Lecture 29. 7. may 2018

7.8.3 Deflection of light

The orbit equation for a free particle with mass $m = 0$ is

$$\frac{d^2u}{d\phi^2} + u = ku^2 \tag{7.116}$$

If light is not deflected it will follow the straight line

$$\cos\phi = \frac{b}{r} = b\,u_0 \tag{7.117}$$

where b is the impact parameter of the path. This is the horizontal dashed line in Figure 7.7. The 0'th order solution (7.117) fullfills

$$\frac{d^2u_0}{d\phi^2} + u_0 = 0 \tag{7.118}$$

Hence it is a solution of (7.116) with $k = 0$.

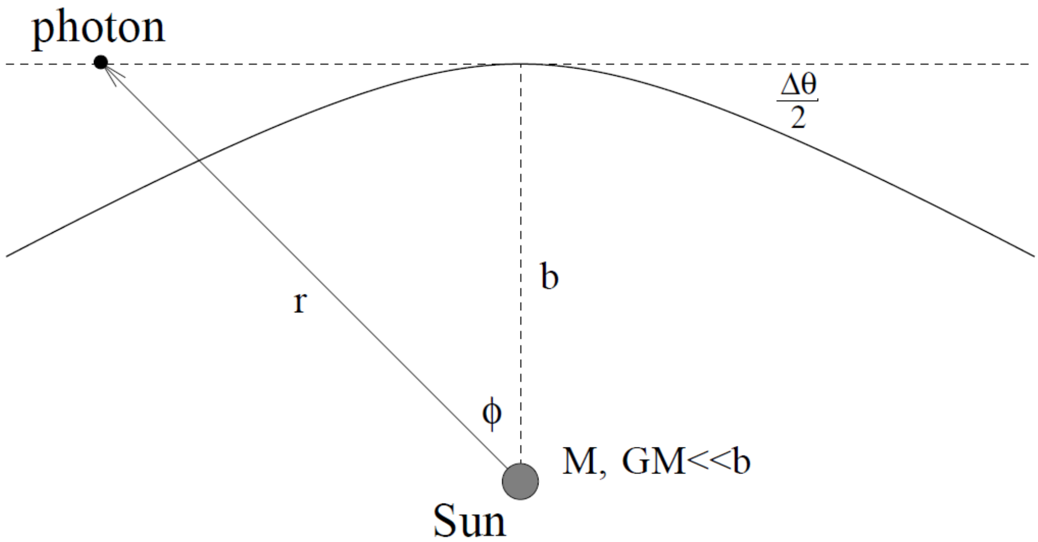

Figure 7.7: Light traveling close to a massive object is deflected.

The perturbed solution is

$$u = u_0 + u_1\ , \quad |u_1| \ll u_0 \tag{7.119}$$

Inserting this into the orbit equation gives

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$$\frac{d^2u_0}{d\phi^2} + \frac{d^2u_1}{d\phi^2} + u_0 + u_1 = ku_0^2 + 2ku_0u_1 + ku_1^2 \tag{7.120}$$

The first and third term at the left hand side cancel each other due to eq. (7.118] and the last term at the right hand side is small to second order in u_1 and will be neglected. Hence we get

$$\frac{d^2u_1}{d\phi^2} + u_1 = ku_0^2 + 2ku_0u_1 \tag{7.121}$$

The last term at the right hand side is much smaller then the first, and will also be neglected. Inserting for u_0 from (7.117) we then get

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{k}{b^2}\cos^2\phi \tag{7.122}$$

This equation has a particular solution of the form

$$u_{1p} = A + B\cos^2\phi \tag{7.123}$$

Inserting this into (7.122) we find

$$A = \frac{2k}{3b^2}\,, \quad B = -\frac{k}{3b^2} \tag{7.124}$$

Hence

$$u_{1p} = \frac{k}{3b^2}\left(2 - \cos^2\phi\right) \tag{7.125}$$

The general solution of the homogeneous equation is

$$u_{1h} = C_1\cos\phi + C_2\sin\phi \tag{7.126}$$

The general solution of eq.(7.122) is $u_1 = u_{1h} + u_{1p}$. Hence

$$u = u_0 + u_1 = \left(\frac{1}{b} + C_1\right)\cos\phi + C_2\sin\phi + \frac{k}{3b^2}\left(2 - \cos^2\phi\right). \tag{7.127}$$

The boundary condition $u(0) = 1/b$ gives

$$\frac{1}{b} = \frac{1}{b} + C_1 + \frac{k}{3b^2} \quad \Rightarrow \quad C_1 = -\frac{k}{3b^2}\,. \tag{7.128}$$

Hence

$$u = u_0 + u_1 = \left(\frac{1}{b} - \frac{k}{3b^2}\right)\cos\phi + C_2 \sin\phi + \frac{k}{3b^2}\left(2 - \cos^2\phi\right) \tag{7.129}$$

A correct Newtonian limit requires that $\lim\limits_{k \to 0} u = u_0 = \cos\phi / b$. Hence

$$\frac{\cos\phi}{b} + C_2 \sin\phi = \frac{\cos\phi}{b} \quad \Rightarrow \quad C_2 = 0 \,. \tag{7.130}$$

Thus

$$u = \frac{3b - k}{3b^2}\cos\phi + \frac{k}{3b^2}\left(2 - \cos^2\phi\right). \tag{7.131}$$

Note that in the present case the impact parameter is equal to the radius of the Sun, $b = R_\odot$ and $k = (3/2)R_S$, where R_S is the Schwarzschild radius of the Sun. Hence $k << b$ so we kan neglect k in the first term.

