Lecture 35. 29. May 2018

It follows from Friedmann's equations that

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0 \tag{10.51}$$

which may be written

$$(\rho a^3)^{\cdot} + p(a^3)^{\cdot} = 0 \tag{10.52}$$

Let $V = a^3$ be a comoving volume in the universe and $U = \rho V$ be the energy in the comoving volume. Then we may write

$$dU + pdV = 0 \tag{10.53}$$

This is the first law of thermodynamics for an adiabatic expansion. It follows that the universe expands adiabatically. The adiabatic equation can be written

$$\frac{\dot{\rho}}{\rho + p} = -3\frac{\dot{a}}{a} \tag{10.54}$$

Assuming $p = w\rho$ we get

$$\begin{aligned} \frac{d\rho}{\rho} &= -3(1+w)\frac{da}{a} \\ \ln\frac{\rho}{\rho_0} &= \ln\left(\frac{a}{a_0}\right)^{-3(1+w)} \end{aligned}$$

It follows that

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} \tag{10.55}$$

This equation tells how the density of different types of matter depends on the expansion factor

$$\rho a^{3(1+w)} = constant \tag{10.56}$$

Special cases:

- dust: $w = 0$ gives $\rho_d a^3 = constant$
  Thus, the mass in a comoving volume is constant.

- radiation: $w = \frac{1}{3}$ gives $\rho_r a^4 = constant$
  Thus, the radiation energy density decreases faster than the case with dust when the universe is expanding. The energy in a comoving volume is decreasing because of the thermodynamic work on the surface. In a remote past, the density of radiation must have exceeded the density of dust:

$$\begin{aligned}\rho_{d0}a_0^3 &= \rho_d a^3 \\ \rho_{r0}a_0^4 &= \rho_r a^4 \\ \frac{\rho_r a^4}{\rho_d a^3} &= \frac{\rho_{r0}a_0^4}{\rho_{d0}a_0^3}\end{aligned}$$

  The expansion factor when $\rho_r = \rho_d$:

$$a(t_1) = \frac{\rho_{r0}}{\rho_{d0}}a_0$$

- Lorentz-invariant vacuum: $w = -1$ gives $\rho_\Lambda = constant$.
  The vacuum energy in a comoving volume is increasing $\propto a^3$.

10.4 Some cosmological models

10.4.1 Radiation dominated model

The energy-momentum tensor for radiation is trace free. According to the Einstein field equations the Einstein tensor must then be trace free:

$$\begin{aligned}a\ddot{a} + \dot{a}^2 + k &= 0 \\ (a\dot{a} + kt)^{\cdot} &= 0\end{aligned} \tag{10.57}$$

Integration gives

$$a\dot{a} + kt = B \tag{10.58}$$

Another integration gives

$$\frac{1}{2}a^2 + \frac{1}{2}kt^2 = Bt + C \tag{10.59}$$

The initial condition $a(0) = 0$ gives $C = 0$. Hence

$$a = \sqrt{2Bt - kt^2} \tag{10.60}$$

For $k=0$ we have

$$a=\sqrt{2Bt}\,,\quad \dot{a}=\sqrt{\frac{B}{2t}} \tag{10.61}$$

The expansion velocity reaches infinity at $t=0$, ($\lim_{t\to 0}\dot{a}=\infty$)

10.4.2 Dust dominated model

From the first of the Friedmann equations we have

$$\dot{a}^2+k=\frac{8\pi G}{3}\rho a^2 \tag{10.64}$$

We now introduce a time parameter η given by

$$\frac{dt}{d\eta}=a(\eta)\ \Rightarrow\ \frac{d}{dt}=\frac{1}{a}\frac{d}{d\eta}$$
$$\text{So:}\quad \dot{a}=\frac{da}{dt}=\frac{1}{a}\frac{da}{d\eta} \tag{10.65}$$

We also introduce $A\equiv\frac{8\pi G}{3}\rho_0{a_0}^3$. The first Friedmann equation then gives

$$a\dot{a}^2+ka=\frac{8\pi G}{3}\rho a^3=\frac{8\pi G}{3}\rho_0{a_0}^3=A \tag{10.66}$$

the scale factor of the negatively curved, dust dominated universe model is

$$a(\eta)=\frac{1}{2}\frac{\Omega_{m0}}{1-\Omega_{m0}}(\cosh\eta-1) \tag{10.75}$$

$$t(\eta)=\frac{\Omega_{m0}}{2H_0(1-\Omega_{m0})^{3/2}}(\sinh\eta-\eta) \tag{10.76}$$

