At a radius $r = R_S$ the light cones collapse, and nothing can any longer escape, when we use the Schwarzschild coordinate time.

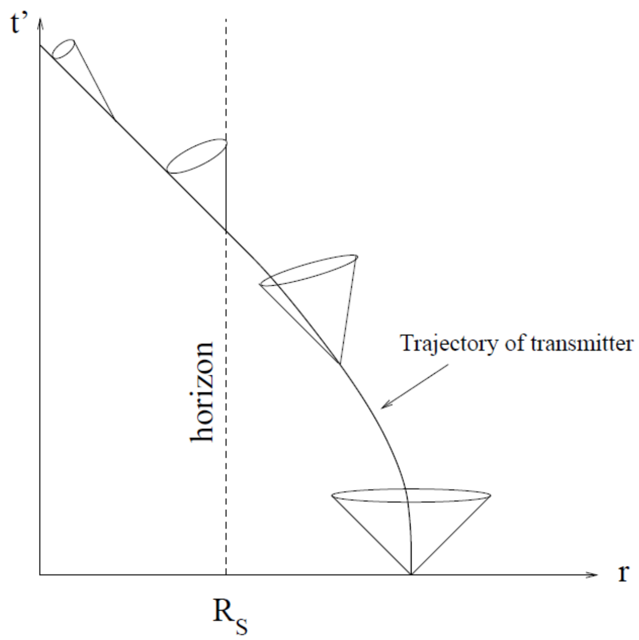

Figure 7.2: Using the ingoing Eddingto n-Finkelstein time coordinate there is no collapse of the light cone at $r = R_S$. Instead we get a collapse at the singularity at $r = 0$. The angle between the left part of the light cone and the t'-axis is always 45 degrees. We also see that once the transmitter gets inside the horizon at $r = R_S$, no particles can escape.

7.4 Analytical extension of the Schwarzschild spacetime

The Schwarzschild coordinates are comoving with a static reference frame outside a spherical mass distribution. If the mass has collapsed to a black hole there exist a horizon at the Schwarzschild radius. As we have seen in section 7.3 there do not exist static observers at finite radii inside the horizon. Hence, the Schwarzschild coordinates are well defined only outside the horizon.

Also the rr-component of the metric tensor has a coordinate singularity at the Schwarzschild radius. The curvature of spacetime is finite here.

Kruskal and Szekeres have introduced new coordinates that are well defined inside as well as outside the Schwarzschild radius, and with the property that the metric tensor is non-singular for all $r > 0$.

In order to arrive at these coordinates we start by considering a photon moving radially inwards. From eq. (7.40) we then have

$$t = -r - R_S \ln \left| \frac{r}{R_S} - 1 \right| + v \tag{7.45}$$

where v is a constant along the world line of the photon. We introduce a new radial coordinate

$$r^* \equiv r + R_S \ln \left| \frac{r}{R_S} - 1 \right| \tag{7.46}$$

Then the equation of the worldline of the photon takes the form

$$t + r^* = v \tag{7.47}$$

The value of the constant v does only depend upon the point of time when the photon was emitted. We may therefore use v as a new time coordinate.

For an outgoing photon we get in the same way

$$t - r^* = u \tag{7.48}$$

where u is a constant of integration, which may be used as a new time coordinate for outgoing photons. The coordinates u and v are the generalization of the **light cone coordinates** of Minkowski spacetime to the Schwarzschild spacetime.

From eqs. (7.47) and (7.48) we get

$$dt = \frac{1}{2}(dv + du) \tag{7.49}$$

$$dr^* = \frac{1}{2}(dv - du) \tag{7.50}$$

and from eq. (7.46)

$$dr = \left(1 - \frac{R_s}{r}\right) dr^* \tag{7.51}$$

Inserting these differentials into eq. (7.38) we arrive at a new form of the Schwarzschild line-element,

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) du\, dv + r^2 d\Omega^2 \tag{7.52}$$