Furthermore the light deflection is determined by considering light far from the symmetry upper symmetry point of the figure. The radius vector is then very close to parallel with the light part. From the figure is then seen that $\phi = \frac{\pi}{2} + \frac{\Delta\phi}{2}$. Hence

$$\cos\phi = \cos\left(\frac{\pi}{2} + \frac{\Delta\phi}{2}\right) = \cos\frac{\pi}{2}\cos\frac{\Delta\phi}{2} - \sin\frac{\pi}{2}\sin\frac{\Delta\phi}{2} \approx -\frac{\Delta\phi}{2} \quad ,$$
$$\sin\phi = \sin\left(\frac{\pi}{2} + \frac{\Delta\phi}{2}\right) = \sin\frac{\pi}{2}\cos\frac{\Delta\phi}{2} + \cos\frac{\pi}{2}\sin\frac{\Delta\phi}{2} \approx 1$$

This means that we can neglect the $\cos^2\phi -$ term inside the parenthesis in eq. (7.131). So we get

$$u \approx -\frac{\Delta\phi}{2b} + \frac{2k}{3b^2} \,. \tag{7.132}$$

The deviation of light is found by taking the limit $r \to \infty$, i.e. $u \to 0$ giving

$$\Delta\phi = \frac{4k}{3b} \,. \tag{7.133}$$

Insering the general relativistic value $k = (3/2)R_S$ finally gives to light from a background star passing the limb of the sun

$$\Delta\phi = 2\frac{R_S}{R_\odot} \,. \tag{7.134}$$

Inserting the values of the Schwarzschild radius and the actual radius of the Sun gives $\Delta\phi = 1.75''$.

8.1 'Surface gravity':gravitational acceleration on the horizon of a black hole

Surface gravity is denoted by κ_1 and is defined by

$$\kappa = \lim_{r \to r_+} \frac{a}{u^t} \qquad a = \sqrt{a_\mu a^\mu} \tag{8.1}$$

where r_+ is the horizon radius, $r_+ = R_S$ for the Schwarzschild spacetime, u^t is the time component of the 4-velocity.

The 4-velocity of a free particle instantanously at rest in the Schwarzschild spacetime:

$$\vec{u} = u^t \vec{e}_t = \frac{dt}{d\tau}\vec{e}_t = \frac{1}{\sqrt{-g_{tt}}}\vec{e}_t = \frac{\vec{e}_t}{\sqrt{1 - \frac{R_S}{r}}} \tag{8.2}$$

The only component of the 4-acceleration different from zero, is a_r. The 4-acceleration:$\vec{a} = \nabla_{\vec{u}}\vec{u} = u^\mu{}_{;\nu} u^\nu \vec{e}_\mu = (u^\mu{}_{,\nu} + \Gamma^\mu{}_{\alpha\nu} u^\alpha) u^\nu \vec{e}_\mu$.

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$$\begin{aligned}
a_r &= (u_{r,\nu} + \Gamma_{r\alpha\nu} u^\alpha) u^\nu \\
&= \underbrace{u_{r,\nu} u^\nu}_{=0} + \Gamma_{rtt} (u^t)^2 \\
&= \frac{\Gamma_{rtt}}{1 - \frac{R_S}{r}} \\
\Gamma_{rtt} &= -\frac{1}{2}\frac{\partial g_{tt}}{\partial r} = -\frac{R_S}{2r^2} \\
a_r &= \frac{\frac{R_S}{2r^2}}{1 - \frac{R_S}{r}} \\
a^r &= g^{rr} a_r = \frac{a_r}{g_{rr}} = (1 - \frac{R_S}{r}) a_r = \frac{R_S}{2r^2}
\end{aligned} \tag{8.3}$$

The acceleration scalar: $a = \sqrt{a_r a^r} = \frac{\frac{R_S}{2r^2}}{\sqrt{1 - \frac{R_S}{r}}}$ (measured with standard instruments: at the horizon, time is not running).

$$\frac{a}{u^t} = \frac{R_S}{2r^2} \tag{8.4}$$

With c:

$$\frac{a}{u^t} = \frac{c^2 R_S}{2r^2} = \frac{GM}{r^2} \tag{8.5}$$

$$\kappa = \lim_{r \to R_S} \frac{a}{u^t} = \frac{1}{2R_S} = \frac{1}{4GM} \tag{8.6}$$

Including c the expression is $\kappa = \frac{c^2}{4GM}$. On the horizon of a black hole with one solar mass, we get $\kappa_\odot = 2 \times 10^{13} \frac{m}{s^2}$.