Integrating eg. (10.69) for $k=0$ leads to an Einstein-deSitter universe

$$a(t)=(\frac{t}{t_0})^{\frac{2}{3}} \tag{10.77}$$

Finally integrating eg. (10.69) for $k>0$ gives, in a similar way as for $k<0$

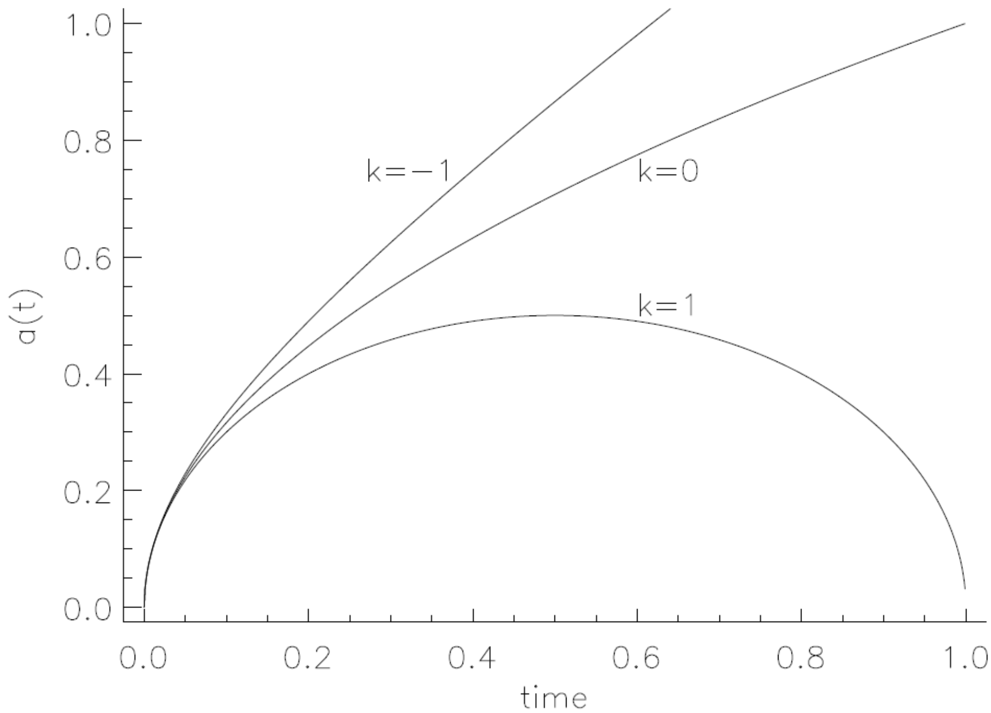
$$a(\eta)=\frac{1}{2}\frac{\Omega_{m0}}{1-\Omega_{m0}}(1-\cos\eta) \tag{10.78}$$

$$t(\eta)=\frac{\Omega_{m0}}{2H_0(\Omega_{m0}-1)^{3/2}}(\eta-\sin\eta) \tag{10.79}$$

We see that this is a parametric representation of a cycloid.

In the Einstein-deSitter model the Hubble factor is

$$\boxed{H=\frac{\dot{a}}{a}=\frac{2}{3}\frac{1}{t}\,,\quad t=\frac{2}{3}\frac{1}{H}=\frac{2}{3}t_H} \tag{10.80}$$



We now consider a flat Friedmann-Lemaitre universe model with dust and a positive cosmological constant. Using that for dust $\rho a^3 = \rho_0$, Friemann's 1. Equation then takes the form

$$a\dot{a}^2 = K + \frac{\Lambda}{3}a^3 \quad , \quad K = \frac{8\pi G\rho_0}{3} .$$

Introducing a new function *y* by $a^3 = y^2$ this equation can be written

$$\int \frac{dy}{\sqrt{K + (\Lambda/3)y^2}} = t .$$

Integrating with $a(t_0) = 1$ gives

$$a = A^{1/3} \sinh^{2/3}\left(\frac{t}{t_\Lambda}\right), \qquad A = \frac{\Omega_{M0}}{\Omega_{\Lambda 0}} = \frac{1 - \Omega_{\Lambda 0}}{\Omega_{\Lambda 0}}$$

Here

$$\Lambda = 8\pi G\rho_\Lambda = 3\Omega_{\Lambda 0} H_0^2 \quad , \quad t_\Lambda = \frac{2}{\sqrt{3\Lambda}} = \frac{2}{3}\frac{1}{\sqrt{\Omega_{\Lambda 0}}}\frac{1}{H_0} .$$

Inserting $\Omega_{\Lambda 0} = 0.7$ and $H_0 = 20$ km/s per million light years gives $t_\Lambda \approx 12 \cdot 10^9$ years.