The metric is still not well behaved at the horizon. Introducing the coordinates

$$U = -e^{-\frac{u}{2R_s}} \tag{7.53}$$

$$V = e^{\frac{v}{2R_s}} \tag{7.54}$$

gives

$$UV = -e^{\frac{v-u}{2R_s}} = -e^{\frac{r^*}{R_s}} = -\left|\frac{R_s}{r} - 1\right| e^{\frac{r}{R_s}} \tag{7.55}$$

and

$$du\, dv = -4R_s^2 \frac{dU\, dV}{UV} \tag{7.56}$$

The line-element (7.52) then takes the form

$$ds^2 = -\frac{4R_s^3}{r} e^{-\frac{r}{R_s}} dU\, dV + r^2 d\Omega^2 \tag{7.57}$$

This is the first form of the Kruskal-Szekeres line-element. Here is no coordinate singularity, only a physical singularity at $r = 0$.

We may furthermore introduce two new coordinates

$$T = \frac{1}{2}(V + U) = \left(\frac{r}{R_s} - 1\right) e^{\frac{r}{2R_s}} \sinh \frac{t}{2R_s} \tag{7.58}$$

$$Z = \frac{1}{2}(V - U) = \left(\frac{r}{R_s} - 1\right) e^{\frac{r}{2R_s}} \cosh \frac{t}{2R_s} \tag{7.59}$$

Hence

$$V = T + Z \tag{7.60}$$

$$U = T - Z \tag{7.61}$$

giving

$$dU dV = dT^2 - dZ^2 \tag{7.62}$$

Inserting this into eq. (7.57) we arrive at the second form of the Kruskal-Szekeres line-element

$$ds^2 = -\frac{4R_s^3}{r} e^{-\frac{r}{R_s}} \left(dT^2 - dZ^2\right) + r^2 d\Omega^2 \tag{7.63}$$

The inverse transformations of eqs. (7.58) and (7.59) is

$$\left(1 - \frac{r}{R_s}\right) e^{\frac{r}{R_s}} = T^2 - Z^2 \tag{7.64}$$

$$\tanh \frac{t}{2R_s} = \frac{T}{Z} \tag{7.65}$$

Note from eq. (7.63) that with the Kruskal-Szekeres coordinates T and Z the equation of the radial null geodesics has the same form as in flat spacetime

$$Z = \pm T + \text{constant} \tag{7.66}$$

7.5 Embedding of the Schwarzschild metric

We will now look at a static, spherically symmetric space. A curved simultaneity plane ($dt = 0$) through the equatorial plane ($d\theta = 0$) has the line element

$$ds^2 = g_{rr}dr^2 + r^2 d\phi^2 \tag{7.67}$$

with a radial coordinate such that a circle with radius r has a circumference of length $2\pi r$.

We now embed this surface in a flat 3-dimensional space with cylinder coordinates (z, r, ϕ) and line element

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \tag{7.68}$$

The surface described by the line element in (7.67) has the equation $z = z(r)$. The line element in (7.68) is therefore written as

$$ds^2 = [1 + (\frac{dz}{dr})^2]dr^2 + r^2 d\phi^2 \tag{7.69}$$

Demanding that (7.69) is in agreement with (7.67) we get

$$g_{rr} = 1 + (\frac{dz}{dr})^2 \Leftrightarrow \frac{dz}{dr} = \pm\sqrt{g_{rr} - 1} \tag{7.70}$$

Choosing the positive solution gives

$$\boxed{dz = \sqrt{g_{rr} - 1}dr} \tag{7.71}$$

In the Schwarzschild spacetime we have

$$g_{rr} = \frac{1}{1 - \frac{R_S}{r}} \tag{7.72}$$

Making use of this we find z:

$$z = \int_{R_S}^{r} \frac{dr}{\sqrt{1 - \frac{R_S}{r}}} = \sqrt{4R_S(r - R_S)} \tag{7.73}$$

This is shown in Figure 10.6 including negative values of *z*.

