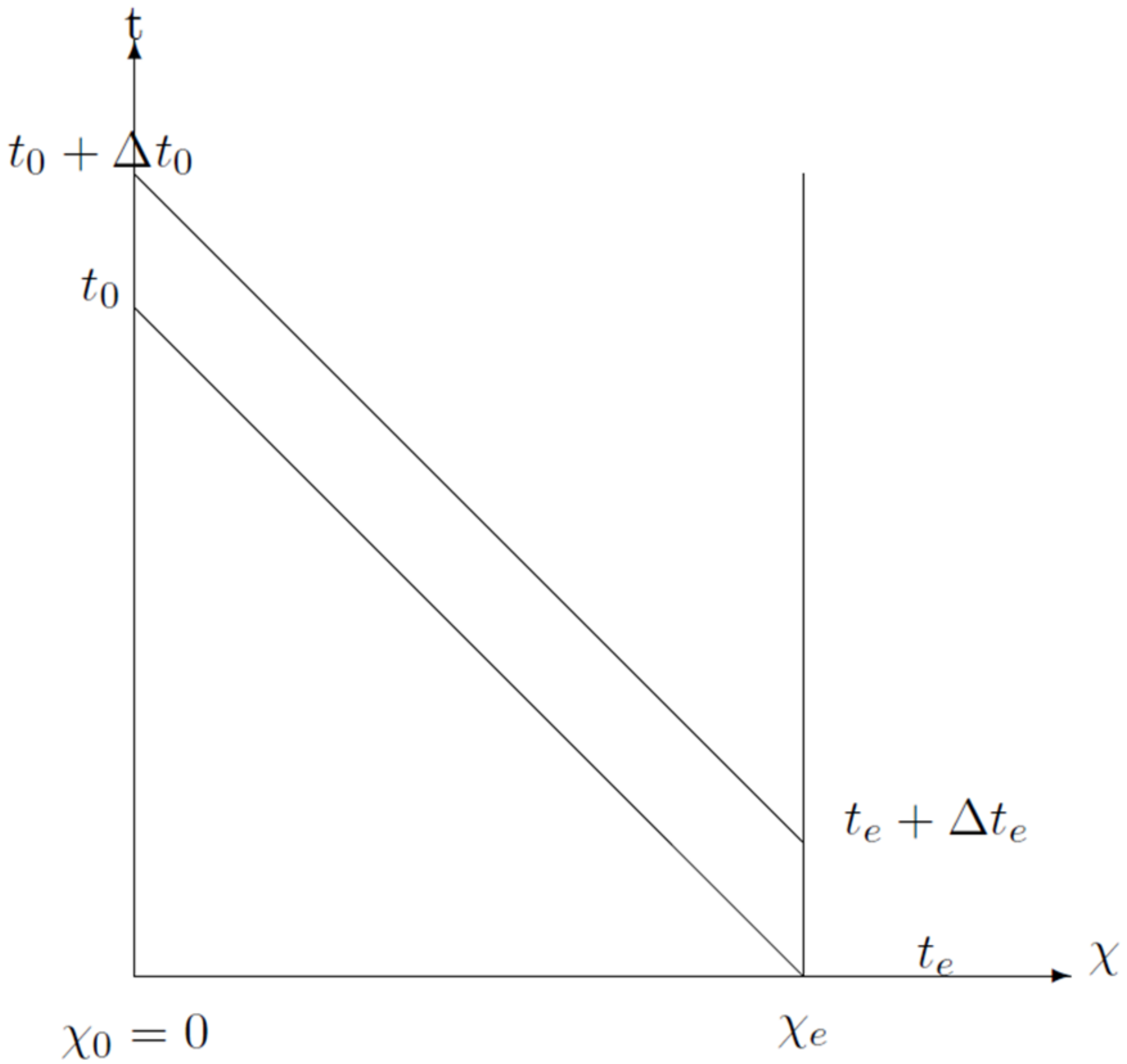
Lecture 34. 28. May 2018

10.3.2 Cosmological redshift of light

Δt_e : the time interval in transmitter-position at transmission-time
Δt_0 : the time interval in receiver-position at receiving-time