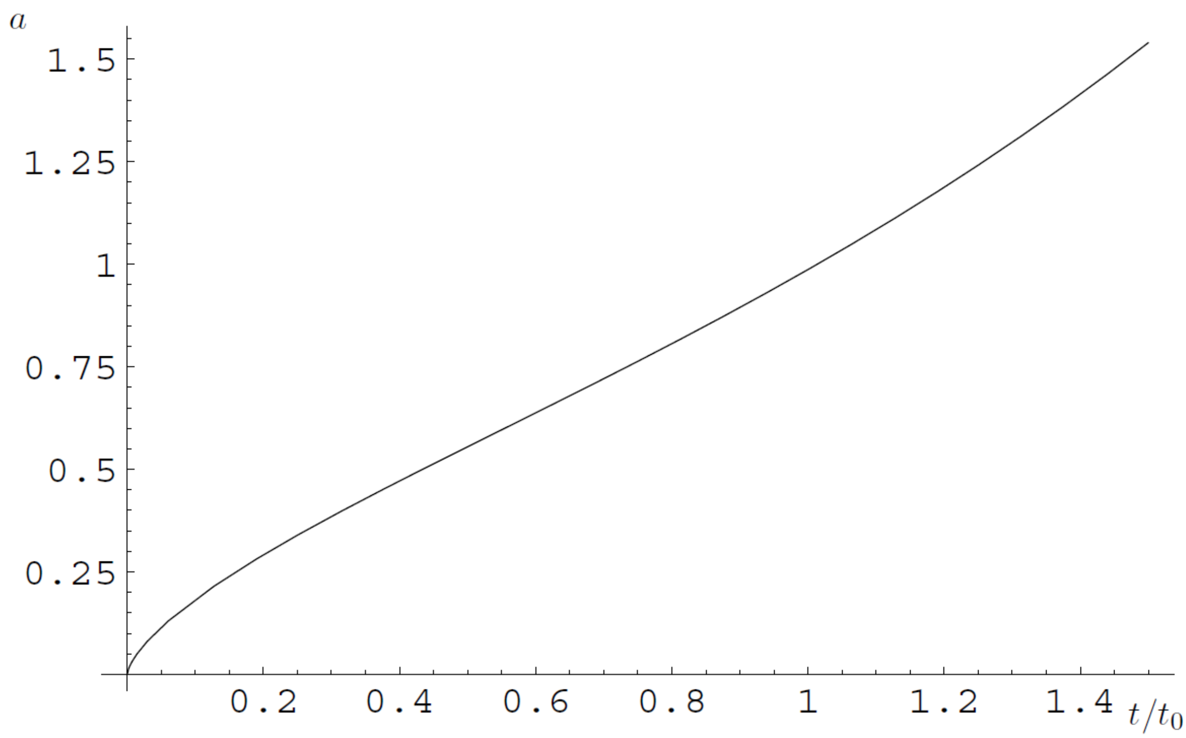


Figure 10.5: The expansion factor as function of cosmic time in units of the age of the universe.

The age t_0 of the universe is found from $a(t_0) = 1$, which by use of the formula $arc\tanh x = arc\sinh(x/\sqrt{1-x^2})$, leads to the expression

$$t_0 = t_\Lambda arc\tanh\sqrt{\Omega_{\Lambda 0}} \tag{10.99}$$

The values above give $t_0 \approx 14.5 \cdot 10^9$ years.

A dimensioness quantity representing the rate of change of the cosmic expansion velocity is the deceleration parameter, which is defined as $q = -\ddot{a}/aH^2$. For the present universe model the deceleration parameter as a function of time is

$$q = \frac{1}{2}[1 - 3\tanh^2(t/t_\Lambda)] \tag{10.103}$$

The transition from deceleration to acceleration happened when $q = 0$ giving

$$t_1 = t_\Lambda arc\tanh(1/\sqrt{3}) \tag{10.104}$$

or expressed in terms of the age of the universe

$$t_1 = \frac{arc\tanh(1/\sqrt{3})}{arc\tanh\sqrt{\Omega_{\Lambda 0}}} t_0 \tag{10.105}$$

The corresponding cosmic red shift is

$$z(t_1) = \frac{a_0}{a(t_1)} - 1 = \left(\frac{2\Omega_{\Lambda 0}}{1-\Omega_{\Lambda 0}}\right)^{1/3} - 1 \tag{10.106}$$

Inserting $\Omega_{\Lambda 0} = 0.7$ gives $t_1 = 0.54 t_0$ and $z(t_1 = 0.67$.

For this universe model the emission time for light with redshift *z* is

$$t_e = t_0 \frac{\text{arcsinh}\sqrt{\dfrac{\Omega_{\Lambda 0}}{(1-\Omega_{\Lambda 0})(1+z)^{3/2}}}}{\text{arcsinh}\sqrt{\dfrac{\Omega_{\Lambda 0}}{(1-\Omega_{\Lambda 0})}}} .$$

Inserting $\Omega_{\Lambda 0} = 0.7$ and $t_0 = 13.7$ gives for emission time in billion years

$$t_e = 11.3 \text{arsinh} \frac{1.53}{\left(1+z^{3/2}\right)} .$$