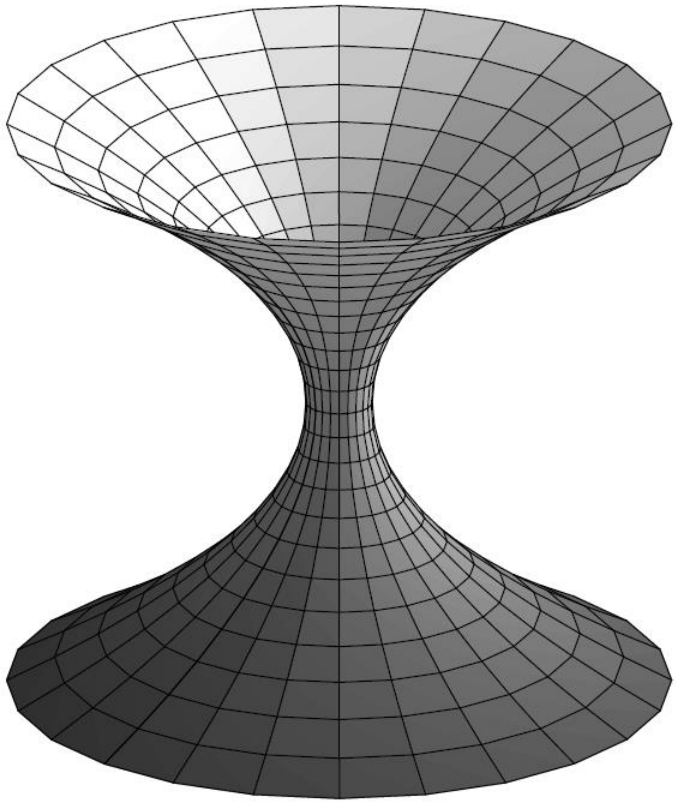

Figure 10.6: The embedding of a space-like hypersurface of the Schwarzschild spacetime. Depicted is Flamm's parabola which is two such hypersurfaces glued together along the horizon.

7.7 Particle trajectories in Schwarzschild 3-space

$$\begin{aligned} L &= \frac{1}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \\ &= -\frac{1}{2}\left(1 - \frac{R_s}{r}\right)\dot{t}^2 + \frac{\frac{1}{2}\dot{r}^2}{1 - \frac{R_s}{r}} + \frac{1}{2} r^2 \dot{\theta}^2 + \frac{1}{2} r^2 \sin^2\theta \dot{\phi}^2 \end{aligned} \tag{7.77}$$

Since t is a cyclic coordinate

$$-p_t = -\frac{\partial L}{\partial \dot{t}} = \left(1 - \frac{R_s}{r}\right)\dot{t} = constant = E \tag{7.78}$$

where E is the particle's energy as measured by an observer "far away" ($r \gg R_s$). Also ϕ is a cyclic coordinate so that

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = r^2 \sin^2\theta \dot{\phi} = constant \tag{7.79}$$

where p_ϕ is the particle's orbital angular momentum.

Making use of the 4-velocity identity $\vec{U}^2 = g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu = -1$ we transform the above to get

$$-\left(1 - \frac{R_s}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{R_s}{r}} + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2 = -1 \tag{7.80}$$

which on substitution for $\dot{t} = \frac{E}{1 - \frac{R_s}{r}}$ and $\dot{\phi} = \frac{p_\phi}{r^2 \sin^2\theta}$ becomes

$$-\frac{E^2}{1 - \frac{R_s}{r}} + \frac{\dot{r}^2}{1 - \frac{R_s}{r}} + r^2 \dot{\theta}^2 + \frac{p_\phi^2}{r^2 \sin^2\theta} = -1 \tag{7.81}$$

Now, refering back to the Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{X}^\mu}\right) - \frac{\partial L}{\partial X^\mu} = 0 \tag{7.82}$$

we get, for θ

$$\begin{aligned} (r^2 \dot{\theta})^\bullet &= r^2 \sin\theta \cos\theta \dot{\phi}^2 \\ &= \frac{p_\phi^2 \cos\theta}{r^2 \sin^3\theta} \end{aligned} \tag{7.83}$$

Multiplying this by $r^2\dot{\theta}$ we get

$$(r^2\dot{\theta})(r^2\dot{\theta})^{\bullet} = \frac{\cos\theta\dot{\theta}}{\sin^3\theta}p_\phi^2 \tag{7.84}$$

which, on integration, gives

$$(r^2\dot{\theta})^2 = k - \left(\frac{p_\phi}{\sin\theta}\right)^2 \tag{7.85}$$

where k is the constant of integration.