Light follows curves with $ds^2 = 0$, with $d\theta = d\phi = 0$ we have :

$$dt = -a(t)d\chi \tag{10.16}$$



Integration from transmitter-event to receiver-event :

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = -\int_{\chi_e}^{\chi_0} d\chi = \chi_e$$
$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} = -\int_{\chi_e}^{\chi_0} d\chi = \chi_e \ ,$$

which gives

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a} - \int_{t_e}^{t_0} \frac{dt}{a} = 0 \tag{10.17}$$

or

$$\int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a} - \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a} = 0 \tag{10.18}$$

Under the integration from t_e to $t_e + \Delta t_e$ the expansion factor $a(t)$ can be considered a constant with value $a(t_e)$ and under the integration from t_0 to $t_0 + \Delta t_0$ with value $a(t_0)$, giving:

$$\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_0}{a(t_0)} \tag{10.19}$$

Δt_0 and Δt_e are intervals of the light at the receiving and transmitting time. Since the wavelength of the light is $\lambda = c\Delta t$ we have:

$$\frac{\lambda_0}{a(t_0)} = \frac{\lambda_e}{a(t_e)} \tag{10.20}$$

This can be interpreted as a "stretching" of the electromagnetic waves due to the expansion of space. The cosmological redshift is denoted by z and is given by:

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{a(t_0)}{a(t_e)} - 1 \tag{10.21}$$

Using $a_0 \equiv a(t_0)$ we can write this as:

$$1 + z(t) = \frac{a_0}{a} \tag{10.22}$$

10.3.4 Isotropic and homogeneous universe models

We will discuss isotropic and homogenous universe models with perfect fluid and a non-vanishing cosmological constant Λ. Calculating the components of the Einstein tensor from the line-ement (10.14) we find in an orthonormal basis

$$E_{\hat{t}\hat{t}} = \frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} \tag{10.31}$$

$$E_{\hat{m}\hat{m}} = -\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2}. \tag{10.32}$$

The components of the energy-momentum tensor of a perfect fluid in a comoving orthonormal basis are

$$T_{\hat{t}\hat{t}} = \rho, \qquad T_{\hat{m}\hat{m}} = p. \tag{10.33}$$

Hence the $\hat{t}\hat{t}$ component of Einstein's field equations is

$$3\frac{\dot{a}^2 + k}{a^2} = 8\pi G\rho + \Lambda \tag{10.34}$$

$\hat{m}\hat{m}$ components:

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 8\pi Gp - \Lambda \tag{10.35}$$

where ρ is the energy density and p is the pressure. The equations with vanishing cosmological constant are called the Friedmann equations. Inserting eq. (10.34) into eq. (10.35) gives:

$$\ddot{a} = -\frac{4\pi G}{3}a(\rho + 3p) \tag{10.36}$$

If we interpret ρ as the mass density and use the speed of light c, we get

$$\ddot{a} = -\frac{4\pi G}{3}a(\rho + 3p/c^2) \tag{10.37}$$

Inserting the gravitational mass density ρ_G from eq.(9.21) this equation takes the form

$$\ddot{a} = -\frac{4\pi G}{3}a\rho_G \tag{10.38}$$

Let us consider some simple universe models.