Because of the spherical geometry we are free to choose a coordinate system such that the particle moves in the equatorial plane and along the equator at a given time $t = 0$. That is $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$ at time $t = 0$. This determines the constant of integration and $k = p_\phi^2$ such that

$$(r^2\dot{\theta})^2 = p_\phi^2\left(1 - \frac{1}{\sin^2\theta}\right) \tag{7.86}$$

The RHS is negative for all $\theta \neq \frac{\pi}{2}$. It follows that the particle cannot deviate from its original (equatorial) trajectory. Also, since this particular choice of trajectory was arbitrary we can conclude, quite generally, that any motion of free particles in a spherically symmetric gravitational field is planar motion.

7.7.1 Motion in the equatorial plane

$$-\frac{E^2}{1-\frac{R_s}{r}} + \frac{\dot{r}^2}{1-\frac{R_s}{r}} + \frac{p_\phi^2}{r^2} = -1 \tag{7.87}$$

that is

$$\dot{r}^2 = E^2 - \left(1 - \frac{R_s}{r}\right)\left(1 + \frac{p_\phi^2}{r^2}\right) \tag{7.88}$$

This corresponds to an energy equation with an effective potential $V(r)$ given by

$$\begin{aligned} V^2(r) &= \left(1 - \frac{R_s}{r}\right)\left(1 + \frac{p_\phi^2}{r^2}\right) \\ \dot{r}^2 + V^2(r) &= E^2 \\ \Rightarrow V &= \sqrt{1 - \frac{r_s}{r} + \frac{p_\phi^2}{r^2} - \frac{R_s p_\phi^2}{r^3}} \\ &\approx 1 - \frac{1}{2}\frac{R_s}{r} + \frac{1}{2}\frac{p_\phi^2}{r^2} \end{aligned} \tag{7.89}$$

Newtonian potential V_N is defined by using the last expression in

$$V_N = V - 1 \Rightarrow V_N = -\frac{GM}{r} + \frac{p_\phi^2}{2r^2} \tag{7.90}$$

The possible trajectories of particles in the Schwarzschild 3-space are shown schematically in Figure 7.5 as functions of position and energy of the particle in the Newtonian limit.

To take into account the relativistic effects the above picture must be modified. We introduce dimensionless variables

$$X = \frac{r}{GM} \quad \text{and} \quad k = \frac{p_\phi}{GMm} \tag{7.91}$$

The potential $V^2(r)$ now take the form

$$V = \left(1 - \frac{2}{X} + \frac{k^2}{X^2} - \frac{2k^2}{X^3}\right)^{1/2} \tag{7.92}$$

For r equal to the Schwarzschild radius ($X = 2$) we have

$$V(2) = \sqrt{1 - 1 + \frac{k^2}{4} - \frac{2k^2}{8}} = 0 \tag{7.93}$$

For $k^2 < 12$ particles will fall in towards $r = 0$. The relativistic an Newtonian graphs are shown in Figure 10.3.

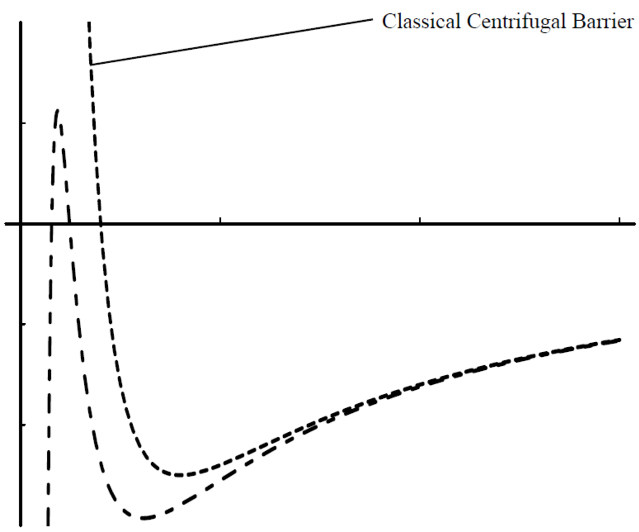

Figure 10.3: The graphs of the two potentials $V(r)$ and $V_N(r)$. Notice how the Newtonian potential has a centrifugal barrier for small r.