1. Empty, flat universe model with vanishing cosmological constant: $\rho = p = k = \Lambda = 0$. Then eq.(10.34) gives $\dot{a} = 0$. Integrating with the normalization $a(t_0) = 1$ gives $a(t) = 1$. The line-element then takes the form

   $$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2$$

   This represents the Minkowski spacetime in spherical coordinates.
2. Empty universe model with vanishing cosmological constant: $\rho = p = \Lambda = 0,\ k \neq 0$. Then eq.(10.34) gives $\dot{a}^2 + k = 0$. This requires $k = -1$. For an expanding universe model we then get $\dot{a} = 1$. Integrating with the normalization $a(t_0) = 1$ gives $a(t) = t/t_0$. The line-element then takes the form

$$ds^2 = -c^2dt^2 + \left(\frac{t}{t_0}\right)^2 \left(\frac{dr^2}{1+kr^2} + r^2 d\Omega^2\right).$$

The universe model represented by this line-element is called *the Milne universe model*.

Applying the coordinate transformation

$$ct = \sqrt{c^2T^2 - R^2} \quad , \quad r = \frac{ct_0 R}{\sqrt{c^2T^2 - R^2}} \, ,$$

transforms the line-element to the form

$$ds^2 = -c^2\, dT^2 + dR^2 + R^2\, d\Omega^2 \, ,$$

which represents the Minkowski spacetime. *When there exists a coordinate transformation between two line-elements they represent the same spacetime in two coordinate systems, which may be comoving with different reference frames.* The coordinate *R* is comoving with a static reference frame, SR. The coordinate *r* is comoving with another reference frame, RF. We can find the motion of the reference particles of RF relative to those of SR as follows.

Solving the last of the two transformation equations with respect to *R* gives

$$R = \frac{rcT}{\sqrt{c^2t_0^2 + r^2}} \, .$$

The reference particles of RF have *r* = constant. Hence for these particles *R* increases linearly with *T*. This means that the frame in which *r* is comoving, is expanding with a constant expansion velocity. Hence the Milne universe is nothing but the Minkowski spacetime as described from an expanding reference frame.

3. Expanding, flat, empty (?) universe model with positive cosmological constant: $\rho = p = k = 0,\ \Lambda > 0$. For this universe model eq.(10.34) reduces to

$$H = \frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}} \, .$$

Hence the Hubble parameter is constant. Integration with $a(t_0) = 1$ gives

$$a(t) = e^{H(t-t_0)} \, .$$

The line-element takes the form

$$ds^2 = -c^2\, dt^2 + e^{2H(t-t_0)}\left(dr^2 + r^2\, d\Omega^2\right).$$

The spacetime represented by this line-element is called *the De Sitter spacetime*. It was represented by De Sitter in 1917 as a static and spherically symmetric solution of Einsteins's equations with a cosmological constant for empty space. Five years later is was shown that the reference particles of the static frame were not freely moving, and that when transforming the solution to a coordinate system comoving with freely moving reference particles, one obtained the line-element above. Also Lemaitre showed in 1933 that the cosmological constant could be interpreted to represent the constant energy density of Lorentz Invariant Vacuum Energy, LIVE.

In a universe dominated by a Lorentz-invariant vacuum the acceleration of the cosmic expansion is

$$\ddot{a}_v = \frac{8\pi G}{3} a\rho_v > 0, \tag{10.40}$$

i.e. *accelerated expansion*. This means that vacuum acts upon itself with repulsive gravitation.

The field equations can be combined into

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{10.41}$$

where ρ_m is the density of matter, $\Lambda = 8\pi G\rho_\Lambda$ where ρ_Λ is the vacuum energy with constant density. $\rho = \rho_m + \rho_\Lambda$ is the total mass density. Then we may write

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{10.42}$$

The critical density ρ_{cr} is the density in a universe with euclidean spacelike geometry, $k = 0$, which gives

$$\rho_{cr} = \frac{3H^2}{8\pi G} \tag{10.43}$$

We introduce the relative densities

$$\Omega_m = \frac{\rho_m}{\rho_{cr}}, \qquad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}} \tag{10.44}$$

Furthermore we introduce a dimensionless parameter that describes the curvature of 3-space

$$\Omega_k = -\frac{k}{a^2H^2} \tag{10.45}$$

Eq. (10.42) can now be written

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1 \tag{10.46}$$